CHAPTER- 3

Ultra metric spaces and some fixed point theorems

In 1970, Van Roovij[23] initiated the study of ultra metric spaces. We introduce the concept of convex structure, J-convex structure, strong convex structure and quasi convex structure on ultra metric spaces and discuss their basic properties. Also, we extend fixed point theorems in these convex structures.

3.1. Convex ultra metric spaces

We introduce the concept of convex ultra metric space and discuss some basic properties. If (X, d) is an ultra metric space, then a mapping $W:X\times X\times [0, 1] \rightarrow X$ satisfy the convex structure conditions

$$d(u, W(x, y, \lambda)) \leq \max\{\lambda d(u, x), (1-\lambda)d(u, y)\}$$

Eqn.3.1

for each $(x, y, \lambda) \in X\times X\times [0, 1]$ and for all $u \in X$.

Then by taking $u=x$ and $x\neq y$ in Eqn.3.1 we get

$$d(x, W(x, y, \lambda)) \leq (1-\lambda)d(x, y).$$

Eqn.3.2

Similarly by taking $u = y$ and $x \neq y$ in Eqn.3.1 we get

$$d(y, W(x, y, \lambda)) \leq \lambda d(x, y).$$

Eqn.3.3

Then for $x \neq y$, we have $d(x, y) \leq \max\{d(x, W(x, y, \lambda)), d(W(x, y, \lambda), y)\}$

$$\leq \max\{ (1-\lambda) d(x, y), \lambda d(x, y)\}$$

$$= d(x, y) \max\{(1-\lambda), \lambda\} = \lambda d(x, y) \text{ or } (1-\lambda)d(x, y).$$

Then by taking $\lambda = \frac{1}{2}$, $d(x, y) \leq \frac{1}{2}d(x, y)$. This implies that $d(x, y) = 0$ for $x \neq y$. This is
impossible in an ultra metric space. Therefore Eqn. 3.1 is not valid for u=x or u=y. This motivates us to define the following structure in an ultra metric space.

**Definition 3.1.1:** Let \((X, d)\) be an ultra metric space and \(I= [0, 1]\). A mapping \(W:X \times X \times [0, 1] \rightarrow X\) is said to be a convex ultra structure on \((X, d)\) if for each \((x, y, \lambda) \in X \times X \times [0, 1]\) the following hold.

(i) \(d(u, W(x, y, \lambda)) \leq \max\{ \lambda d(u, x), (1-\lambda)d(u, y) \}\) for every \(u \in X \setminus \{x, y\}\).

(ii) \(d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y)\) for every \(u \in \{x, y\}\).

If \(W\) is a convex ultra structure on \((X, d)\), then the triplet \((X, d, W)\) is called a convex ultra metric space.

**Proposition 3.1.2:** Let \((X, d, W)\) be a convex ultra metric space. Suppose \(u, x, y \in X\) and \(0 \leq \lambda \leq 1\). Then

(i) \(d(x, W(x, y, \lambda)) \leq (1-\lambda)d(x, y)\).

(ii) \(d(y, W(x, y, \lambda)) \leq \lambda d(x, y)\).

(iii) \(d(u, W(x, y, 0)) \leq d(u, y)\).

(iv) \(d(u, W(x, y, 1)) \leq d(u, x)\).

(v) \(d(u, W(x, y, 1-\lambda)) \leq \max\{\lambda d(u, y), (1-\lambda)d(u, x)\}\).

(vi) \(d(u, W(x, y, \frac{1}{2})) \leq \frac{1}{2} \max\{d(u, y), d(u, x)\}\).

(vii) \(d(x, W(x, y, \frac{1}{2})) \leq \frac{1}{2} d(x, y)\).

(viii) \(d(y, W(x, y, \frac{1}{2})) \leq \frac{1}{2} d(x, y)\).

(ix) \(d(x, y) = d(x, W(x, y, \lambda)) + d((W(x, y, \lambda), y)\).
Proof: By taking $u=x$ in Definition 3.1.1, we get
\[ d(x, W(x, y, \lambda)) \leq \max \{ \lambda d(x, x), (1-\lambda) d(x, y) \} = (1-\lambda) d(x, y). \] This proves that
\[ d(x, W(x, y, \lambda)) \leq (1-\lambda) d(x, y). \]

Similarly by taking $u=y$ in Definition 3.1.1, we get
\[ d(y, W(x, y, \lambda)) \leq \max \{ \lambda d(y, x), (1-\lambda) d(y, y) \} = \lambda d(y, x). \]

When $\lambda=0$ Definition 3.1.1 becomes
\[ d(u, W(x, y, 0)) \leq \max \{ 0 \cdot d(u, x), (1-0) \cdot d(u, y) \} = d(u, y). \]
Similarly, when $\lambda=1$ Definition 3.1.1 becomes
\[ d(u, W(x, y, 1)) \leq \max \{ 1 \cdot d(u, x), (1-1) \cdot d(u, y) \} = d(u, y). \]

Assume that $\lambda=1-\lambda$ in Definition 3.1.1 we get
\[ d(u, W(x, y, (1-\lambda))) \leq \max \{ (1-\lambda) d(u, x), (1-(1-\lambda)) d(u, y) \} = \max \{ (1-\lambda) d(u, x), \lambda d(u, y) \}. \]

By taking $\lambda=\frac{1}{2}$ in Definition 3.1.1, we get
\[ d(u, W(x, y, \frac{1}{2})) \leq \max \{ \frac{1}{2} d(u, x), (1-\frac{1}{2}) d(u, y) \} = \frac{1}{2} \max \{ d(u, x), d(u, y) \}. \]

