1.1 History of Reynolds number

The non-dimensional parameter called Reynolds number was discovered by all time famous Irish engineer and physicist Osborne Reynolds of the University of Manchester in 1883. Osborne Reynolds was born in Belfast, Ireland on August 23, 1842. Reynolds early work was on magnetism and electricity but soon concentrated on hydraulics and hydrodynamics. He also worked on electromagnetic properties of the sun and of comets, and considered tidal motions in rivers. In the 1870’s Reynolds turned his powerful intellect to the problem of propeller cavitation. After 1873, Reynolds concentrated mainly on fluid dynamics and it was in this area that his contributions were of world leading importance. Reynolds also studied the interaction of air and water, built model ships, and figured out how to make screw propulsion more efficient. This work led to a better understanding of pumps, and in 1875 Reynolds even anticipated multi-stage steam turbines of the Parson’s type. His work led to election into the Royal Society in 1877. But it was his publications in 1883(an experimental investigation of the circumstances which determine whether the motion of water in parallel channels shall be direct or sinuous), 1885(the law of resistance in parallel channels), and 1889(hydrodynamic stability) concerning hydrodynamics that brought him universal recognition. Reynolds identified the fundamental dimensionless parameter that characterizes the behavior of flowing fluids known appropriately as Reynolds number. It was the ratio that shows the effect of viscosity in a given medium that governs the switch between laminar and turbulent flow. The author (of Biography in Encyclopaedia Britannica) wrote:-

*His studies of condensation and heat transfer between solids and fluids brought radical revision in boiler and condenser design, while his work on turbine pumps permitted their rapid development.*

Reynolds also formulated the theory of lubrication(1886) and developed the standard mathematical framework used in fluid mechanics. Not only did Reynolds develop the fundamental theory, but also applied it to a variety of engineering problems ranging from the condensation of steam and gas flow to rivers and tidal motion. During his period of office, Reynolds produced a stream of first rank papers on all kinds of physical and engineering phenomena. His contributions to Engineering Science were pivotal. The many formal honours that followed are a measure of his distinction-he was elected an Honorary Fellow of Queens in 1882, a Fellow of the Royal Society in 1877(from which he
received the Royal Medal in 1888), an honorary LL.D. of the University of Glasgow in 1884, President of the Manchester Literary and Philosophical Society in 1888. He received the Dalton Medal in 1903. Just before his retirement, his collected works, “Papers on Mechanical and Physical Subjects”, were published in three volumes by Cambridge University Press.

It is impossible to review the vast range of Reynolds contribution in few pages; not only did he develop fundamental theory, but he applied the principles to a wide range of engineering problem including ship propulsion, pumps, turbines, estuaries of rivers, cavitations, condensation of steam, thermodynamics of gas flow, rolling friction and lubrication (developed the theory of lubrication in 1886). Osborne Reynolds 37 year career dedicated to the science of engineering served as a bridge between the self-taught gentlemen engineers and scientists who come before, and the university-educated specialists, Ernest Rutherford, took up Reynolds position at Owens College in 1907. By the beginning of the 1900’s Reynolds health began to fail and retired in 1905, as a Prof. of Mechanical Engineering, Manchester, England. Not only did he deteriorate physically but also mentally, which was sad to see in so brilliant a man who was hardly 60 years old. He died on February 21, 1912 at Watchet, Somerset, England.

This non-dimensional parameter ‘R’ is important in wind tunnel experiments since it relates to the aerodynamic properties of lifting surfaces (like airfoils) when extrapolating from small wind tunnel test models to full-size wings. Reynolds discovered that, if the same atmospheric pressure were used for experiments with wind tunnel model as a full-size aircraft would encounter under actual conditions, the experimental results would be invalid. In fluid mechanics, the Reynolds number is the ratio of inertial forces \((v \rho L)\) to viscous forces \((\mu / L)\) and is used for determining whether a flow will be laminar or turbulent. It is the most important dimensionless number in fluid dynamics and provides a criterion for determining dynamic similitude. When two similar objects in perhaps different fluids with possibly different flow rates have similar fluid flow around them, they are said to be dynamically similar.

In order for results obtained with a scale model in wind tunnel experiments to be valid, the Reynolds number needs to be the same under wind tunnel conditions and in regular atmospheric conditions. The way to ensure this is to increase the air density inside the tunnel by the same proportion as the model is smaller than the full-size aircraft. In practical terms, if a model is 1/10 the size of a full-size aircraft, then the air density (the number of atmospheres) inside the tunnel must be increased by a factor of 10 to get wind tunnel results that are valid in regular atmospheric conditions with a full-size aircraft.
Reynolds discovered the ratio that has since been called the Reynolds number when examining fluid flow characteristic—how a liquid flows in a pipe or how air flows across an aircraft wing. He demonstrated that the motion of a fluid may be either laminar (in smooth layers) or turbulent, and that the change from a laminar flow to a turbulent flow can happen suddenly. The transition from a smooth laminar flow to a turbulent flow always occurred when the ratio $\rho_1 V D / \mu$ was the same, where $\rho_1$ = density of the fluid, $V$ = velocity, $D$ = pipe diameter, and $\mu$ = fluid viscosity. This ratio is now known as the Reynolds number. The variable that can be adjusted inside a wind tunnel is $\rho_1$ – its density, and it would be adjusted by the same proportion as the model is smaller than the actual aircraft: a $1/10^\text{th}$ model would produce valid results if the atmospheric pressure in the wind tunnel were increased by a factor of 10. In actual subsonic flight, airfoils with low Reynolds number flows are laminar and those with high Reynolds number flows are mostly turbulent, keeping in mind that the Reynolds number is the ratio between density, velocity, diameter, and viscosity (For an airfoil in flight rather than in a wind tunnel, $D$ would be the distance between the leading and trailing edge called the chord length along a flow).

This non-dimensional parameter $Re$, is given as follows for flow through a pipe:

$$Re = \frac{\rho_1 v_s L}{\mu} = \frac{v_s L}{v}$$

where

- $v_s$ = mean fluid velocity or characteristic velocity,
- $L$ = characteristic length scale (such as diameter of a pipe, diameter or length of a body,
- $\mu$ = (absolute) dynamic fluid viscosity (viscosity coefficient),
- $v$ = kinematic fluid viscosity ($v$ = $\mu/\rho_1$),
- $\rho_1$ = fluid density.

Laminar flow occurs at low Reynolds numbers, where viscous forces are dominant and is characterized by smooth, constant fluid motion, while turbulent flow, on the other hand, occurs at high Reynolds numbers and is dominated by inertial forces, producing random eddies, vortices and other flow fluctuations. Reynolds demonstrated, first in the history of fluid mechanics, that the changes from laminar to turbulent flow in a pipe occur when the Reynolds number $Re$ exceeds 2100. The Reynolds number for laminar flow in cylindrical pipes is about 1000. The transition between laminar and turbulent flow is often indicated by a critical Reynolds number, $R_{crit}$, which depends on the exact flow configuration and must be determined experimentally. Within a certain range around this point there is a region of gradual transition where the flow is neither
fully laminar nor fully turbulent, and predictions of fluid behaviour can be difficult. For example, within circular pipes the critical Reynolds number is generally accepted to be 2300, where the Reynolds number is based on the pipe diameter and the mean velocity $v_s$ within the pipe, but engineers will avoid any pipe configuration that falls within the range of Reynolds numbers from about 2000 to 4000 to ensure that the flow is either laminar or turbulent.

1.2 Low Reynolds Number Flow (Slow flow or creeping flow)

Whitehead (1889) attempted to improve Stokes formula for the drag force on a sphere by reintroducing through iteration the inertia terms in the Navier-Stokes equations for slow motion. The method did not succeed as it was not possible to obtain a uniformly valid second approximation for the velocity field. The non-existence of second and higher order approximations to Stokes equation is known as “Whitehead paradox”. Oseen (1910) examined the relative magnitude of the inertia terms neglected by Stokes to those retained viscous terms. He suggested that the inertia terms can no longer be treated small far from the body. Instead of neglecting the inertia terms, he linearized them and the resulting Oseen’s equation

$$\frac{\hat{u}}{\hat{c}_x} = -\frac{1}{\rho_1} \text{grad} p + \nu \nabla^2 \hat{u},$$

provided uniformly valid first approximation for flow past a sphere from which he obtained the drag as

$$F = 6\pi \mu U_a \left[1 + \frac{3}{8} R + O(R^2)\right],$$

where $R = U_a/\nu$ is Reynolds number regarding flow.

1.3 Stokes approximation; Stokes Drag

In physical and biological science, and in engineering, there is a wide range of problems of interest like sedimentation problem, lubrication processes etc. concerning the flow of a viscous fluid in which a solitary or a large number of bodies of microscopic scale are moving, either being carried about passively by the flow, such as solid particles in sedimentation, or moving actively as in the locomotion of micro-organisms. In the case of suspensions containing small particles, the presence of the particles will influence the bulk properties of the suspension, which is a subject of general interest in Rheology. In the motion of
micro-organisms, the propulsion velocity depends critically on their body shapes and modes of motion, as evidenced in the flagellar and ciliary movements and their variations. A common feature of these flow phenomena is that the motion of the small objects relative to the surrounding fluid has a small characteristic Reynolds number Re. Typical values of Re may range from order unity, for sand particles settling in water, for example, down to $10^{-2}$ to $10^{-6}$, for various micro-organisms. In this low range of Reynolds numbers, the inertia of the surrounding fluid becomes insignificant compared with viscous effects and is generally neglected and the Navier-Stokes equations of motion reduce to the Stokes equations as a first approximation. The zero Reynolds number flow is called Stokes flow. All these motions are characterized by low Reynolds numbers and are described by the solution of the Stokes equations. Although the Stokes equations are linear, to obtain exact solutions to them for arbitrary body shapes or complicated flow conditions is still a formidable task. There are only relatively few problems in which it is possible to solve exactly the creeping motion equations for flow around a single isolated solid body. The study of Stokes drag starts some 150 years ago when Stokes (1851) published his celebrated paper, “On the effects of internal friction of fluids on the motion of pendulums” and gave the well known Stokes drag formula

$$F = 6\pi \mu a U.$$  \hfill (1.3.1)

Sir George Gabriel Stokes was the first man after Sir Isac Newton to hold the three positions of Lucasian Professor of mathematics at Cambridge University, secretary, and then president of the Royal Society. Isac Newton recognizing the role of fluid friction propounded his law of viscosity relating shearing strain and thereby laid the corner stone of mathematical theory of viscous flow. Stokes (1851) gave it final shape by deriving the equations of motion of a viscous fluid, now known as the Navier-Stokes equations which in vector form for the steady flow are expressible as

$$\mathbf{u}.\text{grad}\, \mathbf{u} = -(1/\rho)\, \text{grad}\, p + \nu \nabla^2 \mathbf{u} \hfill (1.3.2)$$

where $\mathbf{u}$ is the fluid velocity, $p$ the pressure, $\rho$, the density and $\nu = \mu/\rho$, the kinematic viscosity, $\mu$ being coefficient of viscosity. The above equations are to be supplemented by the equation of continuity which for an incompressible fluid assumes the form

$$\text{div}\, \mathbf{u} = 0.$$  \hfill (1.3.3)
In equations (1.3.2) the left hand side terms are the convective inertia terms and the right hand first and second terms are pressure and viscous terms respectively. The convective inertia terms are seen to be non-linear and are the cause of mathematical difficulties in arriving conveniently at a solution. Exact closed form solutions have only be derived for only a few cases where the inertia terms identically vanish or become simple enough on account of the symmetry of the problem.

For very slow motion, taking \( \mathbf{u} \) to be small, Stokes(1851) neglected the quadratic inertia terms altogether and arrived at the linear equations

\[
0 = -(1/\rho) \nabla p + \nu \nabla^2 \mathbf{u}.
\]

He used the above linearized equations to calculate the viscous damping of a spherical pendulum bob due to air resistance (Stokes, 1851). The force of resistance acting on a sphere of radius ‘a’ was evaluated as \( 6\pi \mu a U \), \( \mu \) is the viscosity and \( U \) the speed of the sphere. In his famous oil-drop experiment Robert Andrew Millikan (1911), the Nobel laureate (1923) in physics for his study on the elementary electronic charge and the photoelectric effect, used the above Stokes resistance law to measure the charge of a single electron. Earlier the resistance law was used to determine the viscosity of a liquid by observing the fall of a sphere in the liquid.

