CHAPTER - 1

INTRODUCTION

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Numerous questions of physical world lead to nonlinear problems. Few of them are - behavior of plastic materials, moments of viscous fluids, chemical reaction; processes in nuclear reactors, nonlinear oscillation in physics, chemistry and biology, gravitational effect of masses in the content of general relativity etc. These nonlinear problems can be reduced to nonlinear operator equations. Here comes fixed point theory. Fixed point theorems constitute an important tool for proving the existence of solutions to such equations. For details, one can refer to Lion [45], Martin [47], Mercier [48], Peitgen [55], Robinson [64], Smart [74], Swaminathan [76], Tarter[79], Waltman [82] and Zeidler [87].

The idea of "Fixed points" was first introduced by Poincare [56] while studying the classical problems of vector distribution on surface. Now Fixed point theory has become an important branch of nonlinear analysis as well as of Topology. The study of fixed point theorems and their application though initiated long ago, still continues to be a highly interesting and useful area of investigations.
A point which is invariant under any transformation is called fixed point. A fixed point \( x \), under any transformation \( T \) is a solution of the functional equation \( T(x) = x \), where \( T : X \rightarrow X \).

Fixed point theorem is a combination of condition on the set \( X \) and the mapping \( T : X \rightarrow X \) which, in turn, assure that \( T \) has atleast one point of \( X \) fixed.

**CONTRACTION MAPPING**

One of the most important theorems of Fixed point theory is Banach contraction theorem. It is proved by Banach as below:

Let \( T \) be a contraction mapping from closed subset \( M \) of complete metric space \( (X,d) \) into itself if there exists a positive real number \( k < 1 \) such that for all \( x,y \in M \),

\[
d(Tx,Ty) \leq k \, d(x,y) \quad (k < 1)
\]

Geometrically we say that for any point \( x \) and \( y \) have images that are closer to the points \( x \) and \( y \). More precisely, the ratio \( d(Tx,Ty) / d(x,y) \) does not exceed by a constant \( k \) which is strictly less than 1. Remarkably, the contraction is thus a Lipschitz mapping with Lipschitz constant \( k < 1 \). Since this contraction was for the first time introduced by Banach, so it is termed as Banach contraction mapping. By using this contraction he proved that this \( T \) has a unique fixed point in complete metric space.
Banach contraction [03] theorem being one of the most important fixed point results has developed separate line of investigation in field of fixed point theory. The first branch was determined by the mapping known as contraction. Banach himself has considered this mapping while proving his fixed point result [03]. Beauty of this mapping lies in the fact that on one side, it gives the existence of a fixed point and on the other side with the help of this mapping in the theorem, Banach has given a systematic procedure for calculating fixed point.

**Commuting Mapping**

Number of authors Rakotch [59], Sehgal [66], Ciric [12], Ekeland [19] etc. have made generalization of Banach fixed point theorem by weakening its hypothesis while retaining the convergence property of successive iterates to the unique fixed point of mapping. Important results concerning contractive mappings have been reviewed by Rhoadas [60].

When many mathematician were seeking the conditions for a pair of mappings which may guarantee the existence and uniqueness of a common fixed point, Kannan [40] defined condition for a common fixed point on complete metric space. Worthwhile contributions were made on common fixed point in the comparable line of approach by several authors Ciric [13], Iseki [30], Kannan ([40], [41])
Rhoadas [60] and Yeh ([85], [86]).

Further, Jungck [32] generalized the Banach contraction principle by introducing a contraction condition for a pair of Commuting Mapping in complete metric space. And in the last few years Jungck's result has been generalized by Chang ([6],[7]), Das and Dabala [14], Fisher [22], Fisher and Sessa [23], Popa [57] Singh and Pant [72], Singh and Singh [73] and Yeh [86] etc.

**NONCOMMUTING MAPPING**

A pair of mappings is said to be Noncommuting mapping if they are not commuting mappings. In 1982, Sessa [67] introduced the concept of weakly commuting mappings, which is a generalization of the concept of commuting mappings, and he and others proved some fixed point theorems for weakly commuting mappings e.g. Jungck ([34], [35], [36]), Popa [58], Rhoades, Park and Moon [62], Rhoades and Sessa [63], Sessa [67], Singh, Ha and Cho [71] etc.