By taking $u=x$ in (vi), we get
\[ d(x, W(x, y, \frac{1}{2})) \leq \max \{ \frac{1}{2} d(x, x), \frac{1}{2} d(x, y) \} = \frac{1}{2} d(x, y). \]
Similarly by taking $u=y$ in (vii), we get
\[ d(y, W(x, y, \frac{1}{2})) \leq \max \{ \frac{1}{2} d(y, x), \frac{1}{2} d(y, y) \} = \frac{1}{2} d(y, x). \]

Since $(X, d, W)$ is a convex ultra metric space, for $x, y \in X$ and $0 \leq \lambda \leq 1$, we obtain
\[ d(x, y) \leq d(x, W(x, y, \lambda)) + d((W(x, y, \lambda), y) \]
\[ \leq \max \{ \lambda d(x, x), (1-\lambda) d(x, y) \} + \max \{ \lambda d(x, y), (1-\lambda) d(y, y) \} \]
\[ = (1-\lambda) d(x, y) + \lambda d(x, y) = d(x, y). \]

This shows that $d(x, y) = d(x, W(x, y, \lambda)) + d((W(x, y, \lambda)), y)$. Hence the proposition is proved. ■
Proposition 3.1.3: Let \((X, d, W)\) be a convex ultra metric space. Suppose \(u, x, y \in X\) and \(0 \leq \lambda \leq 1\). Then

(i) \(d(u, W(x, x, \lambda)) \leq d(u, x)\).

(ii) \(d(u, W(x, x, \lambda)) \leq (1-\lambda)d(u, x)\), when \(0 \leq \lambda \leq \frac{1}{2}\).

(iii) \(d(u, W(x, x, \lambda)) \leq \lambda d(u, x)\), when \(\frac{1}{2} \leq \lambda \leq 1\).

(iv) \(d(x, W(x, x, \lambda)) = 0\).

(v) \(W(x, x, \lambda) = x\).

Proof: In Definition 3.1.1(ii) put \(y = x\), we get
\[
d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, x) = d(u, x).
\]

This proves \(d(u, W(x, x, \lambda)) \leq d(u, x)\).

In Definition 3.1.1(ii) put \(y = x\), we get
\[
d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, x) = (1-\lambda)d(u, x), \text{ when } 0 \leq \lambda \leq \frac{1}{2}.
\]

Similarly in Definition 3.1.1(ii) put \(y = x\), we get
\[
d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, x) \leq \lambda d(u, x), \text{ when } \frac{1}{2} \leq \lambda \leq 1.
\]

In (i) if \(u = x\), then \(d(x, W(x, x, \lambda)) \leq d(x, x) = 0\). This implies \(d(x, W(x, x, \lambda)) = 0\). From this we conclude \(x = W(x, x, \lambda)\).

Example 3.1.4: Consider a linear space \(L\) which is also an ultra metric space with the following properties:

(i) For \(x, y \in L\), \(d(x, y) = d(x-y, 0)\).

(ii) For \(x, y \in L\) and \(0 \leq \lambda \leq 1\),
\[
d(\lambda x + (1-\lambda)y, 0) \leq \max \{\lambda d(x, 0), (1-\lambda)d(y, 0)\} \text{ and } d(\lambda x, \lambda y) \leq \lambda d(x, y).
\]
Suppose $u \neq x$ and $u \neq y$. Then
\[
d(u, W(x, y, \lambda)) = d(u - \lambda x - (1-\lambda)y, 0)
\]
\[
= d(\lambda u + (1-\lambda)u - \lambda x - (1-\lambda)y, 0)
\]
\[
= d(\lambda(u - x) + (1-\lambda)(u - y), 0)
\]
\[
\leq \max\{\lambda d(u - x, 0), (1-\lambda)d(u - y, 0)\}
\]
\[
= \max\{(1-\lambda)d(u, y), \lambda d(u, x)\}.
\] \text{Eqn.3.4}

Suppose $u = x$ and $u \neq y$. Then
\[
d(x, W(x, y, \lambda)) = d(x - \lambda x - (1-\lambda)y, 0)
\]
\[
= d((1-\lambda)x - (1-\lambda)y, 0)
\]
\[
= d((1-\lambda)x, (1-\lambda)y) \leq (1-\lambda)d(x, y)
\]
\[
\leq (1-\lambda)d(u, y) + \lambda d(u, x).
\] \text{Eqn.3.5}

Suppose $u = y$ and $u \neq x$. Then
\[
d(y, W(x, y, \lambda)) = d(y - \lambda x - (1-\lambda)y, 0)
\]
\[
= d((-\lambda x + \lambda y, 0)
\]
\[
= d(\lambda x, \lambda y) \leq \lambda d(x, y)
\]
\[
\leq (1-\lambda)d(u, y) + \lambda d(u, x).
\] \text{Eqn.3.6}

From the equations 3.4, 3.5, 3.6 and Definition 3.1.1, we prove that $(L, d, W)$ is a convex ultra metric space.

\textbf{Definition 3.1.5:} A subset $M$ of a convex ultra metric space $(X, d, W)$ is said to be a convex set in $(X, d, W)$ if $W(x, y, \lambda) \in M$ for all $x, y \in M$ and for all $\lambda \in [0,1]$. 

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Proposition 3.1.6: Let \( \{ K_\alpha : \alpha \in \Delta \} \) be a family of convex subsets of the convex ultra metric space \((X, d, W)\), then \( \bigcap_{\alpha \in \Delta} K_\alpha \) is also a convex subset of \((X, d, W)\).

