CHAPTER - 4

SOME COMMON FIXED POINT RESULTS IN GENERATING SPACES OF QUASI-METRIC FAMILY

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SOME COMMON FIXED POINT RESULTS IN GENERATING SPACES OF QUASI-METRIC FAMILY

There is multitude of common fixed point theorems in complete metric space satisfying various kinds of functional inequalities with commuting mappings. The definition of generating spaces of quasi-metric family was given by Chang et al. [09] and also some properties and examples of it. Consequently, generating spaces of fuzzy quasi-metric family and probabilistic quasi-metric family were also introduced.

This chapter is divided into three sections. In section 4.1 we have proved some fixed point theorems for non-commuting mappings and we have furnished an example for verification of our result. In section 4.2 we have given fuzzy version of section 4.1 and in section 4.3 we have given probabilistic version of section 4.1.

4.1 SOME RESULTS IN GENERATING SPACE OF QUASI-METRIC FAMILY

In this section we have proved some fixed point theorems to ensure common fixed point for noncommuting mapping in generating spaces of quasi-metric family.

Let us recall the following definitions:
DEFINITION 01. Let \( X \) be a non-empty set and \( \{d_\alpha : \alpha \in (0,1]\} \) be a family of mappings \( d_\alpha \) of \( X \times X \) into \( \mathbb{R}^+ \). \((X,d_\alpha : \alpha \in (0,1])\) is called a generating space of quasi-metric family if it satisfies the following conditions:

(4.1.1) \( d_\alpha(x,y) = 0 \) for all \( \alpha \in (0,1] \)
if and only if \( x = y \)

(4.1.2) \( d_\alpha(x,y) = d_\alpha(y,x) \) for all \( x,y \in X \) and \( \alpha \in (0,1] \).

(4.1.3) for any \( \alpha \in (0,1] \), there exist a number \( \mu \in (0,1] \)
such that
\[
\min d_\mu(x,z) + d_\mu(z,y), \quad x,y \in X,
\]

(4.1.4) for any \( x,y \in X \), \( d_\alpha(x,y) \) is nonincreasing and
left continuous in \( \alpha \).

DEFINITION 02. Let \( S \) and \( T \) be mappings from a generating space of quasi-metric family \((X,d_\alpha : \alpha \in (0,1])\) into itself. The mapping \( S \) and \( T \) are said to be quasi-compatible if
\[
d_\alpha(STx_n,TSx_n) \to 0 \text{ as } n \to \infty, \alpha \in (0,1]
\]
whenever \( \{x_n\} \) be a sequence in \( X \) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = p \text{ for some } p \in X.
\]

LEMMA 01. Let \( S \) and \( T \) be compatible mappings from a generating space of quasi-metric family \((X,d_\alpha : \alpha \in (0,1])\) into itself. Suppose that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = p \) for some \( p \in X \). Then we have the following:

(4.1.5) \( \lim_{n \to \infty} STx_n = Tp \) if \( T \) is continuous and

(4.1.6) \( STp = TSp \) and \( Sp = Tp \) if \( T \) is continuous.
PROOF. (4.1.5) Suppose that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = p \) for some \( p \in X \). Now, since \( T \) is continuous, we have
\[
\lim_{n \to \infty} STx_n = Tp.
\]
By (4.1.3), we have
\[
d_\alpha(STx_n, Tp) \leq d_\mu(STx_n, TTx_n) + d_\mu(TTx_n, Tp); \quad \mu \in (0, \alpha]
\]
Since \( S \) and \( T \) are quasi-compatible, we have
\[
\lim_{n \to \infty} STx_n = Tp.
\]
(4.1.6) Since \( T \) is continuous, \( \lim_{n \to \infty} STx_n = Tp \)
Hence, by the uniqueness of limit, we have \( Sp = Tp \).
Now, again
\[
d_\alpha(STp, TSp) = \lim_{n \to \infty} d_\alpha(STx_n, TSx_n) = 0.
\]
i.e. \( STp = TSp \).
This completes the proof.