Recently, Jungck [34] proposed a generalization of the concept of weakly commuting mappings, which is called compatible mappings, and he generalized some fixed point theorems of Meir-Keeler type, especially, a theorem of Park Bae [54] and in [38] Jungck Murthy and Cho introduced the concept of compatible of type (A) on metric spaces and obtained some fixed point theorems for this mapping.
In this thesis we have used R-Commuting mapping, Compatible mapping, Compatible mapping of type (A), D-Hybrid Compatible, Baised map, Weakly Baised map and finally $G^*$-uniformly Lipschitzian mapping and $G^*$-uniformly Lipschitzian semigroup.

We have divided the thesis in five chapters and each chapter is subdivided into sections.

**HILBERT SPACE**

An inner product induces a norm on the linear. If such a space is complete, then it is known as a Hilbert space. Every continuous linear functional on a Hilbert space is obtained by taking inner product with a fixed element of that space. This simplifies the considerations of weak convergence and weak boundedness and yields a uniform boundedness principle for continuous linear functional on a Hilbert space.

The result of Von Neumann [52] shows that a Hilbert space is a complete normed space in which the norm satisfies the parallelogram law. The main geometric concept of orthogonality is absent in an abstract Banach space, we shall see that such a concept is possible in Hilbert space. In fact, Hilbert spaces possess many interesting properties of the Euclidean space.

It is easy to see that a subspace of a Hilbert space is a Hilbert space with the induced inner product if and
only if the subspace is closed.

Chapter 2 is entitled as "FIXED POINT RESULTS IN HILBERT AND BANACH SPACES". In this chapter we have proved some fixed point results for three noncommuting maps under compatible of type(A) condition in Hilbert space. Further some results are proved for sequence of mappings with biased map [37] which satisfies Mire Keeler type contraction. Then our results extend the result of Naimpally and Sigh [51].

**SEMIFIELD CONDITION**

A commutative associative topological ring is topological semifield if there is isolation in some set satisfying some axioms (mentioned in chapter). In topological semifield the multiplication requires its partial invertibility. It is from this fact that we name it "semifield". The topological semifield was defined by Anyonovskii, Biltyanskii and Sarymaskov [01]. Any topological semifield contain a topological field isomorphic with the real line.

Iseki [29] has established fixed point theorems under contraction condition in metric space over topological semifield and then number of authors Saltawhite [65], Sharma [68] and Mial [46] etc. have established fixed point theorems under contraction condition with commuting and noncommuting mappings.
In the same manner we have proved some results under Meir-Keeler type contraction with different noncommuting mappings for sequence of mappings in metric space over topological semifield.

Chapter 3 is entitled "FIXED POINTS UNDER SEMIFIELD CONDITION". In this chapter we have proved some fixed point theorems for sequence of mappings under Meir Keeler type contraction condition with noncommuting mappings in metric space over topological semifield. In the later part of the chapter we have obtained some results for sequence of mappings with D-Hybrid compatible condition. We have furnished example in support of our results. Our results generalize the results of Sharma [68] and Miai [46].

**GENERATING SPACE OF QUASI METRIC FAMILY**

Chang et.al. [09] have defined the definition of generating space of quasi metric family and they established some various coincidence point theorems and minimization theorem in such space. It is noted that, let 

\((X,d)\) be a metric space and 

\[d_\alpha(x,y) = d(x,y) \text{ for all } \alpha \in (0,1] \text{ and } x,y \in X,\]

then \((X,d)\) is a generating space of quasi metric family. Further more, every fuzzy metric space and probabilistic metric space, both are generating space of quasi metric family.
It is noted that if \( (X,d_{\alpha} : \alpha \in (0,1]) \) is a generating space of quasi metric family, then there exists a topology \( \mathcal{J}(d_{\alpha}) \) on \( X \) such that \( (X,\mathcal{J}(d_{\alpha})) \) is a Hausdorff topological space and

\[
U(x) = \{ U_x(c,\alpha) : c > 0, \alpha \in (0,1] \}, \quad x \in X,
\]

is a basis of neighborhood of the point \( x \) for the topology \( \mathcal{J}(d_{\alpha}) \), where \( U_x(c,\alpha) = \{ y \in X : d_{\alpha}(x,y) < c \} \).