A better criterion for the neglect of inertia terms emerges if we render the Navier-Stokes equations dimensionless by taking \( U \) to be characteristic velocity, \( L \) as characteristic length and \( \rho \nu U \) as characteristic pressure. Thus, equations (1.3.2) assumes the non-dimensional form

\[
R (\mathbf{u} \cdot \nabla \mathbf{u}) = - \nabla p + \nabla^2 \mathbf{u},
\]

where \( R = UL/\nu \), and the same symbols \( u \) and \( p \) have been retained for the non-dimensional velocity and pressure respectively. Here, Reynolds number ‘\( R \)’ is seen to be a non-dimensional parameter and represents the ratio of inertia term and viscous term. We immediately obtain, on taking \( R \rightarrow 0 \), the non-dimensional Stokes equations

\[
0 = - \nabla p + \nabla^2 \mathbf{u}.
\]

The above analysis reveals that Stokes equations are applicable not only for slow motion (\( U \) small) but also when the particle is of small linear dimension (\( L \) small) or when the fluid is highly viscous (\( \nu \) large i.e., \( \mu \) large, \( \rho \) small). This understanding widens the scope of applications. The study of Stokes flow

An impetus to Stokes flow problems was imparted by the publication of work on flagellar locomotion by Lighthill (1976). Sir Michael James Lighthill is acknowledged as one of the great mathematical scientists of twentieth century. In the footsteps of Newton and Stokes he occupied the chain of Lucasian Professor of mathematics succeeding Paul Dirac the founder of quantum mechanics. He took the innovative step of replacing a long, narrow, cylindrical flagellum by a distribution of Stokeslet and dipoles, solutions of Stokes equations.

Although Stokes equations are only approximations to the full Navier-Stokes equations obtained by complete neglect of non-linear inertia terms, yet they are not devoid of physical significance and depict a true picture of the state of motion in many problems of practical interest, particularly in the fields of biomechanics and chemical engineering where we have to study problems such as locomotion of micro-organisms, flow of mucus in lungs and movement of tiny particles involving slow motion, small linear dimensions or high viscosity so that the Reynolds number is small for reasons other than the slowness of the motion or the microscopic nature of the body. Thus flight of an object through rarefied air high above earth’s surface may represent a very viscous flow even though the air through which the object passes has a very low viscosity, because its density is correspondingly much lower. Of course, in this case, the dimension of the object is large compared with the mean-free path of the air molecules. Otherwise the continuum hypothesis is invalid. The linear Stokes equations are simple in appearance. But this simple appearance proves to be deceptive when we attempt to solve them analytically. The task becomes all the more stupendous as we shall see in the next section if we use the solution to evaluate the important quantity viz; drag force on a body moving in the viscous fluid.
Historical description of work on Stokes drag

All these motions are characterized by low Reynolds numbers and are described by the solution of the Stokes equations. Although the Stokes equations are linear, to obtain exact solutions to them for arbitrary body shapes or complicated flow conditions is still a formidable task. There are only relatively few problems in which it is possible to solve exactly the creeping motion equations for flow around a single isolated solid body. Stokes(1851) calculated the flow around a solid sphere undergoing uniform translation through a viscous fluid whilst Oberbeck(1876) solved the problem in which an ellipsoid translates through liquid at a constant speed in an arbitrary direction. Edwards(1892), applying the same technique, obtained the solution for the steady motion of a viscous fluid in which an ellipsoid is constrained to rotate about a principal axis. The motion of an ellipsoidal particle in a general linear flow of viscous fluid at low Reynolds number has been solved by Jeffery(1922), whose solution was also built up using ellipsoidal harmonics. The analysis described by Jeffery extended further by Taylor(1923). Goldstein(1929a) obtained a force on a solid body moving through viscous fluid. Goldstein(1929b) studied the problem of steady flow of viscous fluid past a fixed spherical obstacle at small Reynolds numbers. In this paper, Goldstein obtained the expression of drag on fixed sphere. Lamb(1932) gave the solution for the general ellipsoid. Lighthill(1952) studied the problem of squirming motion of nearly spherical deformable bodies through liquids at very small Reynolds number. Aoi(1955) computed the drag experienced by a spheroid and obtained the general formula for the drag. He concluded that the pressure drag and frictional drag experienced by a spheroid contribute to the total drag in a definite ratio which is independent of the Reynolds number. The prolate and oblate spheroid has been treated by him as a detailed case study. Saffman(1956) observed the problem of small moving spheroidal particles in a viscous liquid and shown that the rate of orientation of a particle would then be independent of its size, and verified the prediction experimentally. Hill and Power(1956) have obtained arbitrarily closed approximations of drag by proving a complimentary pair of extremum principles for a Newtonian viscous fluid in quasi-static flow.

Stokes flow of an arbitrary body is of interest in biological phenomena and chemical engineering. In fact, the body with simple form such as sphere or ellipsoid is less encountered in practice. The body, which is presented in science and technology, often takes a complex arbitrary form. For example, under normal condition, the erythrocyte(red blood cell) is a biconcave disk in shape, which can easily change its form and present different contour in blood motion due to its deformability. In second half of twentieth century, a considerable progress has been made in treating the Stokes flow of an arbitrary body. Payne and Pell(1960) used the methods of generalized axially symmetric potential
theory to calculate the flow past a class of axi-symmetric bodies, including the lens, ellipsoid of revolution, spindle, and two separated spheres. Breach (1961) tackled the problem of slow flow past ellipsoids of revolution. Brenner (1963) gave the general expression for Stokes resistance over an arbitrary particle. Brenner and Cox (1963) obtained the expression of resistance to a particle of arbitrary shape in translational motion at small Reynolds number. Tuck (1964, 1970) developed a method for a simple problem in potential theory and is applied to a problem in Stokes flow, yielding a procedure for obtaining the Stokes drag on a blunt slender body of arbitrary shape. Acrivos and Taylor (1964) presented the general solution of the creeping flow equations for the motion of an arbitrary particle in an unbounded fluid in terms of spherical coordinates. They derived the force exerted on the particle, and the particular case of a slightly but otherwise arbitrarily deformed sphere was treated by them. Brenner (1964a, b, c, 1966a, b) further presented a theoretical calculation of the low Reynolds number resistance of a rigid, slightly deformed sphere to translational and rotational motions in an unbounded fluid. In which, he derived explicit expressions, to the first-order in the small parameter characterizations, which relates the Stokes resistance dyadic with the torque dyadic and the location of the centre of hydrodynamic stress of the particle to its geometry. Cox (1965) generalized the results given by Brenner and Cox (1963). Shi (1965) generalized the results of Proudman and Pearson (1957) and Kaplun and Lagerstrom (1957) for a sphere and a cylinder to study an ellipsoid of revolution of large aspect ratio with its axis of revolution perpendicular to the uniform flow at infinity. Matunobu (1966) used the Stokes equations for creeping flow to obtain steady flow of an incompressible, viscous fluid of infinite extent past a liquid drop which deviates slightly from a sphere and includes fully circulating flow. The expression of drag force experienced by the drop was derived by him. O’Brien (1968) found the form factors for deformed spheroids in Stokes flow. Taylor (1969) studied the motion of axi-symmetric bodies in viscous fluids. Chester et al. (1969) obtained the approximate expression of drag on sphere up to higher order beyond the first term given by Stokes (1851) in an incompressible viscous fluid at low Reynolds number. Batchelor (1970) has studied Stokes flow past a slender body of arbitrary (not necessarily circular) cross-section. Gautesen and Lin (1971) studied the problem of creeping flow past a radially deforming sphere within the framework of the Stokes approximation. They have shown that, independent of the deformation, the viscous drag equals twice the pressure drag. Lai and Mockros (1972) have used the Stokes linearized equations of motion to calculate the flow field generated by a spheroid executing axial translatory oscillations in an infinite, otherwise still, incompressible, viscous fluid. They expressed the fluid resistance on the spheroid as the sum of an added mass effect, a steady-state drag and an effect due to the history of the motion. Lin and Gautesen (1972) studied the problem of creeping flow of an incompressible viscous fluid past a deforming sphere for
all values of Reynolds number. They found the expression of drag up to the order of second power of Reynolds number ‘$R$’. **Morrison** (1972) derived the force on an accelerating body in an axisymmetric slow viscous flow which is valid for any axisymmetrical body, irrespective of the conditions at the surface. **Takagi** (1973) employed tangent-sphere coordinates to obtain the slow viscous flow due to the translational motion of torus without central opening along the axis of symmetry. The drag exerted on it is shown to be given by $D = 5.6 \pi \mu a$, where ‘$a$’ is the diameter of its generating circle. The problem of Stokes flow past a pervious sphere with a source at its centre was first considered by **Datta** (1973), who obtained the solution by linearizing the Navier-Stokes equation under the assumption that the interaction between the source flow and the Stokes flow is small. **Naruse** (1975) studied the low Reynolds number flow of an incompressible fluid past a body by solving the Navier-Stokes equations, on the basis of the method of matched asymptotic expansions. It is shown that, when the shape of the body is symmetric with respect to a point, the force on the body is determined to the order of $Re$ squared times $\log Re$, where $Re$ denotes the Reynolds number. The problem of steady flow at small Reynolds number past a pervious sphere with a source at its centre was considered again by **Datta** (1976).

Stokes flow past slender, prolate and oblate bodies of revolution has been studied by several investigators. For example, **Cox** (1970), **Keller and Rubinow** (1976), **Geer** (1976), **Johnson and Wu** (1979, 1980) and **Sellier** (1999) and many others have presented general theories for creeping motion of long slender bodies and slender particles in a viscous fluid. **Chwang and Wu** (part 1, 1974; part 2, 1975; part 4, 1976), **Chwang** (part 3, 1975), **Jhonson and Wu** (part 5, 1979) and **Huang and Chwang** (part 6, 1986), in a series of work over low Reynolds number hydromechanics, have explored the fundamental singular solution of the Stokes equation to obtain solutions for several specific body shapes translating and rotating in a viscous fluid. Regarding the distribution range of the singularities, it was pointed out that some results for plane-symmetric bodies in a potential flow may also be valid in all types of Stokes flow. By providing the exact solution of the Stokes equation in an elegant, closed form, the singularity method proved to be a useful alternative to the more standard methods of solution. Unfortunately, one cannot, in a straight forward manner, generalize this approach to the systems of many particles or to particles in the vicinity of a wall. **Usha and Nigam** (1976) obtained the expression of Stokes drag over deformed sphere (up to the order of $O(\varepsilon)$, where $\varepsilon$ is deformation parameter) by the help of integral equation simulation. **Alawneh and Kanwal** (1977) obtained closed form solutions for various boundary value problems in mathematical physics by considering suitable distributions of the Dirac delta function and its derivatives on lines and curves. **Sthapit and Datta** (1977) studied the problem of Stokes flow past a radially deforming
sphere incorporating the effect of the rate of deformation through the non-linear inertia terms. They obtained the expression of drag for small values of time.