DEFINITION 03. Let \( S \) and \( T \) be mappings from a generating space of quasi-metric family \((X, d_\alpha: \alpha \in (0,1])\) into itself. The mappings \( S \) and \( T \) are said to be compatible of type (A) if:
\[
d_\alpha(TSx_n, SSx_n) = 0 \quad \text{and} \quad d_\alpha(STx_n, TTx_n) = 0
\]
whenever \( \{x_n\} \) be a sequence in \( X \) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = p \text{ for some } p \in X.
\]

LEMMA 02. Let \( S \) and \( T \) are mappings from a generating space of quasi-metric family \((X, d_\alpha: \alpha \in (0,1])\) into itself. If \( S \) and \( T \) are compatible of type (A) for any \( \alpha \in (0,1] \) and for \( \mu \in (0, \alpha] \). Then \( STp = TTp = TSp = SSP \).
PROOF. Suppose \( \{x_n\} \) be a sequence in \( X \) defined by \( x_n = p \) as \( n \to \infty \) and \( Sx_n = Tp \).

Then we have \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = Sp \).

Since \( S \) and \( T \) have compatible of type (A), we have

\[
d_\alpha(STp, TTp) = \lim_{n \to \infty} d_\alpha(STx_n, TTx_n) = 0 ; \alpha \in (0,1].
\]

Hence we have \( STp = TTp \).

Similarly, we have \( TSp = SSp \).

But \( Tp = Sp \)

\( \to \) \( TTp = TSp \).

Therefore \( STp = TTp = TSp = SSp \).

REMARK 01. Quasi-compatible pair of maps are compatible of type (A) but converse is not always true.

THEOREM A[*]. Let \( S, T \) and \( G \) are mappings from generating space of quasi-metric family into itself satisfying the following conditions:

(4.1.7) \( S(X) \subseteq G(X) \) and \( T(X) \subseteq G(X) \),

(4.1.8) \( d_\alpha(Sx, Ty) \leq h \max \{d_\alpha(Sx,Gy), d_\alpha(Gx, Ty), d_\alpha(Gx, Gy)\} \)

for all \( x, y \in X \) and \( \alpha \in (0,1] \), where \( 0 \leq h < 1/2 \),

(4.1.9) \( G \) is continuous,

(4.1.10) the pairs \( \{S, G\} \) and \( \{T, G\} \) are quasi compatible on \( X \).

Then \( S, T \) and \( G \) have common fixed point.


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PROOF. Let $x_0$ be any point of $X$.

Since $S(X) \subseteq G(X)$ and $T(X) \subseteq S(X)$ and $TG(X) \subseteq GG(X)$, so there exist $x_1$ and $x_2$ in $X$ such that

$$GGx_1 = GGx_0 \text{ and } GGx_2 = TGx_1.$$ 

In general

$$GGx_{2n+1} = SGx_{2n} \text{ and } GGx_{2n+2} = TGx_{2n+1}$$

for $n = 0, 1, 2, 3, \ldots$.

Let

$$d_n = d_\alpha(GGX_{2n}, GGx_{2n+1}) = d_\mu(GGX_{n}, GGx_{n+1})$$

for all $\alpha \in (0, 1]$ and $\mu \in (0, \alpha]$.