Chapter 4 is entitled "SOME COMMON FIXED POINT RESULT IN GENERATING SPACE OF QUASI METRIC FAMILY". In this chapter we have proved some fixed point results for noncommuting maps. Further, in the later part of the chapter we have also given fuzzy and probabilistic version. Further more we have furnished an example which verifies our results.

MINIMIZATION THEOREM AND FIXED POINT

In the last few years, Caristi's fixed point theorem [05] and Ekeland's variational principle [20] have been very useful tools in nonlinear analysis, optional control theory, geometric theory of Banach space. Downing and Kirk [17], Park [53] and others generalizes this theorem for single valued mapping and the multivalued versions of the theorem were obtained by Chang and Luo [08]. Later, Takahashi [77] proved a non convex
A minimization theorem in complete metric space and Jung et al. [31] established a Takahashi type minimizations theorem [77] in the complete fuzzy metric space and obtained the analogous of Dowling Kirk's fixed point theorem [17].

Recently, many great developments have been made in the theory and application of 2-metric. In this chapter we introduce the concept of generating space of quasi 2-metric family.

Above mentioned application aspects of minimization theorem naturally provide two investigations. One - to construct a nonconvex minimization theorem for generating space of quasi 2-metric family. Second to prove a fixed point theorem with the application of such minimization theorem.

Chapter 5 is entitled "MINIMIZATION THEOREM IN GENERATING SPACE OF QUASI 2-METRIC FAMILY ". In this chapter, we have introduced the concept of generating space of quasi metric family by the inspiration of 2-metric space and generating space of quasi metric family and proved some fixed point theorems. In the later part of the chapter we have proved minimization theorem and as an application of this minimization theorem we have given fixed point theorem. At the end of the chapter we have given fuzzy version of minimization theorem.
Probably the fixed point theorem most frequently cited in analysis is the "Banach contraction mapping Principle" which asserts that if \((M,d)\) is complete metric space and if \(T\) is self mapping of \(M\) which satisfies for fixed \(k < 1\) and all \(x,y \in M\),

\[d(Tx, Ty) \leq k d(x, y),\]

then \(T\) has a unique fixed point in \(M\), and more over for each \(x \in M\) the Picard iterates \(\{T^n(x)\}\) converge to this fixed point.

Within the context of complete metric space, the assumption \(k < 1\) is crucial even to the existence part of the result. But, within more general yet quite natural setting an elaborate fixed point theory exists for the case \(k = 1\). This wider class of mappings are called nonexpansive mappings.

The possible answer to an important question in the theory of nonexpansive have been studied extensively:

Given a Banach space \(X\) and a nonempty (and, generally, bounded closed convex) subset \(K\) of \(X\), what further assumption on \(K\) (or \(X\)) guarantee the existence of fixed points for every nonexpansive self mapping of \(K\)?

We have some definitions and results which have been evolved during the study of the above question. These include the existence of common fixed point for uniformly
k Lipschitzian mapping for \( k > 1 \), and certain existence results to wider classes of mapping and different space structure, e.g. \( G^*\)-Uniformly Lipschitzian mapping in the \( p\)-Uniformly Banach space, which more general than Lipschitzian and \( k\)-Lipschitzian mapping. Also be extend these results for \( G^*\)-Uniformly Lipschitzian semigroup these are more general than Lipschitzian and \( k\)-Lipschitzian semigroup of mappings in \( p\)-Uniformly Banach space.

Chapter 6 is entitled "FIXED POINT RESULTS FOR \( G^*\)-UNIFORMLY LIPSCHITZIAN MAPPING". In this chapter we have defined \( G^*\)-uniformly Lipschitzian mapping which is much more general than uniformly \( K\)-Lipschitzian and proved some fixed point theorems in \( p\)-uniformly Banach space and Banach space. Also we have defined \( G^*\)-uniformly Lipschitzian semigroup and proved fixed point theorems in \( p\)-uniformly Banach space and Banach space.

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