In a series of papers, Gluckman et al. (1971, 1972) developed a new numerical method for treating the slow viscous motion past finite assembles of particles of arbitrary shape, termed the multipole representation technique. The approach is based on the theory that the solution for any object conforming to a natural coordinate system in a particle assemblage can be approximated by a truncated series of multi-lobular disturbances in which the accuracy of the representation is systematically improved by the addition of higher-order multipoles. For example, for a system of spherical particles the solution is found in terms of Legendre functions. Youngren and Acrivos (1975) used the boundary-element method to calculate hydrodynamic forces and torques acting on spheroidal and cylindrical particles in a uniform and simple shear flow. They expressed the solution of Stokes equations in the form of linear integral equations for the Stokeslet distribution over the particle surface. The required density of the Stokeslets, identical with the surface stress forces can be obtained numerically by reducing the integral equations to a system of linear algebraic equations. The technique has been successfully tested against the analytical solutions for spheroidal particles in a shear flow. Rallison and Acrivos (1978) applied a similar method in order to determine the deformation and condition of break-up in shear of a liquid drop suspended in another liquid of different viscosity. This very general method, which can be used in the case of bodies of arbitrary shape, so far has not been used to analyze flow fields around systems of particles. Liao and Krueger (1980) calculated the multipole expansion of slow viscous flow about spheroids of different sizes. Harper (1983) derived few theorems for the hydrodynamic image of an axially symmetric slow viscous (Stokes) flow in a sphere which is impermeable and free of shear stress. He establishes a second theorem in a sense in which such a flow past an arbitrary rigid surface or shear-free sphere becomes, on inversion in an arbitrary sphere with centre on the axis of symmetry, a flow past the rigid or shear-free inverse of the surface or sphere. Barshinger and Geer (1984) have tackled the problem of Stokes flow past a thin oblate body of revolution for the special case of an axially incident uniform flow. They have solved the Stokes equations asymptotically as the thinness ratio \( \varepsilon \) of the body approaches zero. The total net force experienced by the body and the point wise stress on the body surface are computed and discussed by them. Fischer et al. (1984) calculated the total force exerted on the isolated rigid obstacle in three dimensional space and in two dimensions placed in the
stationary flow of an incompressible viscous fluid with the help of matched asymptotic expansions. Wu (1984) proposed a new method of the line distribution of discrete singularities and continuous singularities to solve the Stokes flow that passes the arbitrary non-slender prolate axisymmetrical body. Wu applied this method to calculate the drag factor and the pressure distribution for the Cassini prolate oval as an example of the non-slender prolate arbitrary body. Wu and Qing (1984) have proposed the same singularity method to treat the creeping motion of the arbitrary prolate axisymmetrical body. They obtained analytic expressions in closed form and numerical results for the prolate spheroid and Cassini oval for the flow field. Tun (1984) calculated the drag factor and the pressure distribution for the Cassini oval as an example of the nonslender prolate arbitrary axisymmetric body using the same method of line distribution of discrete singularities and continuous singularities. Dabros (1985) attempted to find the hydrodynamic forces and velocities of arbitrary shaped particles, placed in an arbitrary flow field, particularly in the vicinity of the wall, using a singular point solution as the base function. Zhu and Wu (1985) used the method of continuous distribution of singularities to treat the Stokes flow of the arbitrary oblate axi-symmetric body. Kim (1985) illustrated the advantages of new forms of Faxen laws with derivatives of finite order for prolate spheroids. Kim (1986) obtained the disturbance velocity fields due to translational and rotational motions of an ellipsoid in a uniform stream, constant vorticity and constant rate-of-strain required in fundamental studies of behaviour of suspensions by the singularity method.

Ramkissoon (1986) examined the Stokes flow past a fluid spheroid whose shape deviates slightly from that of a sphere. To the first order in the small parameter characterizing the deformation, an exact solution has been obtained. As an application, he discussed the drag experienced by a fluid oblate spheroid and deduced the some well known cases. Leith (1987) extended the Stokes law on a sphere to a nonspherical object by allocating the interaction of the fluid with the object into its interaction with two analogous spheres, one with the same projected area and one with the same surface area as the object. He used this approach to characterized dynamic shape factor for objects whose shape factors are already exists in the literature. He reported the shape factor for a sphere, cylinders, prisms, spheroids and double conicals. Weinheimer (1987) presented the comparative analysis between measured drag forces on cylinders and disks with those computed for Stokes flow around equivalent spheroids in both axial and transverse flow situations. With the help of Stokes drag expression, he calculated the terminal velocities of ice crystals falling in the atmosphere with major dimensions of up to a few tens of microns. Yuan and Wu (1987) obtained
the analytic expressions in closed form for flow field by distributing continuously the image Sampsonlets with respect to the plane and by applying the constant density, the linear and the parabolic approximation. They calculated the drag factor of the prolate spheroid and the Cassini oval for different slender ratios and different distances between the body and the plane. **Power and Miranda** (1987) have successfully given the Fredholm integral equation representation of second kind for Stokes resistance problems i.e. when the velocity of particle is known, and the forces and moments are to be found. They represented the velocity as a double layer integral to which they added a Stokeslet and a Rotlet, both located at the centre of the body. Equating the representation to the given velocity resulted in a Fredholm integral equation of the second kind in the double layer density, and a numerical solution became possible after relating the Stokeslet and Rotlet strengths(force and moment) to the unknown double-layer density. **Lawrence and Weinbaum** (1988) presented the more general analysis of the unsteady Stokes equations for the axisymmetric flow past a spheroidal body to elucidate the behaviour of the force at arbitrary aspect ratio. These results are used to propose an approximate functional form for the force on an arbitrary body in unsteady motion at low Reynolds number. **Karrila and Kim** (1989) showed the completion of the double layer representation by **Power and Miranda** (1987) to be one of many possible completions. They suggested the same representation for the velocity as did by **Power and Miranda** (1987) and discuss various completions, suggesting one which is advantageous to an iterative numerical process for multiparticle systems. Both these completions are successful because, as observed by **Power and Miranda** (1987) and previously by **Ladyzhenskaya** (1963), the double layer representation alone is able to represent flow fields that correspond to the total force and total moment equal to zero. Authors like **Hsu and Ganatos** (1989) and **Tran-Cong and Phan-Thien** (1989) solved the Fredholm integral equation of first kind which is known to be ill-posed problem without evaluating the eigen function as formulated by **Ladyzhenskaya** (1963). A much more extensive review over numerical methods may be found in the paper of **Weinbaum and Ganatos** (1990) and to the paper of **Karrila and Kim** (1989). **Chester** (1990) considered the motion of a body through a viscous fluid at low Reynolds number. He derived general formulae for the force and couple acting on a body of arbitrary shape and implemented over to reduce some special cases. **Liron and Barta** (1992) have presented a new singular boundary-integral equation of the second kind for the stresses on a rigid particle in motion in Stokes flow. They also produced the forces and moments on the particle with the help of generalized Faxen law. **Chang et al.** (1992) have studied axi-symmetric viscous flow around ellipsoids of circular section in detail by the method of matched asymptotic expansion and a deterministic hybrid vortex method. **Piquet and Queutey** (1992) have investigated the computation of incompressible three-dimensional viscous flow past prolate spheroid by using iterative fully
decoupled technique based on the fully elliptic mode. **Claeys and Brady** (1993) have presented a new simulation method for low-Reynolds-number flow problems involving elongated particles in an unbounded fluid. They have applied the methodology to prolate spheroids and found it efficient and accurate by comparison with other numerical methods for Stokes flow. **Hubbard and Douglas** (1993) presented simple and accurate method of estimating the translational hydrodynamic friction on rigid Brownian particles of arbitrary shape. **Keh and Tseng** (1993) presented a combined analytical and numerical study for Stokes flow caused by an arbitrary body of revolution and calculated drag on prolate and oblate Cassini ovals. **Lowenberg** (1993) computed the Stokes resistance, added mass, and Basset force numerically for finite-length, circular cross-section cylinders using a boundary integral formulation. In this study, he found analytical formulas for the Stokes force, added mass, Basset force of spheroids which contrasted with the numerical results for cylinders of the aspect ratio in the range: $0.01 \leq a/b \leq 100$. He concluded that for some of these parameters, significant differences persist for disk and rod shaped particles. **Feng and Wu** (1994) gave the convergent results for prolate Cassini ovals by using a method of combined analytic-numerical method. **Douglas et al.** (1994) calculated the translational friction coefficient and the capacitance of a variety of objects with a probabilistic method involving hitting the probed objects with random walks launched from an enclosing spherical surface. **Zhou and Pozrikidis** (1995) implemented the method of fundamental solutions to compute Stokes flow past or due to the motion of solid particles. The computed locations and strengths of the singularities have been compared by them with those corresponding to exact discrete and continuous singularity representations, and the computed force and torque exerted on the particles are compared with exact values available from analytical solutions. **Brenner** (1996) studied the hydrodynamic Stokes resistance on non-spherical particles.