Suppose $x_{2n}, x_{2n+1}$ satisfy (4.1.8), then for all $\alpha \in (0, 1)$$$

$$d_\alpha(SGx_{2n}, TGx_{2n+1}) \leq h \max\{d_\alpha(SGx_{2n}, GGx_{2n+1}),$$

$$d_\alpha(GGX_{2n}, TGx_{2n+1}), d_\alpha(GGX_{2n}, GGx_{2n+1})\}$$

$$= h \max\{d_\alpha(GGX_{2n+1}, GGx_{2n+1}),$$

$$d_\alpha(GGX_{2n}, GGx_{2n+2}), d_\alpha(GGX_{2n}, GGx_{2n+1})\}$$

$$\leq h \max\{d_\mu(GGX_{2n}, GGx_{2n+1})$$

$$+ d_\mu(GGX_{2n+1}, GGx_{2n+2}), d_\alpha(GGX_{2n}, GGx_{2n+1})\}$$

Thus by (4.1.4), we have

$$d_\alpha(SGx_{2n}, TGx_{2n+1}) \leq h \{d_\mu(GGX_{2n}, GGx_{2n+1})$$

$$+ d_\mu(GGX_{2n+1}, GGx_{2n+2})\}$$

$$= d_\alpha(GGX_{2n}, GGx_{2n+2}) \leq h \{d_\mu(GGX_{2n}, GGx_{2n+1})$$

$$+ d_\mu(GGX_{2n+1}, GGx_{2n+2})\}$$

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\[ d_{2n+1} \leq h(d_{2n} + d_{2n+1}) \]

\[ d_{2n+1} = \frac{h}{1-h} d_{2n} \]

\[ d_{2n+1} \leq d_{2n} \]

Similarly

\[ d_{2n} \geq d_{2n-1} \]

Thus \( \{d_{2n}\} \) be monotone decreasing and hence converge to zero.

Therefore \( \{Gx_{2n}\} \) is a Cauchy sequence and converge to \( Gp \) and hence to \( p \) in \( X \). Since \( \{SGx_{2n}\} \) and \( \{TGx_{2n}\} \) are subsequence of \( \{Gx_{2n}\} \) and so converges to same point \( p \).

Now by Lemma 01 we obtain

\[ SGp = TSp \] and \( Sp = Gp \)

Similarly

\[ TGp = GTp \] and \( Tp = Gp \)

and hence \( Sp = Tp = Gp \).

Also \( Sp = p = Gp = Tp \) as \( Gp = p \).

Hence \( p \) is common fixed point of \( S, T \) and \( G \).

This completes the proof.

**COROLLARY 01.** Let \( S, T \) and \( G \) are mappings from generating space of quasi-metric family into itself satisfying

(4.1.7), (4.1.8), (4.1.9) and

(4.1.11) the pairs \( \{S,G\} \) and \( \{T,G\} \) are compatible of type (A).

Then \( S, T \) and \( G \) have common fixed point.
PROOF. Similar to the proof of the Theorem A by using the Lemma 02.

4.2 Version in Fuzzy Metric Spaces

A fuzzy number is a mapping $x : \mathbb{R} \to [0,1]$. The set
\[
\{x\}_{\alpha} = \{p \in \mathbb{R} : x(p) \geq \alpha\}
\]
is called a $\alpha$-level set of fuzzy number $x$, for any $\alpha \in (0,1)$.

A fuzzy number $x$ is said to be convex if $x, t \in \mathbb{R}$,
\[
r \leq s \leq t, \min \{x(r), x(t)\} \leq x(s).
\]

If there exist a point $p \in \mathbb{R}$ such that $x(p) = 1$, then fuzzy number is normal.

If a fuzzy number $x$ is upper semicontinuous, convex and normal, then the $\alpha$-level set of $x$ is a closed interval $[a^\alpha, b^\alpha]$, i.e. $\{x\}_{\alpha} = [a^\alpha, b^\alpha]$, $\alpha \in (0,1)$, where $a^\alpha = -\infty$ and $b^\alpha = \infty$ are admissible.

A fuzzy number is called nonnegative if $x(p) = 0$ for all $p < 0$. The fuzzy number $\theta$ is defined by $\theta(p) = 1$ for $p = 0$ and $\theta(p) = 0$ for $p \neq 0$. Let $G$ be the set of all nonnegative upper semicontinuous normal convex fuzzy numbers and
\[
L, R : [0,1] \times [0,1] \to [0,1]
\]
be two functions such that they are nondecreasing in both arguments, symmetric and $L(0,0) = 0$, $R(1,1) = 1$. Let $X$ be a nonempty set and $d : X \times X \to G$ be the mapping and denote
\[
(4.2.1) \quad \{d(x,y)\}_{\alpha} = [\lambda^\alpha_{\alpha}(x,y), \rho^\alpha_{\alpha}(x,y)],
\]
for $x, y \in X, \alpha \in (0,1)$. 