Proof: Let \( M = \bigcap_{\alpha \in \Delta} K_\alpha \). If \( M = \emptyset \), then \( M \) is trivially a convex subset of \((X, d, W)\).

Suppose that \( M \neq \emptyset \), let \( x, y \in M \) and \( 0 \leq \lambda \leq 1 \). Since each \( K_\alpha \) is a convex subset of \((X, d, W)\), \( W(x, y, \lambda) \in K_\alpha \) for every \( \alpha \in \Delta \) and hence \( W(x, y, \lambda) \in M \). This proves that \( M \) is a convex subset of \((X, d, W)\).

Proposition 3.1.7: Let \((X, d, W)\) be a convex ultra metric space. For every \( z \in X \) and \( r > 0 \)

(i) The balls \( S(z, r) \) and \( S[z, r] \) of \((X, d, W)\) are convex subsets of \((X, d, W)\).

(ii) If \( x, y \in S(z, r) \) and \( 0 \leq \lambda \leq 1 \) then \( S(x, r) = S(y, r) = S(W(x, y, \lambda), r) = S(z, r) \).

Proof: Let \( x, y \in S(z, r) \) and \( 0 \leq \lambda \leq 1 \). Then \( d(x, z) < r \) and \( d(y, z) < r \). Suppose \( z \neq x \) and \( z \neq y \).

By using Definition 3.1.1,

\[
d(z, W(x, y, \lambda)) \leq \max \{ \lambda d(z, x), (1-\lambda)d(z, y) \} \]

\[
< \max \{ \lambda r, (1-\lambda)r \} < r.
\]

Suppose \( z = x \) and \( z \neq y \). Then \( d(z, W(x, y, \lambda)) \leq (1-\lambda)d(z, y) < r \).

Suppose \( z = y \) and \( z \neq x \). Then \( d(z, W(x, y, \lambda)) \leq \lambda d(z, y) < r \).

The above arguments show that \( W(x, y, \lambda) \in S(z, r) \) and hence \( S(z, r) \) is convex. Analogously it can be proved that \( S[z, r] \) is convex.

Let \( x, y \in S(z, r) \) and \( \lambda \in I \). In convex ultra metric space \( S(x, r) = S(y, r) = S(z, r) \).

Since \( S(z, r) \) is convex, \( W(x, y, \lambda) \in S(z, r) \). Hence \( S(x, r) = S(y, r) = S(z, r) = S(W(x, y, \lambda), r) \).

This proves the proposition.
**Proposition 3.1.8:** If \((X, d, W)\) is a convex ultra metric space then it is a convex metric space.

**Proof:** Let \((X, d, W)\) be a convex ultra metric space and \(x,y \in X\) for \(0 \leq \lambda \leq 1\),

\[
d(u, W(x, y, \lambda)) \leq \max \{\lambda d(u, x), (1-\lambda)d(u, y)\}, \text{ for every } u \in X \setminus \{x, y\}
\]

\[
\leq \lambda d(u, x) + (1-\lambda)d(u, y).
\]

This shows that \((X, d, W)\) is a convex metric space. 

\(\blacksquare\)

**Proposition 3.1.9:** Let \((X, d, W)\) be a convex ultra metric space. Let \(A \subseteq X\).

If \((X, d, W)\) has the Property(C) as convex metric space, then for every subset \(A\) of \(X\),

\[A_c = \{x \in X : R_x(A) = R(A)\}\]

is a nonempty set, closed and convex.

**Proof:** Since \((X, d, W)\) is a convex ultra metric space, by using Proposition 3.1.8 \((X, d, W)\) is a convex metric space. If the convex metric space \((X, d, W)\) has the Property(C), then by applying Lemma 2.1.4, \(A_c = \{x \in X : R_x(A) = R(A)\}\) is a nonempty set, closed and convex.

\(\blacksquare\)

**Proposition 3.1.10:** Let \(M\) be a nonempty compact subset of a convex ultra metric space \((X, d, W)\) and \(K\) be the least closed convex set containing \(M\). If the diameter \(\delta(M)\) is positive, then there exists an element \(u \in K\) such that \(\sup \{d(x, u) : x \in M\} < \delta(M)\).

**Proof:** Since \((X, d, W)\) is a convex ultra metric space, by using Proposition 3.1.8 \((X, d, W)\) is a convex metric space. Then \(K\) is also a convex set in the convex metric space. By applying Lemma 2.1.5, there exists an element \(u \in K\) such that \(\sup \{d(x, u) : x \in M\} < \delta(M)\).

\(\blacksquare\)
**Definition 3.1.11:** A convex ultra metric space \((X, d, W)\) is said to have the normal structure if for each closed bounded convex subset \(A\) of \((X, d, W)\) which contains at least two points, there exists \(x \in A\) which is not a diametral point of \(A\).

**Theorem 3.1.12:** Suppose that a convex ultra metric space \((X, d, W)\) has the Property(C). Let \(K\) be a nonempty bounded closed convex subset of \((X, d, W)\) with normal structure. If \(T\) is a nonexpansive mapping of \(K\) into itself, then \(T\) has a fixed point in \(K\).

**Proof:** Since \((X, d, W)\) is a convex ultra metric space, by using Proposition 3.1.8, \((X, d, W)\) is a convex metric space. Then \(K\) is also a bounded closed convex subset of convex metric space \((X, d, W)\) with normal structure. Since \(T\) is a nonexpansive mapping of \(K\) into itself, then by applying Lemma 2.1.6, \(T\) has a fixed point in \(K\). \(\blacksquare\)

**Definition 3.1.13:** Let \(K\) be a compact convex ultra metric space. Then a family \(\mathcal{F}\) of nonexpansive mappings \(T\) of \(K\) into itself is said to have an invariant property in \(K\) if for any compact and convex subset \(A\) of \(K\) such that \(TA \subseteq A\) for each \(T \in \mathcal{F}\), there exists a compact subset \(M \subseteq A\) such that \(TM = M\) for each \(T \in \mathcal{F}\).