**Palaniappan** (1994) investigated the problem of slow streaming flow of a viscous incompressible fluid past a spheroid which departs but little in shape from a sphere with mixed slip-stick boundary conditions. He obtained the explicit expression for the stream function to the first order in the small parameter characterizing the deformation. For validation, he considered the oblate spheroid and evaluated the drag on this non-spherical body. He concluded that the drag in the present case is less than that of the Stokes resistance for a slightly oblate spheroid. **Tanzosh and Stone** (1996) have developed a concise analytic method to investigate the arbitrary motion of a circular disk through unbounded fluid satisfying Stokes equations. Four elementary motions are considered by them: broad side translation, edge wise translation, in-plane rotation and out-of-plane rotation of a disk. They reduced the Stokes equations to a set of dual integral expressions relating to the velocity and traction in the plane of the disk. They solved the dual integral equations exactly for each
motion and lead, in turn, to closed-form analytical expressions for the velocity and pressure fields. Ramkissoon (1997) investigated the creeping axisymmetric slip flow past an approximate spheroid whose shape deviates from that of a sphere. He obtained exact solution to the first order in the small parameter characterizing the deformation. As an application, the case of flow past an oblate spheroid has been considered and the drag experienced by it is evaluated and some special well-known cases are deduced as a validation. Datta and Srivastava (1999) developed a new approach to evaluate the Stokes drag force in a simple way on a axially symmetric body with some geometrical constraints placed in axial flow and transverse flow under the no-slip boundary conditions. The results of drag on both the flow situations were successfully tested not only for sphere, prolate and oblate spheroid but also for other bodies like deformed sphere, cycloidal and egg-shaped bodies of revolution with acceptable limit of error. This method has been described in the section 1.5 as the same is exploited here to study the problem of Stokes flow around deformed sphere. Alassar and Badr (1999) have solved the problem of uniform steady viscous flow over an oblate spheroid in the low-Reynolds number range $0.1 \le R \le 1.0$. They have written the full Navier-Stokes equations in the stream function-vorticity form and solved numerically by means of the series truncation method. They considered spheroids having axis ratio ranging from 0.245 to 0.905. They obtained the drag coefficients for oblate spheroid and compared with previous analytical formulae which were based on the solution of the linearized Stokes equations. Datta and Srivastava (2002) obtained the optimum drag profile in axisymmetric Stokes flow under the restrictions of constant volume and constant cross section area by exploiting DS conjecture given by Datta and Srivastava (1999). Zhuang, et al. (2002) proposed a new three-dimensional fundamental solution to the Stokes flow by transforming the solid harmonic functions in Lamb’s solution into expression in terms of oblate spheroidal coordinates. They used the oblate spheroid as model of a variety of particle shapes between a circular disks and a sphere. The effect of various geometric factors on the forces and torques exerted on two oblate spheroids were systematically studied for the first time by them using the proposed fundamental solution. Fonseca and Hermann (2004) have simulated the dynamics of oblate ellipsoids under gravity. They studied the settling velocity and the average orientation of the ellipsoids as a function of volume fraction. Benard et al. (2004) have presented an analytic solution for the motion of a slightly deformed sphere in creeping flows with the assumption of slip on the particle surface. They obtained the explicit expressions for the hydrodynamic force and torque exerted by the fluid on the deformed sphere with the help of perturbation method used previously by Brenner (1964 a, b, c) and Lamb (1932). Palaniappan and Ramkissoon (2005) provided a complete survey of drag formula over axisymmetric particle in Stokes flow. Vafeas and Dassios (2006) have analyzed the low Reynolds number flow of a swarm of ellipsoidal particles.
in an otherwise quiescent Newtonian fluid, that move with constant uniform velocity in an arbitrary direction and rotate with an arbitrary constant angular velocity with an ellipsoid-in-cell model. Mohan and Brenner(2006) have presented a general approach to the solution of the problem of nonisothermal Stokes flow relative to a heat-conducting particle having the shape of a slightly deformed sphere, taking account of Maxwell’s[J.C. Maxwell, Philos.Trans. R. Soc. Lond., 170(1879), pp. 231-256] thermal creep condition at the surface of the particle. Senchenko and Keh(2006) attempted first to obtain analytical approximations for the resistance relations for slightly deformed slip sphere in an unbounded Stokes flow. To the first order in the small parameter characterization of the deformation, they derive expressions for the hydrodynamic force and torque exerted on the particle. They checked the obtained results of axial and transverse drag for the spheroid. All these expressions of drag have been used by us for the validation purposes in the situation of no-slip boundary condition. Tsai et al.(2006) have provided the practical and numerical implementations of the method of fundamental solutions for three-dimensional exterior Stokes problems with quiet far-field condition. This numerical scheme has been checked by them for sphere and rotating dumbbell shaped body. Seddon and Mullin(2007) have presented results from experimental investigations into the motion of a heavy ellipsoid in horizontal rotating cylinder filled with highly viscous fluid. Srivastava(2007) obtained the optimum volume profile in axi-symmetric Stokes flow by exploiting the DS conjecture given by Datta and Srivastava(1999). Scolan and Etienne(part 1 & 2, 2008) have discussed some aspects of the force and moment computations in incompressible and viscous flows on bodies of arbitrary shaped by using the projection techniques developed by Quartapelle and Napolitano[AIAA J. 21(1983), pp. 991-913] without explicitly calculating the pressure. Keh and Chang(2008) presented a combined analytical and numerical study of the slip Stokes flow caused by a rigid spheroidal particle translating along its axis of revolution in a viscous fluid. The drag force exerted on the spheroidal particle by the fluid is evaluated by them with good convergence behaviour for various values of the slip parameter and aspect ratio of the particle. They proved that, for a spheroid with a fixed aspect ratio, its drag force is monotonically decreasing function of the slip coefficient of the particle. Liu et al. (2009a) have extended the force-coupling method, previously developed for spherical particles suspended in a liquid flow for ellipsoidal particles in the Stokes limit. Liu et al.(2009b) have developed a new method to simulate the fully coupled motion involving an ellipsoidal particle and the ambient fluid which reduces a two-phase flow problem into a single-phase fluid flow problem. Chang and Keh(2009) analyzed the steady translation and rotation of a rigid, slightly deformed colloidal sphere in arbitrary directions in a slip viscous fluid in the limit of small Reynolds number. They solved Stokes equations asymptotically using a method of perturbed expansions. They presented the expression of drag
and torque for spheroid up to the order of $o(\varepsilon^2)$, where $\varepsilon$ is the deformation parameter. Bowen and Masliyah (2009) obtained an approximate solution to the equation of motion governing Stokes flow past a number of isolated closed bodies of revolution by the least square fitting of a truncated series expression for the stream function to known boundary conditions. They found reasonably accurate (±5%) estimate for the Stokes resistance on body shapes, such as cylinders and cones, for which the solutions are exceedingly difficult. They applied the computed drag values in determining the limitations of the various empirical expressions used to predict the drag resistance of these geometrically simple bodies. Blake et al. (2010) have considered some of the properties of the S-transform as well as exploiting the special properties associated with Legendre polynomials to generate a range of slender body shape with fixed Stokes drag. Radha et al. (2010) have described a new approximate method to discuss uniform flow past rigid bodies of two different shapes using a complete general solution [Palaniappan et al. (1992); Padmavathi et al. (1998)] of Stokes equations in an incompressible viscous fluid. They have proposed that with this new method approximate values of physical quantities like drag experienced by a rigid body could be obtained in a simple way in accurate manner. Pratibha and Jeffrey (2010) calculated the mobility functions for two unequal spheres at low Reynolds number by using the method of twin multipole expansion. Sherief et al. (2010) have investigated the translational motion of an arbitrary body of revolution in a micropolar fluid by using a combined analytical-numerical method. They have evaluated the drag exerted on a prolate spheroid for various values of the aspect ratio and for different values of the micropolarity parameters. They further applied this technique to the prolate Cassini ovals for justifying good convergence. Datta and Singhal (2011) studied the uniform viscous flow with slip boundary condition under Stokes approximation at low Reynolds number past a pervious sphere with a source at its centre by using the method of matched asymptotic expansions.

**Historical description of work on torque experienced by rotating body**

Stokes flow is becoming increasingly important due to the miniaturization of fluid mechanical parts for example, in micromechanics as well as in nanomechanics. The value of the moment or couple experienced by axially symmetric bodies, rotating steadily in a viscous and incompressible fluid, is needed in designing and calibrating viscometers. Therefore, many attempts have been made to evaluate such a couple for various bodies of revolution. When inertial effects can be validly ignored, so that Stokes linearized theory applies, the solutions have been found for some configurations. These configurations are sphere, spheroids, deformed sphere etc.
Edwardes (1892) studied the steady motion of a viscous liquid in which an ellipsoid was constrained to rotate about a principal axis. Slow rotation of spheroids (including the circular disc) in an infinite fluid was first solved by Jeffrey (1915, 1922) using curvilinear coordinates. His approach was later extended to the spherical lens, torus, and other axi-symmetric shapes. The rotatory oscillations of a sphere about a diameter, in an infinite mass of viscous fluid at rest, have been discussed by Lamb (1932). Sowerby and Rosenhead (1953) determined the couple required to maintain the steady rotation about its axis of symmetry of an oblate spheroid which is placed in a uniform stream of viscous fluid moving parallel to this axis. Kanwal (1955) used the Stokes stream function to obtain the perturbations arising from the slow rotatory and longitudinal oscillations of axi-symmetrical bodies like sphere, an infinite circular cylinder, a prolate spheroid, an oblate spheroid, and a circular disk, in an infinite mass of viscous fluid which is at rest at infinity. Proudman (1956) and Stewartson (1966) analyzed the dynamical properties of a fluid occupying the space between two concentric rotating spheres when the angular velocities of the spheres are slightly different, in other words, when the motion relative to a reference frame rotating with one of the spheres is due to an imposed azimuthal velocity which is symmetric about the equator. Kanwal (1959) attempted and provide an exact analytic solution, amenable to numerical work, of the flow when an infinitesimally thin circular disk of radius $r_0$ is given an impulsive moment $G \delta(t)$: $- \delta(t)$ being DIRAC’s delta function.