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where \( \{d(x,y)\}_\alpha \) is the \( \alpha \)-level set of fuzzy number
\( d(x,y) \in G \) and \( \lambda_\alpha(x,y), \rho_\alpha(x,y) \) are the left and right end
points of the closed interval \( \{d(x,y)\}_\alpha \), respectively.

**DEFINITION 04 [39].** The quadruple \((X,d,L,R)\) is called a
fuzzy metric space if the mapping \( d: X \times X \to G \) satisfies
the following conditions:

\[
\begin{align*}
(4.2.2) & \quad d(x,y) = 0 \text{ if and only if } x = y \\
(4.2.3) & \quad d(x,y) = d(y,x) \text{ for all } x, y \in X, \\
(4.2.4) & \quad \text{For any } x, y, z \in X, \\
(4.2.4a) & \quad d(x,y)(s+t) \geq L(d(x,y)(s), d(z,y)(t)) \\
& \quad \text{whenever } s \geq \lambda_1(x,z), t \geq \lambda_1(z,y) \\
& \quad \text{and } s + t \geq \lambda_1(x,y) \\
(4.2.4b) & \quad d(x,y)(s+t) \leq R(d(x,z)(s), d(z,y)(t)) \\
& \quad \text{whenever } s \leq \lambda_1(x,z), t \leq \lambda_1(z,y) \\
& \quad \text{and } s + t \geq \lambda_1(x,y).
\end{align*}
\]

**PROPOSITION 01 [39].** Let \((X,d,L,R)\) be a fuzzy metric space
with
\[
\lim_{a \to 0^+} R(a,a) = 0, \quad \lim_{t \to 0^+} d_i(x,y)(t) = 0, \quad x, y \in X.
\]
Let \( \{d(x,y)\}_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)] \), then \((X,d_\alpha, \alpha \in (0,1])\)
is a generating space of quasi-metric family.
PROOF. Since \( \lim_{t \to 0} d(x,y)(t) = 0 \), it follows that \( d_{\alpha}(x,y) < \infty \) for all \( \alpha \in (0,1) \). Now we prove that \( [X,d_{\alpha} : \alpha \in (0,1)] \) is a generating space of quasi-metric family. It is clear that \( (X,d_{\alpha} : \alpha \in (0,1)] \) satisfies the condition (4.1.1), (4.1.2), and (4.1.4) in definition 01.

Now we have show that it also satisfies the condition (4.1.3). By the assumption that

\[
\lim_{\alpha \to 0^+} R(\alpha,\alpha) = 0, \quad \text{for any } \alpha \in (0,1),
\]

By the definition of \( d_{\mu} \), it is easy to show that

\[
s \leq \lambda_1(x,z) \text{ and } t \geq \lambda_1(z,y).
\]

(4.2.4 a) If \( s+t \leq \lambda_1(x,y) \), then for any \( c > 0 \) it follows from (4.2.4 b), that

\[
d(x,y)(s+t+2c) \leq R(d(x,z)(s+c),d(z,y)(t+c))\]

\[
\leq R(\mu,\mu) < \alpha.
\]

Hence we have, \( d_{\alpha}(x,y) < 2c + s + t \).

By the arbitrariness of \( c \), we obtain

\[
d_{\alpha}(x,y) \leq s + t = d_{\mu}(x,z) + d_{\mu}(z,y).
\]