**Theorem 3.1.14:** Let \(K\) be a compact convex ultra metric space. If \(\mathcal{F}\) is a family of nonexpansive mappings of \(K\) into itself with invariant property in \(K\), then the family \(\mathcal{F}\) has a common fixed point.

**Proof:** Since \((X, d, W)\) is a convex ultra metric space, by using Proposition 3.1.8 \((X, d, W)\) is a convex metric space. Then \(K\) is also a convex metric space in \((X, d, W)\). If \(\mathcal{F}\) is a family of nonexpansive mappings of \(K\) into itself with invariant property in \(K\), then by applying Lemma 2.1.8, the family \(\mathcal{F}\) has a common fixed point. \(\blacksquare\)
3.2. J-convex ultra metric spaces

In this section, we study the definition of the J-convex ultra structure, its basic properties and establish some fixed point theorems.

**Definition 3.2.1:** Let \((X, d)\) be an ultra metric space. A mapping \(W : X \times X \times (0, 1) \rightarrow X\) is said to be a J-convex ultra structure on \((X, d)\) if the following hold.

(i) \[d(u, W(x, y, \lambda)) \leq \min\{\lambda d(u, x), (1-\lambda)d(u, y)\}\] for each \((x, y, \lambda) \in X \times X \times (0, 1)\) and for all \(u \in X \setminus \{x, y\}\).

(ii) \[d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y)\] for every \(u \in \{x, y\}\).

If \(W\) is a J-convex ultra structure on \((X, d)\), then the triplet \((X, d, W)\) is called a J-convex ultra metric space.

**Proposition 3.2.2:** Let \((X, d, W)\) be a J-convex ultra metric space. Suppose \(u, x, y \in X\) and \(0 < \lambda < 1\). Then

(i) \[d(x, W(x, y, \lambda)) \leq (1-\lambda)d(x, y)\].

(ii) \[d(y, W(x, y, \lambda)) \leq \lambda d(x, y)\].

(iii) \[d(u, W(x, y, \frac{1}{2})) \leq \frac{1}{2} \min\{d(u, y), d(u, x)\}\].

(iv) \[d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))\].

**Proof:** In Definition 3.2.1, put \(u = x\) then

\[d(x, W(x, y, \lambda)) \leq \min\{\lambda d(x, x), (1-\lambda)d(x, y)\} = (1-\lambda)d(x, y)\].

Therefore, \(d(x, W(x, y, \lambda)) \leq (1-\lambda)d(x, y)\).

Similarly by taking \(u = y\) in Definition 3.2.1, we get

\[d(y, W(x, y, \lambda)) \leq \min\{\lambda d(x, y), (1-\lambda)d(y, y)\} = \lambda d(x, y)\].

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When \( \lambda = \frac{1}{2} \), Definition 3.2.1 becomes \( d(u, W(x, x, \frac{1}{2})) \leq \min\{\frac{1}{2}d(u, x), (1-\frac{1}{2})d(u, y)\} = \min\{\frac{1}{2}d(u, x), \frac{1}{2}d(u, y)\} = \frac{1}{2} \min\{d(u, x), d(u, y)\} \).

Since \((X, d, W)\) is a \(J\)-convex ultra metric space, for \( x, y \in X \) and \( 0 \leq \lambda \leq 1 \), we obtain

\[
d(x, y) \leq d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y)
\]

\[
\leq \min\{\lambda d(x, x), (1-\lambda) d(x, y)\} + \min\{\lambda d(x, y), (1-\lambda) d(y, y)\}
\]

\[
= (1-\lambda) d(x, y) + \lambda d(x, y) = d(x, y).
\]

This shows that \( d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y) \).

**Proposition 3.2.3:** Let \((X, d, W)\) be a \(J\)-convex ultra metric space. Suppose \( u, x, y \in X \) and \( 0 < \lambda < 1 \). Then

(i) \( d(u, W(x, x, \lambda)) \leq d(u, x) \).

(ii) \( d(u, W(x, x, \lambda)) \leq \lambda d(u, x) \), when \( 0 \leq \lambda \leq \frac{1}{2} \).

(iii) \( d(u, W(x, x, \lambda)) \leq (1-\lambda)d(u, x) \), when \( \frac{1}{2} \leq \lambda \leq 1 \).

(iv) \( d(x, W(x, x, \lambda)) = 0 \).

(v) \( W(x, x, \lambda) = x \).

**Proof:** In Definition 3.2.1(ii) put \( y = x \), we get

\[
d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, x) = d(u, x).
\]

In Definition 3.2.1(ii) put \( y = x \), we get \( d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, x) = (1-\lambda)d(u, x) \) when \( 0 \leq \lambda \leq \frac{1}{2} \).

Similarly in Definition 3.2.1(ii) put \( y = x \), we get

\[
d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, x) \leq \lambda d(u, x) \) when \( \frac{1}{2} \leq \lambda \leq 1 \).

In (i) if \( u = x \), then \( d(x, W(x, x, \lambda)) \leq d(x, x) = 0 \). This implies \( d(x, W(x, x, \lambda)) = 0 \). From this we conclude \( x = W(x, x, \lambda) \).
Example 3.2.4: Consider a linear space $L$ which is also an ultra metric with the following properties:

(i) For $x, y \in L$, $d(x, y) = d(x - y, 0)$.