Kanwal (1961) has discussed the problem of slow steady rotation of axially symmetric bodies in a viscous fluid. He utilized the same method of generalized axially symmetric potential theory used by Payne and Pell (1960) to derive a relation between the couple experienced by a body and the angular velocity of the body. He also calculated the couple on steadily rotating spindle, torus, and hemisphere and spherical cap as special cases of lens in an incompressible viscous fluid. Brenner (1961) also obtained some general results for the drag and couple on an obstacle which is moving through the fluid. Rubinow and Keller (1961) have considered the force on a spinning sphere which is moving through an incompressible viscous fluid by employing the method of matched asymptotic expansions to describe the asymmetric flow. Breach (1961) utilized the method of perturbation technique developed by Proudman and Pearson (1957) for a sphere and generalized that to apply to all ellipsoids of revolution both prolate and oblate. Lord (1964) calculated the aerodynamic drag torque on a sphere rotating in a rarefied gas through the experimental and theoretical study. Brenner and Sonshine (1964) calculated the torque required to maintain the steady, symmetric rotation of a sphere in a viscous fluid bounded externally by an infinitely long circular cylinder. Brenner (1964a,b,c,d) in a series of papers, presented a theoretical calculation of the low Reynolds number resistance of a rigid, slightly deformed sphere to translational and rotational
motions in an unbounded fluid. O’Neill (1964) gave the expressions of force and couple on slowly moving solid sphere in slow viscous liquid. Childress (1964) has investigated the motion of a sphere moving through a rotating fluid and calculated a correction to the drag coefficient. Cox (1965) considered a single particle of arbitrary shape moving with both translation and rotation in an infinite fluid at very low Reynolds number. He gave the formula for force and couple for cases in which body possesses symmetry properties. He also obtained for both a spheroid and a dumbbell shaped body in pure translation and for a translating rotating sphere as well as dumbbell shaped body in pure rotation. Joseph (1965) prescribed the coupled flow induced by the steady rotation of a fluid saturated; naturally permeable and infinite disk is compared with the flow induced by the rotation of an otherwise impermeable disk over which there is a uniform suction. Brenner (part 6, 1966) provided the formulae of resistance and couple on particle of arbitrary shape. Goldman et al. (1966) calculated the numerical values of forces and torques on sphere of a two-translating sphere system using bipolar coordinates. Wakiya (1967) numerically evaluated the drag and angular velocity experienced by freely rotating spheres and compared with calculated from corresponding approximate formulae known before. Barrett (1967) has tackled the problem of impulsively started sphere rotating with angular velocity $\Omega$ about a diameter. Pearson (1967) has presented the numerical solution for the time-dependent viscous flow between two concentric rotating spheres. He governed the motion of a pair of coupled non-linear partial differential equations in three independent variables, with singular end conditions. He also described the computational process for cases in which one (or both) of the spheres is given an impulsive change in angular velocity - starting from a state of either rest or uniform rotation. Majumdar (1969), has solved, by using bispherical coordinates, the non-axisymmetrical Stokes flow of an incompressible homogeneous viscous liquid in space between two eccentric spheres. It was proved that the resultant force acting upon the spheres is at right angles to the axis of rotation and the line of centres. The effect of the stationary sphere on the force and couple exerted by the liquid on the rotating sphere has been discussed and the results are compared with those of the axi-symmetrical case of Jeffrey (1915). Cooley (1968) has investigated the problem of fluid motion generated by a sphere rotating close to a fixed sphere about a diameter perpendicular to the line of centres in the case when the motion is sufficiently slow to permit the linearization of the Navier-Stokes equations by neglecting the inertia terms. He used a method of matched asymptotic expansions to find asymptotic expressions for the forces and couples acting on the spheres as the minimum clearance between them tends to zero. In his paper, the forces and couples are shown to have the form $a_0 \ln \varepsilon + a_1 + o(\varepsilon \ln \varepsilon)$, where $\varepsilon$ is the ratio of the minimum clearance between the spheres and the radius of the rotating sphere and where $a_0$ and $a_1$ are found explicitly. Waters and King (1970) considered the secondary flow induced by the slow steady rotation of an oblate
or prolate spheroid about its axis of symmetry in a “simple fluid”, which is assumed to have a finite memory. They have presented the analytic solution for secondary flows due to various spheroids including finite disc and a sphere for various elastic parameters. Kanwal (1970) written a note on slow rotation or rotary oscillation of axisymmetric bodies in hydrodynamics and magneto hydrodynamics. Apart from the expression of torque, he also calculated the torque on thin rigid circular disk in conducting fluid. O’Neill and Majumdar (1970) have discussed the problem of asymmetrical slow viscous fluid motions caused by the translation or rotation of two spheres. The exact solutions for any values of the ratio of radii and separation parameters are found by them. Ranger (1971) tackled the problem of axially symmetric flow past a rotating sphere due to a uniform stream of infinity. He has shown that leading terms for the flow consists of a linear superposition of a primary Stokes flow past a non-rotating sphere together with an anti symmetric secondary flow in the azimuthal plane induced by the spinning sphere. Philander (1971) presented a note on the flow properties of a fluid between concentric spheres. This note concerns the flow properties of a spherical shell of fluid when motion is forced across the equator. The fluid under consideration is contained between two concentric spheres which rotate about a diameter with angular velocity $\Omega$. The consequences of the forcing motion across the equator are explored in his work. Cooley (1971) has investigated the problem of fluid motion generated by a sphere rotating close to a fixed sphere about a diameter perpendicular to the line of centres in the case when the motion is sufficiently slow to permit the linearization of the Navier-Stokes equations by neglecting the inertia terms. He used a method of matched asymptotic expansions to find asymptotic expressions for the forces and couples acting on the spheres as the minimum clearance between them tends to zero. In his paper, the forces and couples are shown to have the form $a_0 \ln \varepsilon + a_1 + o(\ln \varepsilon)$, where $\varepsilon$ is the ratio of the minimum clearance between the spheres and the radius of the rotating sphere and where $a_0$ and $a_1$ are found explicitly. Munson and Joseph (1971, part 1 and part 2) have obtained the high order analytic perturbation solution for the viscous incompressible flow between concentric rotating spheres. In second part of their analysis, they have applied the energy theory of hydrodynamic stability to the viscous incompressible flow of a fluid contained between two concentric spheres which rotate about a common axis with prescribed angular velocities. Riley (1972) has discussed the thermal effects on slow viscous flow between rotating concentric spheres. Mena (1972) measured the couple on a sphere in the centre of a finite rotating cylinder over a wide range of Reynolds numbers for both Newtonian and non-Newtonian fluids. Ranger (1973) solved the non symmetrical Stokes flow past a spherical cap by the method of complementary integral representations. He evaluated the drag and couple for hemispherical cap. Using toroidal coordinates, Schneider et al. (1973) have derived an exact solution for the velocity field induced in two immiscible semi-infinite fluids.
possessing a plane interface, by the slow rotation of an axially symmetric body partly immersed in each fluid. They have proved that torque exerted on the rotating body is proportional to the sum of the viscosities. Analytic closed form expressions are derived by them for the torque when the body is either a sphere, a circular disc, or a tangent-sphere dumbbell, and for a hemisphere rotating in an infinite homogeneous fluid. Closed form results are also given for an immersed sphere, tangent to free sphere. In the end, numerical values of the torque are provided for a variety of body shapes and two fluid systems of various viscosity ratios. Takagi (1973) considered the problem of slow viscous flow due to the motion of a closed torus by employing tangent-sphere coordinates. Chwang and Wu (part 1, 1974) considered the viscous flow generated by pure rotation of an axisymmetrical body having an arbitrary prolate form. They derived the velocity field and total torque on rotating axisymmetric body in Stokes flow by distributing rotlets singularity on axis of symmetry. Takagi (1974a) considered the problem of steady rotation of a spherical cap about the line perpendicular to the axis of cap and found the drag and couple over it. In other paper, Takagi (1974b) studied the flow around spinning sphere moving in a viscous fluid and found the expression of drag and couple on sphere by using the method of matched asymptotic expansions. Chwang and Wu (part 2, 1975) employed fundamental singularities to construct exact solutions to a number of exterior and interior Stokes-flow problems for several specific body shapes translating and rotating in a viscous fluid. Munson and Menguturk (1975, part 3) have studied the stability of flow of a viscous incompressible fluid between a stationary outer sphere and rotating inner sphere theoretically and experimentally. Wimmer (1976) has provided some experimental results on incompressible viscous fluid flow in the gap between two concentric rotating spheres. Dorrepaal (1976) analyzed the Ranger’s (1973) solution of the asymmetric Stokes flow past a spherical cap in detail. He derived the formulae for drag and couple valid for all cap angles which are in agreement with the known results in the limiting cases when the cap becomes a sphere and a circular disc. Takagi (1977) studies the steady flow induced by the slow rotation of a solid sphere immersed in an infinite incompressible viscous fluid. He presented the solution in the powers of Reynolds number up to power of fourteen. Ramkisson (1977) examined the Stokes flow due to an axially symmetric body rotating about its axis of symmetry in a micropolar fluid. He derived a simple formula for couple experienced by a body in terms of angular velocity. Drew (1978) has found the force on a small sphere translating relative to a slow viscous flow to order of the ½ power of Re for two different fluid flows far from the sphere, namely pure rotation and pure shear. For pure rotation, the correction of this order to the Stokes drag consists of an increase in the drag. Felderhof (1978) derived Faxen theorems for the force, the torque and the symmetric force dipole moment acting on a spherically symmetric polymer suspended in arbitrary flow. Majumdar and O’Neill (1979) have given the
exact solutions of the Stokes equations for asymmetric flows produced by a closed torus which steadily translates along and rotates about a direction perpendicular to its axis of rotational symmetry in a quiescent viscous fluid. They calculated the force and couple acting on the closed torus. Davis and O’Neill (1979) studied the problem of the slow rotation of a sphere submerged in a fluid with a surfactant surface layer and gave the expression of couple. Shail (1979) solved a Fredholm integral equation of second kind both asymptotically and numerically and computed the resistive torque on the disk. Kim (1980) has calculated the torque and frictional force exerted by a viscous fluid on a sphere rotating on the axis of a circular cone of arbitrary vertex angle about an axis perpendicular to the cone axis in the Stokes approximation. Waters and Gooden (1980) obtained the couple on a rotating oblate spheroid in an elastic-viscous liquid. Schmitz (1980) presented a general theorem for the force multipole moments of arbitrary order in terms of unperturbed fluid velocity field induced in a spherically symmetric particle immersed in a fluid whose motion satisfies the linear Navier-Stokes equation for steady incompressible viscous flow. Davis (1980) considered the problem of calculating the resistive torque on a body rotating slowly with constant angular speed either fully or partially immersed in a liquid with an adsorbed surface. Ivanov and Yanshin (1980) obtained the analytic expressions for the forces and moments acting on symmetrically rotating convex figures of revolution moving in a free molecular flow. Dennis et al. (1981) have investigated the problem of viscous incompressible, rotationally symmetric flow due to the rotation of a sphere with a constant angular velocity about a diameter. The solutions of the finite-difference equations are presented for Reynolds number ranging from 1.0 to 5000. Rao and Iyengar (1981) calculated the couple on slowly rotating spheroid(prolate and oblate) in incompressible micropolar fluid. Smith (1981) investigated the secondary motion due to the slow rotation of an axisymmetric bodies like sphere, spheroid, spherical cap and double sphere in a rotating viscous fluid. Shail and Gooden (1981) extended his own work (Shail, 1979) on the problem of an axisymmetric submerged solid rotating slowly and steadily in a fluid whose surface is covered with a surfactant film. Murray (1982) established an approximated solution, correct to order three in a small parameter, for the velocity field induced in a viscous fluid by the slow rotation of two tori. He calculated the torque acting on both toroidal surfaces. Quartapelle and Napolitano (1983) derived closed form general formulas for the force and moment in terms of the global solenoidal velocity field without reference to the pressure field acting on a rigid body immersed in an incompressible flow. Rao and Iyengar (1983) evaluated the couple on the oscillating spheroid(prolate and oblate) in incompressible micropolar fluid. Yang and Leal (part 1, 1983) considered translation and rotation, each in three mutually orthogonal directions, of particle having an arbitrary orientation relative to the interface. They determined components of the hydrodynamic
resistance tensors which relate the total hydrodynamic force and torque on the particle to its translational and angular velocities for a completely arbitrary translational and angular motion. Bestman (1983) studied the flow of a Newtonian viscous fluid, oscillating with angular velocity $\lambda$ and amplitude $V_{\infty}$, past a rigid sphere rotated slowly with angular velocity $\omega$. Yang and Leal (part 2, 1984) determined the hydrodynamic relationships for the force and torque on the particle at rest in the undisturbed flow field using the method of reflections, from the spatial distribution of Stokeslets, rotlets and higher-order singularities in Stokes flow. Ramkissoon (1984) examined the Stokes flow due to the rotation of an axially symmetric body in couple stress fluids. He derived a simple formula for the couple experienced by the body in terms of the angular velocity. Chakrabarti and Shail (1984) calculated the couple on the rotating solid when body is far from the interface. Dabros (1985) presented a numerical technique which allows one to estimate hydrodynamic forces and torques or translational and angular velocities of particles in a general flow field. Kim (1985) derived new forms for the Faxen law for the force, torque and stresslet on a particle of arbitrary shape and give specific examples for the prolate spheroid. Huang and Chwang (part 6, 1986) applied uniform ring distributions of fundamental singularities like Stokeslets and rotlets for Stokes flow to obtain exact solutions for rotating oblate bodies in an unbounded viscous fluid. They have obtained the expressions of moment on slightly deformed sphere and slender torus rotating about z axis with an angular velocity $\omega$ as special cases. Kim (1986) obtained the disturbance velocity fields due to translational and rotational motions of an ellipsoid in a uniform stream with the help of singularity method. Davis and Brenner (1986) have used the matched asymptotic expansion methods to solve the problem of steady rotation of a tethered sphere at small, non-zero Reynolds numbers. They obtained first order Taylor number correction to both the Stokes-law drag and Kirchhoff’s law couple on the sphere for Rossby numbers of order unity. Gagliardi (1987) has developed the boundary conditions for the equations of motion for a viscous incompressible fluid in a rotating spherical annulus. The solution of the stream and circumferential functions were obtained in the form of a series of powers of the Reynolds number. Transient profiles were obtained for the dimensional torque, dimensionless angular velocity of the rotating sphere, and the dimensionless angular momentum of the fluid. Marcus and Tuckerman (1987, part 1 and 2) have computed numerically the steady and translation simulation of flow between concentric rotating spheres. O’Neill and Yano (1988) derived the boundary condition at the surfactant and substrate fluids caused by the slow rotation of a solid sphere which is partially submerged in the substrate fluid. Yan et al. (1988) studied the problem of Stokes flow induced by the rotation of a prolate axi-symmetric body in the presence of an infinite planar boundary. Yang et al. (1989) have provided the numerical schemes for the problem of the axially symmetric motion of an incompressible viscous fluid in an annulus between two concentric rotating spheres. Howe (1989) discussed the
unsteady forces & couple on the surface of the rigid body. Gagliardi et al.[1990] reported the study of the steady state and transient motion of a system consisting of an incompressible, Newtonian fluid in an annulus between two concentric, rotating, rigid spheres. They solved the governing equations for the variable coefficients by separation of variables and Laplace Transform methods. They presented the results for the stream function, circumferential function, angular velocity of the spheres and torque coefficient as a function of time for various values of the dimensionless system parameters. Chester (1990) derived general formulae for the force and couple acting on a body of arbitrary shape moving through a viscous fluid at low Reynolds number. Gavze (1990) calculated the force and torque acting on accelerating body of arbitrary shape moving at low Reynolds number in a viscous fluid at rest at infinity. Wang (1992) determined the torque on a rotating disk enclosed by a casing in Stokes flow by eigen function expansion and collocation. Chester (1992) derived explicit formulae for the force and couple in terms of the parameters defining the asymptotic field on a moving body of arbitrary shape through viscous fluid. Iosilevskii et al. (1993) investigated the creeping flow engendered by the steady axi-symmetric rotation of a sphere in a transversely-isotropic Newtonian fluid is investigated in the limiting case when the material properties of the fluid are unaffected by the fluid motion. Authors reduced this problem to that of a spheroid rotating about its axis of symmetry in an isotropic fluid by introducing an appropriate coordinate transformation. They obtained closed form results for both velocity field and the couple required to maintain the rotation. Ranger and O’Neill (1993) determined the torque acting on an axi-symmetric solid body which rotates about an axis perpendicular to its axis of symmetry by a method which relaxes one of the three boundary conditions on the body. Loyalka and Griffin (1994) calculated the frictional torque of non-spherical axi-symmeric particles at low Reynolds number under slip boundary conditions. Chakrabarti and Manna (1994) employed a method involving eigen function expansion and collocation to solve the axi-symmetric problem of a slowly and steadily rotating circular disc in a fluid of finite extent. Feng et al. (1995) presented an infinite series solution to the creeping flow equations for the axi-symmetric motion of a sphere of arbitrary size rotating in a quiescent fluid around the axis of circular orifice or a circular disk whose diameters are either larger or smaller than that of the sphere. Howe (1994) developed a relationship between force and moment exerted on a rigid body in unsteady motion in a uniform incompressible, viscous or inviscid fluid. Iyengar and Charya (1995) evaluated the couple experienced by the rotating approximate sphere about its axis of symmetry in an micropolar fluid. Ranger (1996) calculated the drag and couple on solid sphere translating and rotating in a viscous fluid relative to a uniform stream whose speed also decays exponentially with time. Tanzosh and Stone (1996) developed a concise method to investigate the arbitrary motion of a circular disk through an unbounded fluid satisfying
Stokes equation. They considered four elementary motions: broadside translation, edgewise translation, in-plane rotation and out-of-plane rotation of a disk. Author solved the dual integral equations exactly for each motion and lead to closed form analytical expressions for the velocity and pressure fields. Tekasakul et al. (1998) have studied the problem of the rotatory oscillation of an axi-symmetric body in an axi-symmetric viscous flow at low Reynolds numbers. They evaluated numerically the local stresses and torques on a selection of free, oscillating, axi-symmetric bodies in the continuum regime in an axi-symmetric viscous incompressible flow. Shatz (1998a) used indirect boundary element method to compute hydrodynamic torque on full spheroids. Shatz (1998b) calculated the hydrodynamic pressure, torque, and drag on the translating and rotating hemispheroid as a function of hemispheroidal shape.