(4.2.4 b) If \( s+t < \lambda_1(x,y) \) and \( \mu = \lambda_1(x,y) - (s+t) \) then we have

\[
1 = d(x,y)(\lambda_1(x,y)) = d(x,y)(u+s+t)\]

\[
\leq R(d(x,z)(s + \frac{1}{2} u), d(z,y)(t + \frac{1}{2} u))\]

\[
\leq R(\mu,\mu) < \alpha,
\]

which is contradiction. Therefore, the case (4.2.4 b) cannot happen this proves that \( (X,d_{\alpha} : \alpha \in (0,1)] \) satisfies the condition (4.1.3).
THEOREM B [*]: Let \((X,d,L,R)\) be complete fuzzy metric space with
\[
\lim_{t \to \infty} d(x,y)(t) = 0, \text{ for all } x,y \in X \text{ and } \lim_{a \to 0^+} R(a,a) = 0
\]
Let \(S, T\) and \(G\) are mappings from \((x,d_{\alpha})\) \(\alpha \in (0,1]\) into itself satisfying (4.1.7), (4.1.8), (4.1.9) and (4.1.10).
Then \(S, T\) and \(G\) have common fixed point.

PROOF. Similar to the proof of the Theorem A in generating space of quasi metric family on fuzzy metric space.

COROLLARY 02. Let \(S, T\) and \(G\) are mappings from \(
\{(X,d_{\alpha}); \alpha \in (0,1]\}\)
into itself satisfying (4.1.7), (4.1.8), (4.1.9) and (4.1.11). then \(S, T\) & \(G\) have common fixed point.

PROOF. Similar to the proof of corollary 3.1 in generating space of quasi-metric family on fuzzy metric space \(X\).

EXAMPLE 02. Let \(X = [0,1]\) and mappings \(S, T\) and \(G:[0,1] \to [0,1]\) is defined as
\[
Sx = \frac{x^3}{2}, \ Tx = x^2, \ Gx = x \text{ and } x_n = 1/n.
\]
Also \(d(x,y) = \frac{|x-y|}{1+|x-y|}\).
Then \((T,S)\) is quasi-compatible and hence compatible of type \((A)\) satisfying the conditions of Theorem A and Corollary 01 thus 0 is a unique fixed point.

DEFINITION 05. Let \((X,d,L,R)\) be a fuzzy metric space and \(E,F\) be two mappings from \(X\) into itself. Then \(E,F\) are said to \(R\)-weakly commutative if for some positive \(R\),
\[d(Ex,Fy) \leq Rd(Ex,Fy),\]
for all \(x,y \in X\).

THEOREM C.[*] Let \((X,d,L,R)\) be complete generating space of quasi-metric family on fuzzy metric space \(X\) with
\[
\lim_{t \to \infty} d(x,y)(t) = 0, \quad x,y \in X \quad \text{and} \quad \lim_{a \to 0^+} R(a,a) = 0.
\]
Let \(E,F,G\) be mapping from \(X\) into itself satisfying the following conditions:
\[
(4.2.5) \quad E(X) \subseteq G(X) \quad \text{and} \quad F(X) \subseteq G(X)
\]
\[
(4.2.6) \quad d_\alpha(Ex,Fy) \leq h \max\{d_\alpha(Gx,Gy),d_\alpha(Ex,Gy), d_\alpha(Gx,Fy)\}
\]
where \(0 \leq h < 1\), \(a \in (0,1]\) and \(x,y \in X\).
\[
(4.2.7) \quad (E,G) \quad \text{and} \quad (F,G) \quad \text{are \(R\)-weakly commuting maps.}
\]
\[
(4.2.8) \quad G \quad \text{is continuous.}
\]
Then there exists unique common fixed point of \(E,F,G\).

PROOF. Let \(x_0\) be any point of \(X\). From (4.2.5) we have
\[EG(X) \subseteq GG(X) \quad \text{and} \quad FG(X) \subseteq GG(X).
\]
Therefore there exists \(x_1\) and \(x_2\) in \(X\) such that
\[GGx_1 = EGx_0 \quad \text{and} \quad GGx_2 = FGx_1.
\]
In general
\[GGx_{2n+1} = EGx_{2n} \quad \text{and} \quad GGx_{2n+2} = FGx_{2n+1}
\]
for \(n = 0, 1, 2, \ldots\).

Let \( d_n = d_\alpha(GGx_{n+1}, GGx_n) \).