(ii) For $x, y \in L$ and $0 < \lambda < 1$, $d(\lambda x + (1 - \lambda) y, 0) \leq \min\{\lambda d(x, 0), (1 - \lambda) d(y, 0)\}$ and

$$\min\{d(-\lambda x, \lambda y), d(\lambda x, \lambda y)\} \leq \lambda d(x, y).$$

Let $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$ for all $x, y \in X$ and $\lambda$ with $0 < \lambda < 1$.

Suppose $u \neq x$ and $u \neq y$. Then

$$d(u, W(x, y, \lambda)) = d(u - \lambda x - (1 - \lambda)y, 0)$$

$$= d(\lambda(x - y), 0)$$

$$= d(\lambda(x - y), 0)$$

$$\leq \min\{\lambda d(x, 0), (1 - \lambda) d(y, 0)\}$$

$$= \min\{(1 - \lambda)d(u, y), \lambda d(u, x)\}. \tag{Eqn.3.7}$$

Suppose $u = x$ and $u \neq y$. Then

$$d(x, W(x, y, \lambda)) = d(x - \lambda x - (1 - \lambda)y, 0)$$

$$= d((1 - \lambda)x - (1 - \lambda)y, 0)$$

$$= d((1 - \lambda)x, (1 - \lambda)y)$$

$$\leq (1 - \lambda)d(x, y). \tag{Eqn.3.8}$$

Suppose $u = y$ and $u \neq x$. Then

$$d(y, W(x, y, \lambda)) = d(y - \lambda x - (1 - \lambda)y, 0)$$

$$= d((- \lambda x + \lambda y, 0) = d(\lambda x, \lambda y) \leq \lambda d(x, y). \tag{Eqn.3.9}$$

From the Eqn.3.7, Eqn.3.8, Eqn.3.9 and Definition 3.3.1, we prove that $(L, d, W)$ is a $J$-convex ultra metric space.
**Definition 3.2.5:** A subset $M$ of J-convex ultra metric space $(X, d, W)$ is said to be a convex set in $(X, d, W)$ if $W(x, y, \lambda) \in M$ for all $x, y \in M$ and for all $\lambda \in (0, 1)$.

**Proposition 3.2.6:** Let $\{K_{\alpha} : \alpha \in \Delta\}$ be a family of convex subsets of the J-convex ultra metric space $(X, d, W)$, then $\bigcap_{\alpha \in \Delta} K_{\alpha}$ is also a convex subset in $(X, d, W)$.

**Proof:** Let $M = \bigcap_{\alpha \in \Delta} K_{\alpha}$. If $M = \emptyset$, then $M$ is a convex subset of $(X, d, W)$. Suppose $M \neq \emptyset$. Let $x, y \in M$ and $\lambda \in (0, 1)$. Since $K_{\alpha}$ is a convex subset of $(X, d, W)$, by using Definition 3.2.5, $W(x, y, \lambda) \in K_{\alpha}$ for every $\alpha \in \Delta$ and hence $W(x, y, \lambda) \in M$. This proves that $M$ is a convex subset of $(X, d, W)$.

**Proposition 3.2.7:** Let $(X, d, W)$ be a J-convex ultra metric space. Then the balls $S(z, r)$ and $S[z, r]$ of $(X, d, W)$ are convex subsets of $(X, d, W)$ for every $z \in X$ and $r > 0$.

**Proof:** Let $x, y \in S(z, r)$ and $0 < \lambda < 1$. Then $d(x, z) < r$ and $d(y, z) < r$. Suppose $z \neq x$ and $z \neq y$. Then using Definition 3.2.1,

$$d(z, W(x, y, \lambda)) \leq \min \{ \lambda d(z, x), (1-\lambda)d(z, y) \} < \min \{ \lambda r, (1-\lambda)r \} < r.$$  

Suppose $z = x$ and $z \neq y$. Then $d(z, W(x, y, \lambda)) \leq (1-\lambda)d(z, y) < r$.

Suppose $z = y$ and $z \neq x$. Then $d(z, W(x, y, \lambda)) \leq \lambda d(z, y) < r$.

The above arguments show that $W(x, y, \lambda) \in S(z, r)$ that implies $S(z, r)$ is convex. Analogously it can be proved that $S[z, r]$ is convex.

**Proposition 3.2.8:** Let $(X, d, W)$ be a J-convex ultra metric space and $z \in X$, $r > 0$. If $x, y \in S(z, r)$ and $\lambda (0 < \lambda < 1)$. Then $S(x, r) = S(y, r) = S(W(x, y, \lambda), r) = S(z, r)$. 

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Proof: Let \( x, y \in S(z, r) \) and \( \lambda \in (0, 1) \). In the J-convex ultra metric space \( S(x, r)=S(y, r)=S(z, r) \). Since \( S(z, r) \) is convex, \( W(x, y, \lambda) \in S(z, r) \). Hence \( S(x, r) = S(y, r) = S(z, r) = S(W(x, y, \lambda), r) \). This proves the proposition.

Definition 3.2.9: A J-convex ultra metric space \( (X, d, W) \) is said to have the Property(JCU) if every bounded decreasing sequence of nonempty closed convex subsets of \( (X, d, W) \) has nonempty intersection.

Proposition 3.2.10: Let \( (X, d, W) \) be a J-convex ultra metric space. Let \( A \subseteq X \).
If \( (X, d, W) \) has the Property(JCU), then \( A_c = \{ x \in X : R_x(A) = R(A) \} \) is a nonempty set, closed and convex.