Datta and Srivastava (1999) advanced a new approach to evaluate the drag force in a simple way on a restricted axially symmetric body placed in longitudinal stream and transverse stream when the flow is governed by Stokes equations. They also proposed that this analysis may be extended to calculate the couple on a body rotating about its axis of symmetry. Datta and Srivastava (2000) calculated the couple on slowly rotating porous sphere with fluid source at its centre about a diameter. They found that the couple decreases on account of increasing source parameter. Pregnalato et al. (2001) investigated the flow past a rotating sphere numerically using a spectral element/ spectral direct numerical simulation. The effect of sphere rotation on transition regimes is analyzed for low Reynolds number. Arbaret et al. (2001) presented the effect of shape and orientation on rigid particle rotation and matrix deformation in simple shear flow. Datta and Pandya (2001) gave a simple integral formula to evaluate the torque on a slowly rotating axi-symmetric body partially immersed in a viscous fluid covered by an adsorbed surface film. Galdi (2002) analyzed the problem of steady flow of a Navier-Stokes fluid around a rotating obstacle. Kim and Choi (2002) conducted the numerical simulations for laminar flow past a sphere rotating in the stream wise direction, in order to investigate the effect of the rotation on the characteristics of flow over the sphere. Takasakul and Loyalka (2003) discussed the rotary oscillations of several axi-symmetric viscous flows with slip by employing numerical technique based on Green’s function. Benard et al. (2004) obtained the hydrodynamic drag force and torque exerted by the fluid on the deformed sphere and expressed explicitly for a translational and rotational deformed sphere. Kitauchi et al. (2004) obtained an analytic solution of steady linear viscous flow on a spherical cap rotating about its centre. Shatz (2004) calculated the torque on rotating full spheroid as well as hemispheroid by appropriate choice of Green’s function. Liu et al. (2004) have developed a very efficient numerical method based on the finite difference technique for solving time-dependent non-linear flow problems. They have applied this method to study the unsteady axi-symmetric isotherm flow of an incompressible viscous fluid in a spherical shell with a stationary inner sphere.
and a rotating outer sphere. **Ifidon** (2004) numerically investigated the problem of determining the induced steady axially symmetric motion of an incompressible viscous fluid confined between two concentric spheres, with the outer sphere rotating with constant angular velocity and the inner sphere fixed for large Reynolds numbers. **Scolan** (2005) determined the hydrodynamic force and moment on bodies in viscous incompressible flow with the use of conformal mappings. **Senchenko and Keh** (2006) derived expressions for the hydrodynamic force and torque exerted on the particle translating and rotating Stokes flow under slip boundary conditions. The expressions reduced further to those for no-slip boundary conditions as slip coefficient tends to zero. **Davis** (2006) obtained the expression for force and torque on a rotating sphere close to and within a fluid-filled rotating sphere. **Ragazzo and Tabak** (2007) extended the **Howe** (1989) formulas for the force and torque on a rigid body. **Yu et al.** (2007) have studied the rotation of a single spheroid in a planar Couette flow as a model for simple shear flow numerically with the distributed Lagrangian multiplier based fictitious domain method. **Marcello** (2008) has introduced new exact analytic solutions for the rotational motion of a axially symmetric rigid body having two equal principal moments of inertia and subjected to an external torque which is constant in magnitude. **Kitauchi and Ikeda** (2009) obtained an analytic solution of two dimensional, steady, linear, viscous flow on a polar cap-the polar region of a sphere that lies above (or below) a given plane normal to the rotation axis-rotating about its centre. **Chang and Keh** (2009) analyzed the problem of steady translation and rotation of a rigid, slightly deformed colloidal sphere in arbitrary (longitudinal and transverse) directions in a viscous fluid within the limit of small Reynolds number. After deriving the general expressions of hydrodynamic drag and torque, they applied these results for special cases of prolate and oblate spheroids and calculated these expressions up to the second order of deformation parameter. **Wan and Keh** (2009) studied the problem of the rotation of a rigid particle of revolution about its axis in a viscous fluid theoretically under the slip boundary conditions. They calculated the hydrodynamic torque on rotating slip spheroid. **Chang and Keh** (2010) studied analytically and numerically the problem of rotation of a rigid spheroidal particle about its axis of revolution in a viscous fluid in the steady limit of negligible Reynolds number. They found the hydrodynamics torque on spheroids in both conditions (slip and no-slip) and evaluated the numerical values of torque with respect to various aspect ratio as well as slip parameter. **Lukerchenko et al.** (2010) have evaluated the drag torque, drag force and magnus force acting on a rotating prolate spheroid experimentally. **Ashmawy** (2011) has discussed the rotational motion of an arbitrary axisymmetric body in a viscous fluid using a combined analytical-numerical technique under slip boundary condition. The couple exerted on a prolate and oblate spheroid and on prolate and oblate Cassini ovals is evaluated for various values of the aspect ratio and for different values of slip parameter by the author.
1.4 Oseen’s approximation; Oseen’s drag

In the year 1910, Carl Wilhelm Oseen (1927) proposed Oseen’s approximation to treat problems in which a flow field involves a small disturbance of a constant mean flow, as in a stream of liquid. His work was based on the experiments of Sir George Gabriel Stokes (1851), who had studied a sphere of radius ‘a’ falling in a fluid of viscosity ‘μ’. Oseen developed a correction terms, which included inertial factors, for the velocity used in Stokes calculations, to solve the problem. Oseen’s approximation leads to an improvement to Stokes approximation. Oseen’s drag formulation can be used in connection with flow of fluids under various special conditions, such as: containing particles, sedimentation of particles, centrifugation or ultracentrifugation of suspensions, colloids, and blood through isolation of tumors and antigens [Fung (1997)]. The fluid does not even have to be a liquid, and the particles do not need to be solid. It can be used in a number of applications, such as smog formation and atomization of liquids. Blood flow in small vessels, such as capillaries, is characterized by small Reynolds and Womersley numbers. At small Reynolds and Womersley numbers, the viscous effects of the fluid become predominant. Understanding the movement of these particles is essential for drug delivery and studying metastasis movements of cancers.

Oseen considered the sphere to be stationary and the fluid to be flowing with a velocity ‘U’ at an infinite distance from the sphere. Inertial terms were neglected in Stokes approximation. It is a limiting solution when the Reynolds number tends to zero. When the Reynolds number is small and finite, correction for the inertial term is required. Oseen substituted the following velocity values into Navier-Stokes equations

\[ u = U + u', \ v = v', \ w = w', \]  

\[ \text{(1.4.1)} \]

on neglecting the quadratic terms in the primed quantities leads to the derivation of Oseen’s approximation or Oseen’s equation (vector form, Happel and Brenner, 1964)

\[ U \text{grad} \quad u = \left( \frac{1}{\rho_1} \right) \text{grad} \quad p + v \text{ \nabla}^2 u, \ \text{div} \quad u = 0. \]  

\[ \text{(1.4.2)} \]

When Stokes equations was solved on the basis of Oseen’s approximation, the resultant hydrodynamic drag (force) is given by [Happel and Brenner, page 44, eq.(2-6.5), 1964]

\[ F = 6 \pi \mu a U \left[ 1 + \frac{3}{8} R + O \left( R^2 \right) \right]. \]  

\[ \text{(1.4.3)} \]
where \( R = \frac{Ua}{\nu} \) is particles Reynolds number. This Oseen’s drag differs from Stokes drag by a factor \( 1 + (3/8)R \).

**Modifications to Oseen’s approximation**

As the fluid near the sphere is almost at rest, and in that region inertial force is negligible and Stokes equation (or Stokes approximation) is well justified. Far away from the sphere, the flow velocity approaches \( U \) and Oseen’s approximation is still needs to be more accurate. In 1957, **Proudman and Pearson** solved the Navier-Stokes equations and gave an improved Stokes solution in the neighbourhood of the sphere and an improved Oseen’s solution at infinity, and matched the two solutions in a supposed common region of their validity. They obtained the revised expression of drag in terms of Reynolds number ‘\( R \)’

\[
F = 6 \pi \mu a U \left[ 1 + \frac{3}{8} R + \frac{9}{40} R^2 \log R + O\left( R^2 \right) \right],
\]

(1.4.4)

where \( R = \frac{Ua}{\nu} \) is particles Reynolds number.

There are not many workers who contributed on Oseen’s approximation in various types of flows. Not all, but the mention of few may be worth full here amongst the others **Kaplun**(1957), **Kaplun and Lagerstrom**(1957), **Lagerstrom and Cole**(1955), **Chang**(1960), **Brenner**(1961), **Chester**(1962), **Krasovitskaya et al.**(1970), **Dyer and Ohkawa**(1992).

**1.5 Methods (Evaluation of drag)**

**A. Direct Integration of Stress**

The force \( F_i \) in \( i \) direction on the particle with surface \( S \) is given by the integral

\[
F_i = \int_S t_{ji} n_j dS \quad \left( n_j = \text{unit normal} \right)
\]

(1.5.1)

where

\[
t_{ji} = -p + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

(1.5.2)
is the stress tensor, \( \mu \) the fluid viscosity. Here the velocity \( u_k \) satisfies the Stokes equations (in Cartesian tensor notations)

\[
\begin{align*}
\mu \nabla^2 u_k &= \nabla p \\
u_{k,k} &= 0
\end{align*}
\]

(1.5.3)

where \( p \) is pressure, and the boundary conditions

\[
u_k = U_k \text{ on surface } S,
\]

\[
u_{k,p} \to 0 \text{ as } |x_i| \to \infty.
\]

The equations are valid for small Reynolds number \( \text{Re} = \rho UL/\mu \), where \( \rho \) is density, \( U \) characteristic speed and \( L \) a characteristic length of the problem.

For the general case of an arbitrary particle, an integral equation formulation is most suitable to determine velocity field \( u_k \). Depending upon the nature of the problem, we can formulate it in terms of different kinds of integral equations.

**B. Fredholm’s Integral Equation of Second Kind**

The straight forward way to achieve this is through the generalized Green’s integral formula for the Stokes equations which for the exterior flow may be represented as

\[
u_k(x) = -\frac{1}{8\pi\mu} \int_S \delta_{jk}(x, y) t_{ji} (y) n_i dS - \frac{1}{8\pi\mu} \int_S u_j(y) \delta_{jk}(x, y) n_i dS
\]

(1.5.4)

\[
p(x) = -\frac{1}{8\pi\mu} \int_S p_j(x, y) t_{ji} (y) n_i dS - \frac{1}{4\pi} \int_S u_j(y) \frac{\partial p_j(x, y)}{\partial x_i} n_i dS
\]

(1.5.5)

where

\[
u_{j,k}(x, y) = \frac{\delta_{jk}}{r} + \frac{(x_j - y_j)(x_k - y_k)}{r^3}
\]

(1.5.6a)
\[
t_{ji} = \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \tag{1.5.6b}
\]

\[
p_j(x, y) = \frac{2\mu (x_j - y_j)}{r^3} \tag{1.5.6c}
\]

\[
t_{jki}(x, y) = -\frac{6\mu (x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5} \quad \text{with} \quad r = |x - y|. \tag{1.5.6d}
\]

Without going into details, we give the solutions in terms of double layer potential as

\[
U_k(x) = \left\{ M_{jk}(x, y)g_j(y) \, dS \right\}
\]

\[
p(x) = -\frac{1}{4\pi} \int_S \frac{\partial p_j(x, y)}{\partial x_i} g_j(y) n_i \, dS \tag{1.5.7}
\]

\[
M_{jk}(x, y) = \frac{3}{4\pi} \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5} \tag{1.5.8}
\]

and

\[
\frac{\partial p_j(x, y)}{\partial x_i} = 2\mu \left[ \frac{\delta_{ij}}{r} - \frac{(x_i - y_i)(x_j - y_j)}{r^3} \right], \tag{1.5.9}
\]

where

\[
g_j(y) \quad \text{is the unknown density function determined through the integral equation}
\]

\[
U_k = \frac{1}{2} g_k(x) + \int_S M_{jk}(x, y)g_j(y) \, ds, \quad x, y \in S. \tag{1.5.10}
\]
The existence of the solution can also be established through above formulation.