Then as above we can show that \( \{GGx_n\} \) is a Cauchy sequence for \( \alpha \in (0,1) \) as in Theorem A and so converge to \( Gp \) and hence to \( p \) in fuzzy metric space \( X \).

Since \( \{EGx_{2n}\} \) and \( \{FGx_{2n}\} \) are subsequences of \( \{GGx_n\} \) and so converges to same point. Since \( G \) is continuous therefore

\[ EGx_{2n} \to Ep, \]
\[ FGx_{2n} \to Fp \] and
\[ GGx_{2n} \to Gp \] as \( n \to \infty \).

Now by the definition of \( R \)-weakly commuting

\[ d_\alpha(EGx_{2n}, GGx_{2n}) \leq R d_\alpha(Ex_{2n}, Gx_{2n}) \]
\[ d_\alpha(Ep, Sp) \leq R d_\alpha(p, p) = 0 \quad \text{as } n \to \infty \]

Similarly,

\[ d_\alpha(Fp, Cp) \leq R d_\alpha(p, p) = 0 \quad \text{as } n \to \infty \]

Thus from the above inequality we have \( Ep = Gp = Fp \) and hence \( Ep = Fp = Gp = p \).

Now we establish the uniqueness of the fixed point

Let \( p \) and \( q \) are two fixed points of \( E, F \) and \( G \) such that

\[ d_\alpha(p, q) > 0 \] then we have by (4.2.6) and definition of \( R \)-weakly commutativity

\[ d_\alpha(p, q) = d_\alpha(Ep, Fq) \leq h \max \{d_\alpha(Gp, Gq), d_\alpha(Ep, Gq), \]
\[ d_\alpha(EP, Fq)\} \]
\[ = h \max \{d_\alpha(p, q), d_\alpha(p, q), d_\alpha(p, q)\} \]
\[ \leq h d_\alpha(p, q) \]
\[ \leq (1-h) d_\alpha(p, q) \leq 0 \]
\[ d_\alpha(p, q) \leq 0 \]

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which is contradiction. Thus \( \{ p \} \) is unique common fixed point of \( E, F \) and \( G \).

**REMARK 02.** If \( \alpha \) is fixed in \( (0,1] \), we obtain the following

**THEOREM D [\(*\).** Let \( (X,d,L,R) \) be complete fuzzy metric space with

\[
\lim_{t \to 0} d(x,y)(t) = 0, \quad x,y \in X \quad \text{and} \quad \lim_{a \to 0^+} R(a,a) = 0.
\]

Let \( E,F,G \) be mapping from \( X \) into itself satisfying the following conditions:

(4.2.9) \quad \( E(X) \subseteq G(X) \) and \( F(X) \subseteq G(X) \)

(4.2.10) \quad \( d(Ex,Fy) \leq h \max \{ d(Gx,Gy), d(Ex,Gy), d(Gx,Fy) \} \)

where \( 0 \leq h < 1 \) and \( x,y \in X \).

(4.2.11) \quad (E,G) \) and \( (F,G) \) are \( R \)-weakly commuting maps.

(4.2.12) \quad G \) is continuous.

Then there exists unique common fixed point of \( E,F \) and \( G \).

### 4.3 VERSION IN PROBABILISTIC METRIC SPACES

In this section, we obtain the results corresponding to the Theorem A and Theorem B in probabilistic metric space.

Throughout this section, we denote the set of all left-continuous distribution functions by \( \Delta \).