Proof: Let \( x \in A \). For every positive integer \( n \), define
\[
A_n(x) = \{ y \in A : d(x, y) \leq R(A) + \frac{1}{n} \}
\]
and \( C_n = \bigcap_{x \in A} A_n(x) \). Since \( x \in A_n(x) \), \( A_n(x) \) is a nonempty set and closed. Let \( y, z \in A_n(x) \) and \( \lambda \in (0, 1) \).
\[
d(x, W(y, z, \lambda)) \leq \min \{ \lambda d(x, y), (1-\lambda)d(x, z) \}
\]
\[
\leq \min \{ \lambda(R(A) + \frac{1}{n}), (1-\lambda)(R(A) + \frac{1}{n}) \} \leq R(A) + \frac{1}{n}.
\]
Therefore \( W(y, z, \lambda) \in A_n(x) \) and hence \( A_n(x) \) is a convex set.

We claim that \( \{ C_n \} \) is a bounded decreasing sequence of nonempty, closed and convex subset.

Let \( \varepsilon > 0 \), then by the definition of \( R(A) \), there exists \( y \in A \) with \( R_y(A) < R(A) + \varepsilon \) and by the definition of \( R_y(A) \), \( d(x, y) \leq R_y(A) \) for every \( x \in A \).
\[
< R(A) + \varepsilon \text{ for every } x \in A.
\]
As ε is arbitrary for every \( x \in A \), \( d(x, y) \leq R(A) + \varepsilon < R(A) + \frac{1}{n} \). Hence \( y \in A_n(x) \) for every \( x \in A \) and hence \( y \in \bigcap_{x \in A} A_n(x) = C_n \). Thus, \( C_n \) is a nonempty set. Since each \( A_n \) is a closed set \( C_n \) is also closed. By Proposition 3.2.6, \( C_n \) is a convex set. Hence, \( C_n \) is a nonempty, closed and convex set for all \( n \).

Let \( z \in C_{n+1} \). Then \( z \in A_{n+1}(x) \) for every \( x \in A \), \( d(x, z) \leq R(A) + \frac{1}{n+1} \leq R(A) + \frac{1}{n} \).

Therefore \( z \in A_n(x) \) for every \( x \in A \) which implies that \( z \in C_n \). Now we prove that \( A_c = \bigcap_{n=1}^{\infty} C_n \). Since the space \((X, d, W)\) has the Property (JCU), \( \bigcap_{n=1}^{\infty} C_n \) is a nonempty set.

Let \( x \in \bigcap_{n=1}^{\infty} C_n \). Then \( x \in A_n(y) \) for every \( y \in A \) and for every \( n \). Now \( d(x, y) \leq R(A) + \frac{1}{n} \) for every \( y \in A \) and for every \( n \). Thus \( R_x(A) \leq R(A) \). By the definition of \( R(A) \), we have \( R(A) \leq R_x(A) \). Hence \( R(A) = R_x(A) \) which implies that \( x \in A_c \).

Conversely, let \( x \in A_c \), then \( R_x(A) = R(A) \). Now, for every \( y \in A \), \( d(x, y) \leq R_x(A) = R(A) \leq R(A) + \frac{1}{n} \) for every \( n \). This shows that \( x \in A_n(y) \) for every \( y \in A \) and for every \( n \).

This completes the proof.

**Proposition 3.2.11:** Let \( M \) be a nonempty compact subset of a J-convex ultra metric space \((X, d, W)\) and \( K \) be the least closed convex set containing \( M \). If the diameter \( \delta(M) \) is positive, then there exists an element \( u \in K \) such that \( \sup \{d(x, u): x \in M\} < \delta(M) \).

**Proof:** Since \((X, d, W)\) is a J-convex ultra metric space, by using Proposition 3.2.1(ii) \((X, d, W)\) is a convex ultra metric space. Then \( K \) is also a convex set in the convex ultra metric space in \((X, d, W)\). By applying Proposition 3.1.10, there exists an element \( u \in K \) such that \( \sup \{d(x, u): x \in M\} < \delta(M) \).
**Definition 3.2.12:** A J-convex ultra metric space \((X, d, W)\) is said to have the normal structure if for each closed bounded convex subset \(A\) of \(X\) which contains at least two points, there exists \(x \in A\) which is not a diametral point of \(A\).

**Definition 3.2.13:** A J-convex ultra metric space is said to be a strictly convex if for any \(x, y \in X\) and \(\lambda \in (0, 1)\), there exists an unique \(z \in X\) such that \(\lambda d(x, y) = d(z, y)\) and \((1-\lambda) d(x, y) = d(x, z)\).

**Theorem 3.2.14:** Suppose that a convex ultra metric space \((X, d, W)\) has Property(C). Let \(K\) be a nonempty bounded closed convex subset of \((X, d, W)\) with normal structure. If \(T\) is a nonexpansive mapping of \(K\) into itself, then \(T\) has a fixed point in \(K\).

**Proof:** Since \((X, d, W)\) is a J-convex ultra metric space, by using Proposition 3.2.1(ii) \((X, d, W)\) is a convex ultra metric space. Then \(K\) is also a bounded closed convex subset of the convex ultra metric space \((X, d, W)\) with normal structure. Since \(T\) is a nonexpansive mapping of \(K\) into itself, then by applying Theorem 3.1.12 \(T\) has a fixed point in \(K\). ■

**Definition 3.2.15:** Let \(K\) be a compact J-convex ultra metric space. Then a family \(F\) of nonexpansive mappings \(T\) of \(K\) into itself is said to have an invariant property in \(K\) if for any compact and convex subset \(A\) of \(K\) such that \(TA \subseteq A\) for each \(T \in F\), there exists a compact subset \(M \subseteq A\) such that \(TM = M\) for each \(T \in F\).