C. Integral Equation of the First Kind

The double layer formulation is not convenient because eigen functions are in general not available. Therefore, we shall now present the solution in terms of a single layer potential, thereby getting on integral equation of the first kind. The approach has been found useful by Youngren and Acrivos (1975) for computational purpose. It has also been found particularly suitable for treating the more general low Reynolds number problem, in combination with singular perturbation technique by Fischer (1982).

Without going into details, we present below the integral equation

\[ U_k = -\frac{1}{8\pi \mu} \int_S u_{jk}(x,y)f_j(y)\,ds, \quad x, y \in S. \]  

(1.5.11)

Knowing the unknown density \( f_j(y) \), we get in this case

\[ u_k(x) = -\frac{1}{8\pi \mu} \int_S u_{jk}(x,y)f_j(y)\,ds. \]  

(1.5.12)

For those, who are further interested in integral equation formulation of Stokes flow, mention may be made of the works of Power and Miranda (1987). After determining \((u_k, p)\), we make use of the integral (1.4.1) (by utilizing (1.4.2)) to evaluate the drag force. We can easily visualize that it is a lengthy and cumbersome process of evaluation of drag. The derivations in this section are based on the theory of hydrodynamic potential as developed in the book by Ladyzhenskaya (1969).

D. Method of Singularities

Of the few analytical methods available for solving Stokes-flow problems, one is the boundary-value method, which is based on choice of an appropriate co-ordinate system to facilitate separation of the variables for the body geometry in question. Another is the singularity method, whose accuracy depends largely on whether the correct types of singularity are used and how their spatial distributions are chosen. The boundary-value method seems to have been widely adopted in practice, more so than the singularity method. In the literature, the most important exact solutions of Stokes-flow problems are those found by using the classical treatment of the motion of ellipsoids by Oberbeck (1876), Edwards (1892) and Jeffery (1922) (see also Lamb, 1932, p. 604); all these studies are based on the use of ellipsoidal co-ordinates and on some rather
sophisticated analysis of ellipsoidal harmonics. These solutions were not derived until the pioneering work of Chwang and Wu (part 2, 1975) appeared.

Actually, the singularity method has been known since the pioneering work of Lorentz (1897), Oseen (1927), and Burger (1938). It has been further developed and applied in the recent studies of slender-body theory for low-Reynolds-number flows by Hancock (1953), Broersma (1960), Tuck (1964, 1970), Taylor (1969), Batchelor (1970a,b), Tillett (1970), Cox (1970, 1971), Blake and Chwang (part I, 1974), Blake (part II, 1974) and others. Through these investigations the relative simplicity and effectiveness of the method have gradually become more recognized. Nevertheless, it is felt that the potential power of the singularity method has not been fully explored for the general case of arbitrary body shapes as well as for the special case of slender bodies. The primary difficulty is the lack of general knowledge about the types of singularity required and their distribution densities, which are dictated by the specific body shape and different free-stream velocity profiles. It is thought that further development of the method can be greatly enhanced by accumulating a number of exact solutions for several representative cases, since useful information could be extracted from these solutions to guide more general theories.

**Constructive Method of Singularities in Hydrodynamics**

We construct the solution by determining the appropriate distribution of singularities of the Stokes equations. The basic singularities of the Stokes equations are: stokeslet, stresslet, rotlet, Stokes doublet, potential doublet etc. The advantage of this method is that the drag force can be obtained easily by integrating the stokeslet distribution alone, therefore the cumbersome process of finding stress then integrating it is by passed. The drag force (in closed forms) on a prolate spheroid in uniform stream using singularity distribution, was obtained by Chwang and Wu (part 2, 1975) as follows

**Axial Flow**

\[
F_x = 16\pi\mu Uae^3 \left[ -2e + \left( 1 + e^2 \right) L \right]^{-1}, \quad L = (1 + e)/(1 - e).
\]  

(1.5.13a)

**Transverse Flow**

\[
F_y = 32\pi\mu Uae^3 \left[ 2e + \left( 3e^2 - 1 \right) L \right]^{-1}, \quad L = (1 + e)/(1 - e).
\]  

(1.5.13b)
The drag force (in closed forms) on an oblate spheroid in uniform stream using singularity distribution, was obtained by Chwang and Wu (part 2, 1975) as follows:

**Axial Flow**

\[
F_x = 8\pi \mu Uae^3 \left[ e^{\sqrt{(1-e^2)}-\left(1-2e^2\right)} \sin^{-1}e \right]^{-1}.
\] (1.5.14a)

**Transverse Flow**

\[
F_y = 16\pi \mu Uae^3 \left[ -e^{\sqrt{(1-e^2)}+\left(1+2e^2\right)} \sin^{-1}e \right]^{-1}.
\] (1.5.14b)

These values are in agreement with those of Oberbeck (1876) and Jeffery (1922). Drag force on spheroid in a variety of flows have been calculated with this method by Chwang and Wu (part 2, 1975).

**E. DS-Conjecture (Datta and Srivastava, 1999)**

**Axial Flow**

Let us consider the axially symmetric body of characteristic length L placed along its axis (x-axis, say) in a uniform stream U of viscous fluid of density \(\rho_1\) and kinematic viscosity \(\nu\). When Reynolds number \(UL/\nu\) is small, the steady motion is governed by Stokes equations (Happel and Brenner 1964),

\[
0 = -\left(\frac{1}{\rho_1}\right) \text{grad} p + \nu \nabla^2 \mathbf{u}, \quad \text{div} \mathbf{u} = 0,
\] (1.5.15)

subject to the no-slip boundary condition.
For the case of a sphere of radius R, the solution is easily obtained and on evaluating the stress, the drag force F comes out as \( F = \frac{2}{9} \pi \mu U \int_0^\pi R \sin^3 \alpha \, d\alpha = \lambda R \), \( (1.5.16) \)

\[
\lambda = 6\pi \mu U. \tag{1.5.17}
\]
This shows that the drag force increases linearly with the radius of the sphere. In other words, the difference between drag force on two spheres of radii \( y \) and \( y + dy \) is given by

\[
dF = \lambda \, dy.
\]

(A.5.18)

A sphere of radius ‘\( b \)’ is obtained by rotating the curve \( x = b \cos t, \ y = b \sin t (0 \leq t \leq \pi) \) about the \( x \)-axis and the force \( F = \lambda b \) is obtained from (A.5.18) as

\[
\int_0^b \lambda \, dy
\]

exhibiting that the force system \( dF \) may be considered as lying in the \( xy \) plane. The element force \( dF \) may be decomposed into two parts \((1/2) \ dF\), each acting over the upper half and lower half; \((1/2) \ dF\) on the upper half acts at a height \( y \)(say) above the \( x \)-axis. The total force \( F/2 \) on the upper half, may be considered as made up of these differential forces \( dF/2 \) acting over elements corresponding to a system of half spheres of radii increasing from 0 to \( b \) and spread over from \( A \) to \( A'(\text{figure 1(a)}) \). The moment of this force system(taken to be in the \( xy \) plane) about \( O \), provides

\[
h \left( \frac{F}{2} \right) = M = \frac{1}{2} \int_0^b y \ dF = \frac{1}{2} \lambda \int_0^b y \ dy = \frac{1}{4} \lambda b^2,
\]

or

\[
F = \frac{1}{2} \frac{\lambda b^2}{h},
\]

(A.5.19)

where ‘\( h \)’ is the height of centroid of the force system. In the case of a sphere of radius ‘\( b \)’ we have \( F = \lambda b \), and so we get from (A.5.19), \( h = b/2 \), as it should be. Next, we can express (A.5.16) also as

\[
F = \int_{\alpha=0}^{\pi} df ,
\]

(A.5.20)

where

\[
df = \frac{3}{4} \lambda R \sin^3 \alpha \, d\alpha ,
\]

(A.5.21)

is the elemental force on a circular ring element at \( P(\text{figure 1(a)}) \) (Happel and Brenner 1964, eq. (4-17.23), p. 122). For the purpose of calculating \( F/2 \), the force on upper half, \((1/2) \ df\) may be taken to be acting at height \( \eta \)(say), above \( x \)-axis, given by

\[\]
Taking $\eta = R/2$, the result is seen to correspond to the value $h = b/2$ confirmed earlier. Thus, we have

$$h = \frac{3\pi}{8} \int_0^\pi R \sin^3 \alpha \, d\alpha. \quad (1.5.22)$$

It is proposed that the formula (1.5.22) holds good for an axially symmetric body also, when $R$ is interpreted as the normal distance $PM$ between the point $P$ on the body and the point of intersection $M$ of the normal at $P$ with axis of symmetry and $\alpha$ as its slope (figure 1(b)). On inserting the value of $h$ from (1.5.22) in (1.5.19), we finally obtained the expression of drag on axially symmetric body in axial flow

$$F_\parallel = \frac{1}{2} \frac{\lambda b^2}{h_\parallel} = \frac{4}{3} \frac{\left(\lambda b^2\right)}{\int_0^\pi R \sin^3 \alpha \, d\alpha} \cdot \frac{\lambda}{U_\parallel}, \quad (1.5.23)$$

where the suffix ‘$\parallel$’ has been introduced to assert that the force is in the axial direction. While using (1.5.23), it should be kept in mind that ‘$b$’ denotes intercept between the meridian curve and the axis of the normal perpendicular to the axis i.e., $b = R$ at $\alpha = \pi/2$.

Sometimes it will be convenient to work in Cartesian co-ordinates. Therefore, referring to the figure 1(b), for the profile geometry, we have

$$y = R \sin \alpha, \quad \tan \alpha = -\left(\frac{dy}{dx}\right)^{-1} = -\frac{dx}{dy} = -x'. \quad (1.5.24)$$
Using above transformation, we may express (1.5.22) as

$$h_\parallel = -\frac{3}{4} \int_0^a \frac{yy''}{(1+y'^2)^{\frac{3}{2}}} \, dx ,$$  \hspace{1cm} (1.5.25)

where $2a_m$ represents the axial length of the body and dashes represents derivatives with respect to $x$. In the sequel, it will be found simpler to work with $y$ as the independent variable. Thus, $h_\parallel$ assumes the form

$$h_\parallel = -\frac{3}{4} \int_0^b \frac{yx'x''}{(1+x'^2)^{\frac{3}{2}}} \, dy ,$$  \hspace{1cm} (1.5.26)

where dashes represents derivatives with respect to $y$.