A function \( \Delta : [0,1] \times [0,1] \to [0,1] \) is called a t-torm if the following conditions are satisfied:


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\[ (4.3.1) \quad \Lambda(a, b) = \Lambda(b, a), \]
\[ (4.3.2) \quad \Lambda(a, 1) = a \]
\[ (4.3.3) \quad \Lambda(a, \Lambda(b, c)) = \Lambda(\Lambda(a, b), c), \]
\[ (4.3.4) \quad \Lambda(a, b) \leq \Lambda(c, d) \text{ for } a \leq c \text{ and } b \leq d. \]

**DEFINITION 06.** A triple \((X, F, \Lambda)\) is called a Menger probabilistic metric space (briefly, a Menger PM-space) if 
\(X\) is a nonempty set, \(\Lambda\) is a t-norm and \(F: X \times X \to D\) is a mapping satisfying the following conditions (we shall denote \(F(x, y)\) by \(F_{x, y}\)): 
\[ (4.3.5) \quad F_{x, y}(t) = 1 \text{ for all } t > 0 \text{ if and only if } x = y. \]
\[ (4.3.6) \quad F_{x, y}(0) = 0, \]
\[ (4.3.7) \quad F_{x, y} = F_{y, x}, \]
\[ (4.3.8) \quad F_{x, y}(s + t) \leq \Lambda(F_{x, z}(s), F_{y, z}(t)) \]
for all \(x, y \in X, s, t \geq 0.\)

**REMARK 03.** It is pointed out if \(\Lambda\) satisfies the condition
\(\sup \Lambda(z, t) = 1,\) then there exists a topology \(\mathcal{J}\) on \(X\) such that \((X, \mathcal{J})\) is a Hausdorff topological space and the family of sets
\[ (\mathcal{U}(p) = \{ U_p(c, \lambda) : c > 0, \lambda \in (0, 1) \}, \ p \in X), \]
is a basis of neighborhoods of the point \(p\) for \(\mathcal{J}\), where
\[ U_p(c, \lambda) = \{ x \in X : F_{x, p}(c) > 1 - \lambda \}. \]

Usually, the topology \(\mathcal{J}\) is called \((c, \lambda)\)-topology on \((X, F, \Lambda)\).
PROPOSITION 02 [09]. Let \((X, P, \Delta)\) be a Menger probabilistic metric space with the \(t\)-norm \(\Delta\) satisfying the condition:

\[
\text{(4.3.9)} \quad \sup_{t<1} \Delta(t, t) = 1.
\]

For any \(\alpha \in (0, 1)\), we define \(d_\alpha : X \times X \to \mathbb{R}^+\) as follows:

\[
\text{(4.3.10)} \quad d_\alpha(x, y) = \inf \{t > 0 : F_{x, y}(t) > 1 - \alpha\}.
\]

Then

\[
\text{(4.3.11)} \quad (X, d_\alpha : \alpha \in (0, 1]) \text{ is a generating space of quasi metric family;}
\]

\[
\text{(4.3.12)} \quad \text{the topology } \mathcal{V}(d_\alpha) \text{ on } (X, d_\alpha : \alpha \in (0, 1]) \text{ coincides with the } (\varepsilon, \lambda)\text{-topology } \mathcal{V} \text{ on } (X, P, \Delta).
\]

**Proof:** \((4.3.11)\) From the definition of \(\{d_\alpha : \alpha \in (0, 1]\}\), it is easy to see that \(\{d_\alpha : \alpha \in (0, 1]\}\) satisfies the conditions \((4.1.1)\) and \((4.1.2)\) in Definition 01. Besides, it follows clearly that \(d_\alpha\) is nonincreasing in \(\alpha\).

Next, we prove that \(d_\alpha\) is left continuous in \(\alpha\). In fact, for any given \(\alpha_1 \in (0, 1]\) and \(\varepsilon > 0\), form the definition of \(d_\alpha\), there exists a \(t_1 > 0\) such that

\[
t_1 < d_{\alpha_1}(x, y) + \varepsilon \quad \text{and} \quad F_{x, y}(t_1) > 1 - \alpha.
\]

Letting \(\delta = F_{x, y}(T_1) - (1 - \alpha_1) > 0\) and \(\lambda \in (\alpha_1 - \delta, \alpha_1]\), we have

\[
1 - \alpha_1 < 1 - \lambda < 1 - (\alpha_1 - \delta) = F_{x, y}(t_1),
\]

which implies that \(t_1 \in \{t > 0 : F_{x, y}(t) > 1 - \lambda\}\). Hence we have

\[
d_{\alpha_1}(x, y) = d_\lambda(x, y) = \inf \{t > 0 : F_{x, y}(t) > 1 - \lambda\}
\]

\[
\leq t_1 < d_{\alpha_1}(x, y) + \varepsilon.
\]

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which shows that $d_\alpha$ is left continuous in $\alpha$.