**Theorem 3.2.16:** Let \(K\) be a compact J-convex ultra metric space. If \(F\) is a family of nonexpansive mappings of \(K\) into itself with invariant property in \(K\), then the family \(F\) has a common fixed point.
**Proof:** Since \((X, d, W)\) is a J-convex ultra metric space, by using Proposition 3.2.1(ii) \((X, d, W)\) is a convex ultra metric space. Then \(K\) is also a convex ultra metric space in \((X, d, W)\). If \(\mathcal{F}\) is a family of nonexpansive mappings of \(K\) into itself with invariant property in \(K\), then by applying Theorem 3.1.14 the family \(\mathcal{F}\) has a common fixed point.

\[\blacksquare\]

### 3.3. Weak convex ultra metric spaces

In this section, we introduce the concept of weak convex structure, its basic properties and establish some fixed point theorems in weak convex pseudo metric spaces.

Let \((X, d)\) be an ultra metric space. The inequality
\[
\max\{\lambda d(u, x), (1-\lambda)d(u, y)\} \leq \max\{d(u, x), d(u, y)\}
\]
holds for all \(u, x, y \in X\) and \(\lambda \in [0, 1]\) in an ultra metric space \((X, d)\). This leads to the following definition.

**Definition 3.3.1:** Let \((X, d)\) be an ultra metric space. A mapping \(W:X \times X \times [0, 1] \to X\) is said to be a weak convex ultra structure on \((X, d)\), if for each \((x, y, \lambda) \in X \times X \times [0, 1]\) and for all \(u \in X\) the condition
\[
d(u, W(x, y, \lambda)) \leq \max\{d(u, x), d(u, y)\}
\]
holds. If \(W\) is a weak convex ultra structure on an ultra metric space \((X, d)\), then the triplet \((X, d, W)\) is called a weak convex ultra metric space.

**Example 3.3.2:** Consider a linear space \(L\) which is also an ultra metric space with the following properties:

(i) For \(x, y \in L\), \(d(x, y) = d(x - y, 0)\).

(ii) For \(x, y \in L\) and \(\lambda (0 \leq \lambda \leq 1)\), \(d(\lambda x + (1-\lambda)y, 0) \leq \max\{d(x, 0), d(y, 0)\}\).

Let \(W(x, y, \lambda) = \lambda x + (1-\lambda)y\) for all \(x, y \in X\) and \(\lambda \in [0, 1]\).
Now \( d(u, W(x, y, \lambda)) = d(u - \lambda x - (1-\lambda)y, 0) \)
\[
= d(\lambda u + (1-\lambda)u - \lambda x - (1-\lambda)y, 0)
\]
\[
= d(\lambda(u-x) + (1-\lambda)(u-y), 0)
\]
\[
\leq \max \{d(u-x, 0), d(u-y, 0)\}
\]
\[
= \max \{d(u, x), d(u, y)\}.
\]

This shows that \((L, d, W)\) is a weak convex ultra metric space.

**Proposition 3.3.3:** Let \((X, d, W)\) be a weak convex ultra metric space. Let \(u, x, y \in X\) and \(0 \leq \lambda \leq 1\). Then

(i) \(d(x, W(x, y, \lambda)) \leq d(x, y)\).

(ii) \(d(y, W(x, y, \lambda)) \leq d(y, x)\).

(iii) \(d(x, W(x, x, \lambda)) = 0\).

(iv) \(x = W(x, x, \lambda)\).

(v) \(d(x, y) = \max \{d(x, W(x, y, \lambda)), d((W(x, y, \lambda), y)\}\}.

**Proof:** By taking \(u = x\) in Definition 3.3.1, we get
\[
d(x, W(x, y, \lambda)) \leq \max \{d(x, x), d(x, y)\} = d(x, y).
\]

Similarly by taking \(u = y\) in Definition 3.3.1, we get
\[
d(y, W(x, y, \lambda)) \leq \max \{d(y, x), d(y, y)\} = d(y, x).
\]

Either in (i) or (ii) we take \(y = x\) we get
\[
d(x, W(x, x, \lambda)) \leq \max \{d(x, x), d(x, x)\} = d(x, x) = 0. \text{ This implies that } d(x, W(x, x, \lambda)) = 0.
\]

From this we conclude \(x = W(x, x, \lambda)\).

Since \((X, d, W)\) is a weak convex ultra metric space, for \(x, y \in X\) and \(\lambda \in [0, 1]\), we obtain
\[
d(x, y) \leq \max \{d(x, W(x, y, \lambda)), d((W(x, y, \lambda), y)\}.
\]
\[ \leq \max\{\max\{d(x, x), d(x, y)\}, \max\{d(x, y), d(y, y)\}\} \]
\[ = \max\{d(x, y), d(x, y)\} = d(x, y). \]

This shows that \(d(x, y) = \max\{d(x, W(x, y, \lambda)), d((W(x, y, \lambda), y))\}.\)

**Definition 3.3.4:** A subset \(M\) of weak convex ultra metric space \((X, d, W)\) is said to be a weak convex set if \(W(x, y, \lambda) \in M\) for all \(x, y \in X\) and for all \(\lambda \in [0, 1]\).

**Proposition 3.3.5:** If \((X, d, W)\) is a convex ultra metric space, then \((X, d, W)\) is a weak convex ultra metric space.