**Transverse flow**

We set up a polar coordinate system $(R, \beta, \gamma)$ with $\beta$ as the polar angle with $y$-axis and $\gamma$ the azimuthal angle in $zx$ plane. Since $y$-axis is not the axis of symmetry for the body we can not make use of circular ring elemental force $(3/4) \lambda R \sin^3 \beta \, d\beta$ corresponding to (1.5.21). But we can easily write down the elemental force on the element $R^2 \sin \beta \, d\beta \, d\gamma$ as

$$\delta f = \frac{3\lambda R}{8\pi} \sin^3 \beta \, d\beta \, d\gamma .$$

Transforming the above to the polar coordinate $(R, \theta , \varphi)$ with the $x$-axis as the polar axis, we have

$$\delta f = \frac{3\lambda R}{8\pi} \left(1-\sin^3 \alpha \cos^2 \varphi\right) \sin \alpha \, d\alpha \, d\varphi ,$$

as the force on the element $R^2 \sin \alpha \, d\alpha \, d\varphi$. On integrating over $\varphi$ from $0$ to $2\pi$, we get

$$df_\perp = \frac{3\lambda R}{8} \left(2-\sin^3 \alpha\right) \sin \alpha \, d\alpha ,$$  \hspace{1cm} (1.5.27)

where the suffix ‘$\perp$’ has been placed to designate the force due to the external flow along the $y$-axis, the transverse direction.
Integrating \( df \) over the surface of the sphere, we get

\[
F_\perp = \frac{3\lambda\pi}{8} \int_0^\alpha \left( 2\sin \alpha - \sin^3 \alpha \right) d\alpha = \lambda R ,
\]

(1.5.28)

Agreeing with the correct value. This suggests we can take the force \( df \) as given by (1.5.27) as the element force on the circular ring element at \( P \). Although the force \( F_\perp \) is along the \( y \) direction, we have reduced it to elemental forces on a system of spheres centered on the \( x \)-axis. Since \( F_\perp \) and \( df \) themselves are scalar quantities, on comparing (1.5.27) and (1.5.21), we can use the analysis as in the axial flow case with ‘\( h \)’ replaced by

\[
h_\perp = \frac{3\pi}{16} R \left( 2\sin \alpha - \sin^3 \alpha \right) d\alpha .
\]

(1.5.29)

Thus, we get from (1.5.19)

\[
F_\perp = \frac{1}{2} \frac{\lambda b^2}{h_\perp} , \text{ where } \lambda = 6\pi\mu U_\perp .
\]

(1.5.30)

We have taken up the class of those axially symmetric bodies which possesses continuously turning tangent, placed in a uniform stream \( U \) along the axis of symmetry (which is \( x \)-axis), as well as constant radius ‘\( b \)’ of maximum circular cross-section at the middle of the body.

In the same manner as we did in axial flow, equation (1.5.22) may also be written in Cartesian form as (in both cases having \( x \) and \( y \) treated as independent)

\[
h_\perp = -\frac{3}{8} \int_a^b \frac{yy'' \left[ 1 + 2(y')^2 \right]}{\left[ 1 + (y')^2 \right]^2} \, dx ,
\]

(1.5.31)

and

\[
h_\perp = -\frac{3}{8} \int_0^b \frac{yx'' \left[ 2 + (x')^2 \right]}{\left[ 1 + (x')^2 \right]^2} \, dy ,
\]

(1.5.32)
In (1.5.31) and (1.5.32), the dashes represents derivative with respect to $x$ and $y$ respectively.

This axi-symmetric body is obtained by the revolution of meridional plane curve (depicted in figure 1(b)) about axis of symmetry which obeys the following limitations:

i. Tangents at the points A, on the $x$-axis, must be vertical,
ii. Tangents at the points B, on the $y$-axis, must be horizontal,
iii. The semi-transverse axis length ‘b’ must be fixed.

The point $P$ on the curve may be represented by the Cartesian coordinates $(x,y)$ or polar coordinates $(r,\theta)$ respectively, $PN$ and $PM$ are the length of tangent and normal at the point $P$. The symbol $R$ stands for the intercepting length of normal between the point on the curve and point on axis of symmetry and symbol $\alpha$ is the slope of normal $PM$ which can be vary from $0$ to $\pi$.

The proposed drag formulae is, of course, subject to restrictions on the geometry of the meridional body profile $y(x)$ of continuously turning tangent implying that $y'(x)$ is continuous together with $y''(x) \neq 0$, thereby avoiding corners or sharp edges or other kind of nodes and straight line portions, $y = ax + b$, $x_1 \leq x \leq x_2$. If such type of cases arises in the body, the contribution of drag corresponding to those parts will be zero and true drag value experienced by the body may not be achieved. Also, it should be noted here that the method holds good for convex axially symmetric bodies which possesses fore-aft symmetry about the equatorial axis perpendicular to the axis of symmetry(polar axis). Apart from this argument, It is interesting to note here that the proposed conjecture is applicable also to those axi-symmetric bodies which fulfils the condition of continuously turning tangent but does not possesses fore-aft symmetry like egg shaped body(Datta and Srivastava 1999). This conjecture is much simpler to evaluate the numerical values of drag than other existing numerical methods like Boundary Element Method(BEM), Finite Element Analysis(FEA) etc. as it can be applied to a large set of convex axi-symmetric bodies possessing fore-aft symmetry about maximal radius situated in the middle of the body for which analytical solution is not available or impossible to evaluate.

1.6 Brief description of present work

In the present work, certain steady low Reynolds number problems involving axially symmetric bodies in an incompressible slow viscous fluid under the limit of Stokes and Oseen’s approximation along with related problems have been investigated.
In chapter-2, the problem of steady Stokes flow past deformed sphere has been dealt in both situations when uniform stream is along the axis of symmetry (axial flow) and is perpendicular to the axis of symmetry (transverse flow). The most general form of deformed sphere, governed by polar equation, 
\[ r = a \left( 1 + \varepsilon \sum_{k=0}^{\infty} d_k P_k(\cos \theta) \right), \]
(where \( d_k \) is shape factor and \( P_k \) is Legendre function of first kind) has been considered for the study. The method used here is based on geometry of axially symmetric bodies developed by author [Datta and Srivastava, 1999] which, in particular, holds good for sphere and class of spheroidal bodies. The general expressions for axial and transverse Stokes drag for deformed sphere has been derived up to the order of \( O(\varepsilon^2) \). The class of oblate axisymmetric bodies is considered for the validation and further numerical discussions. All the expressions of drag are updated up to the order of \( O(\varepsilon^2) \), where \( \varepsilon \) is deformation parameter. In particular, up to the order of \( O(\varepsilon) \), the numerical values of drag coefficients have been evaluated for the various values of deformation parameters(\( \varepsilon \)) and aspect ratio(b/a) for a class of oblate axially symmetric bodies including flat circular disk and compared with some known values already exist in the literature.

In chapter-3, continuing the efforts of chapter 2, the class of prolate axisymmetric bodies is considered for the validation and further numerical discussions. All the expressions of drag are updated up to the order of \( O(\varepsilon^2) \), where \( \varepsilon \) is deformation parameter. In particular, up to the order of \( O(\varepsilon) \) and \( O(\varepsilon^2) \), the numerical values of drag coefficients and their ratio have been evaluated for the various values of deformation parameters(\( \varepsilon \)) and aspect ratio(b/a) for a class of prolate axially symmetric bodies including thin slender elongated body as a special case and compared with some known asymptotic values already exist[Happel & Brenner(1964); Chwang & Wu(1975)] in the literature. Some important applications are also highlighted.

In chapter-4, the problem of steady Stokes flow past dumbbell-shaped axially symmetric isolated body of revolution about its axis of symmetry is considered by utilizing a method [Datta and Srivastava, 1999] based on body geometry under the restrictions of continuously turning tangent on the boundary. The relationship between drag and moment(torque) is established in axial(longitudinal) flow situation. The closed form expression of Stokes drag is then calculated for dumbbell-shaped body in terms of geometric parameters b, c, d and a with the aid of this linear relationship and the formula of torque obtained by (Chwang and Wu, part 1, 1974) with the use of singularity distribution along axis of symmetry. Drag coefficient and moment(torque) coefficient are defined in various forms in terms of dumbbell parameters. Their numerical
values are calculated and depicted in respective graphs and compared with values of same. Some important applications in blood flow are also highlighted.

In chapter-5, the rotation of deformed sphere has been dealt in axial or longitudinal situation in which uniform stream is along the axis of symmetry under the no-slip boundary conditions. The general expression of torque in terms of axial Stokes drag (Srivastava et al. (2012)) is produced with the help of method developed by author (Datta and Srivastava, 1999). The general form of deformed sphere, governed by polar equation, \( r = a \left( 1 + \varepsilon \sum_{k=0}^{\infty} d_k P_k (\cos \theta) \right) \),

(where \( d_k \) is shape factor and \( P_k \) is Legendre function of first kind), is considered for the study. The expression of torque for slowly rotating deformed sphere is derived up to the order of \( O(\varepsilon^2) \). The class of oblate and prolate axisymmetric bodies is considered for the validation and further numerical discussion. All the expressions of moment (torque) coefficients are updated up to the order of \( O(\varepsilon^2) \), where \( \varepsilon \) is deformation parameter. In particular, up to the order of \( O(\varepsilon) \) and \( O(\varepsilon^2) \), the numerical values of moment coefficients and their ratio have been evaluated for the various values of deformation parameters \( \varepsilon \) and aspect ratio \( b/a \) for a class of oblate and prolate axially symmetric bodies including disk and thin slender elongated bodies as a special cases.

In chapter-6, the Oseen’s correction to axial Stokes drag on deformed sphere is presented by using Brenner’s (Brenner, 1961) formula in general first and then applied to prolate and oblate deformed spheroid up to the second order of deformation parameter. Numerical values of Oseen’s correction is obtained with respect to deformation parameter ‘\( \varepsilon \)’ and Reynolds number ‘\( R \)’. The corresponding variations are depicted in figures. Some particular cases of needle shaped body and flat circular disk are considered and found to be in good agreement with those exist in the literature. The important applications are also highlighted.

In chapter-7, Brenner’s formula (Brenner, 1961) for Oseen’s correction to Stokes drag on axially symmetric particle placed in uniform stream parallel to its axis of symmetry is advanced to transverse flow situation. For this, a linear relationship between axial and transverse Stokes drag (Datta and Srivastava, 1999) is utilized and applied to a class of deformed sphere. The general expressions of Oseen’s correction to Stokes drag are further utilized over perturbed prolate and oblate spheroid and corrected to first order of Reynolds number ‘\( R \)’ and second order of deformation parameter ‘\( \varepsilon \)’. The respective variation of Oseen’s drag with respect to deformation parameter ‘\( \varepsilon \)’ and low
Reynolds number ‘R’ are depicted through graphs and compared with some known values. Some important applications are also highlighted.

In chapter-8, the problem of a pervious sphere carrying a fluid sink at its centre and rotating with slow uniform angular velocity $\Omega$ about a diameter is studied. The analysis reveals that only the azimuthal component of velocity exists and is seen that the effect of sink is to decrease it. The general expression of angular velocity as a function of sink parameter ‘$s=Q/va$’ and radial length ‘$r$’ is evaluated and it is found that for specific values of ‘$s$’ and $r \geq 1$, angular velocity decreases to zero. The torque experienced by the rotating sphere is calculated. Limiting cases, for low and large values of sink parameter ‘$s$’ are discussed and results obtained here are compared with some known values.

In chapter-9, the problem of concentric pervious spheres carrying a fluid sink at their centre and rotating slowly with different uniform angular velocities $\Omega_1$, $\Omega_2$ about a diameter has been studied. The analysis reveals that only azimuthal component of velocity exists and the torque, rate of dissipated energy is found analytically in the present situation. The expression of torque on inner sphere rotating slowly with uniform angular velocity $\Omega_1$, while outer sphere also rotates slowly with uniform angular velocity $\Omega_2$, is evaluated. The special cases like, (i) inner sphere is fixed (i.e. $\Omega_1 = 0$), while outer sphere rotates with uniform angular velocity $\Omega_2$, (ii) outer sphere is fixed (i.e. $\Omega_2 = 0$), while inner sphere rotates with uniform angular velocity $\Omega_1$, (iii.) inner sphere rotates with uniform angular velocity $\Omega_1$, while outer rotates at infinity with angular velocity $\Omega_2$ have been deduced. The corresponding variation of torque with respect to sink parameter has been shown via figures.

All the tables are provided in the end of every chapters followed by concerned figures.