Finally, we prove that $(X,d_\alpha : \alpha \in (0,1])$ also satisfies the condition (4.1.3).

By the condition (4.3.9), for any given $\alpha \in (0,1]$, there exists an $\mu \in (0,\alpha]$ such that
\[ \Delta(1-\mu, 1-\mu) > 1-\alpha. \]

Letting $d_\mu(x,z) = \sigma$ and $d_\mu(z,y) = \beta$, from (4.3.10), for any given $c > 0$, we have
\[ F_{x,z}(\sigma+c) > 1-\mu, \quad F_{z,y}(\beta+c) > 1-\mu \]
and so
\[ F_{x,y}(\sigma+\beta+2c) \geq \Delta(F_{x,z}(\sigma+c), F_{z,y}(\beta+c)) \geq \Delta(1-\mu, 1-\mu) > 1-\alpha. \]

Hence we have
\[ d_\alpha(x,y) = (\sigma+\beta+2c) = d_\mu(x,z) + d_\mu(z,y) + 2c. \]

By the arbitrariness of $c > 0$, we have
\[ d_m(x,y) \leq d_\mu(x,z) + d_\mu(z,y). \]

(4.3.12) To prove the conclusion (4.3.12), it is enough to prove that for any $c > 0$ and $\alpha \in (0,1]$,
\[ d_\alpha(x,y) > c \text{ if and only if } F_{x,y}(c) > 1-\alpha. \]

In fact, if $d_\alpha(x,y) < c$,
by (4.3.10), we have $F_{x,y}(c-\mu) > 1-\alpha$.

Conversely, if $F_{x,y}(c) > 1-\alpha$, since $F_{x,y}$ is a left continuous distribution function, there exists an $\mu > 0$ such that $F_{x,y}(c-\mu) > 1-\alpha$, and so $d_\alpha(x,y) \leq c-\mu < c$.

This completes the proof.
THEOREM E. Let \((X,F,\Delta)\) be complete Menger probabilistic metric spaces with the \(t\)-norm and \(\Delta\) satisfying the condition 4.3.1 to 4.3.4. Let \(S, T\) and \(G\) are two self mappings satisfying

\[
\text{(4.3.13)} \quad S(X) \preceq G(X) \text{ and } T(X) \preceq G(X),
\]

\[
\text{(4.3.14)} \quad \inf \{t > 0 : \pi_{FSx,Ty}(t) > 1-\alpha\} \\
< h \max \{\inf \{t > 0 : \pi_{FSx,Gy}(t) > 1-\alpha\}, \inf \{t > 0 : \pi_{FGx,Ty}(t) > 1-\alpha\}, \inf \{t > 0 : \pi_{FGx,Gy}(t) > 1-\alpha\}\}
\]

for all \(x, y \in X\) and \(\alpha \in (0,1]\), where \(0\leq h < 1/2\),

\[
\text{(4.3.15)} \quad G \text{ is continuous},
\]

\[
\text{(4.3.16)} \quad \text{the pairs } \{S,G\} \text{ and } \{T,G\} \text{ are quasi-compatible on } X.
\]

Then \(S, T\) and \(G\) have common fixed point.

PROOF. Similar to the proof of the Theorem A in generating space of Quasi metric family on complete Menger probabilistic metric spaces.

COROLLARY 03. Let \((X,F,\Delta)\) be complete Menger probabilistic metric spaces with the \(t\)-norm \(\Delta\) satisfying the \((4.3.9)\). Let \(S, T\) and \(G\) are two self mappings satisfying \((4.3.13)\), \((4.3.14)\), \((4.3.15)\) and \((4.1.11)\). Then \(S, T\) and \(G\) have common fixed point.

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