**Proof:** Let \((X, d, W)\) be a convex ultra metric space. Let \(x, y \in X\) and \(\lambda (0 \leq \lambda \leq 1)\). Then for all \(u \in X\),
\[ d(u, W(x, y, \lambda)) \leq \max\{\lambda d(u, x), (1-\lambda)d(u, y)\} \leq \max\{d(u, x), d(u, y)\}. \]
This shows that \((X, d, W)\) is a strong convex ultra metric space.

**Proposition 3.3.6:** Let \((X, d, W)\) be a weak convex ultra metric space and \(z \in X\) and for every \(r > 0\).

(i) Let \(\{K_\alpha : \alpha \in \Delta\}\) be a family of convex subsets of weak convex ultra metric space \((X, d, W)\), then \(\bigcap_{\alpha \in \Delta} K_\alpha\) is also a weak convex subset of \((X, d, W)\).

(ii) All the open and closed balls are weak convex.

(iii) \(S(x, r) = S(y, r) = S(W(x, y, \lambda), r) = S(z, r)\) if \(x, y \in S(z, r), z \in X\) and \(r > 0\).

**Proof:** Let \(M = \bigcap_{\alpha \in \Delta} K_\alpha\). If \(M = \emptyset\), then \(M\) is a weak convex ultra subset of \((X, d, W)\).

Suppose that \(M \neq \emptyset\). Let \(x, y \in M\) and \(\lambda \in \Delta\). Since each \(K_\alpha\) is a weak convex subset, \(W(x, y, \lambda) \in K_\alpha\) for every \(\alpha \in \Delta\). Hence \(W(x, y, \lambda) \in M\). This proves that \(M\) is a weak convex subset of \((X, d, W)\).
Let $x, y \in S(z, r)$ and $0 \leq \lambda \leq 1$. Then $d(x, z) < r$ and $d(y, z) < r$. Suppose $z \neq x$ and $z \neq y$. By using Definition 3.3.1,

$$d(z, W(x, y, \lambda)) \leq \max\{d(z, x), d(z, y)\} < \max\{r, r\} = r.$$  

The above arguments show that $W(x, y, \lambda) \in S(z, r)$ and hence $S(z, r)$ is weak convex. Analogously it can be proved that $S[z, r]$ is weak convex.

Let $x, y \in S(z, r)$ and $\lambda \in I$. In the weak convex ultra metric space $S(x, r) = S(y, r) = S(z, r)$. Since $S(z, r)$ is weak convex, $W(x, y, \lambda) \in S(z, r)$. Hence $S(x, r) = S(y, r) = S(z, r) = S(W(x, y, \lambda), r)$. ■

**Definition 3.3.7:** A weak convex ultra metric space $(X, d, W)$ is said to have the Property(WCU) if every bounded decreasing sequence of nonempty closed weak convex subsets of $(X, d, W)$ has nonempty intersection.

**Proposition 3.3.8:** Let $(X, d, W)$ be a weak convex ultra metric space. Let $A \subseteq X$. If $(X, d, W)$ has the Property(WCU), then $A^c = \{x \in X : R_x(A) = R(A)\}$ is a nonempty set, closed and weak convex.

**Proof:** Let $x \in A$. For every positive integer $n$, define

$$A_n(x) = \{y \in A : d(x, y) \leq R(A) + \frac{1}{n}\} \quad \text{and} \quad C_n = \bigcap_{x \in A} A_n(x).$$

Since $x \in A_n(x)$, $A_n(x)$ is a nonempty set and closed. Let $y, z \in A_n(x)$ and $\lambda \in [0, 1]$.

$$d(x, W(y, z, \lambda)) \leq \max\{d(x, y), d(x, z)\} \leq \max\{(R(A) + \frac{1}{n}), (R(A) + \frac{1}{n})\} = R(A) + \frac{1}{2n}.$$  

Therefore $W(y, z, \lambda) \in A_n(x)$ and hence $A_n(x)$ is a weak convex set.

We claim that $\{C_n\}$ is a bounded decreasing sequence of nonempty, closed and weak convex subsets.
Let $\varepsilon > 0$, then by the definition of $R(A)$, there exists $y \in A$ with $R_y(A) \leq R(A) + \varepsilon$ and by the Definition of $R_y(A)$, $d(x, y) \leq R_y(A)$ for every $x \in A$.

$$R(A) + \varepsilon \text{ for every } x \in A.$$  

Since $\varepsilon$ is arbitrary for every $x \in A$, $d(x, y) \leq R(A) + \varepsilon < R(A) + \frac{1}{n}$ for every $x \in A$. Hence $y \in A_n(x)$ for every $x \in A$ which implies that $y \in \bigcap_{x \in A} A_n(x) = C_n$. Thus $C_n$ is a nonempty set.

Since each $A_n$ is a closed set $C_n$ is also a closed set. By Proposition 3.3.6(i), $C_n$ is a weak convex set. Hence $C_n$ is a nonempty, closed and weak convex set for all $n$.

Let $z \in C_{n+1}$. Then $z \in A_{n+1}(x)$ for every $x \in A$, $d(x, z) \leq R(A) + \frac{1}{n+1} \leq R(A) + \frac{1}{n}$. Therefore $z \in A_n(x)$ for every $x \in A$ which implies that $z \in C_n$. Now we prove that $A_c = \bigcap_{n=1}^{\infty} C_n$. Since the space $(X, d, W)$ has the Property (WCU), $\bigcap_{n=1}^{\infty} C_n$ is a nonempty set.

Let $x \in \bigcap_{n=1}^{\infty} C_n$. Then $x \in A_n(y)$ for every $y \in A$ and for every $n$. Now $d(x, y) \leq R(A) + \frac{1}{n}$ for every $y \in A$ and for every $n$. This implies that $R_x(A) \leq R(A)$. By the definition of $R(A)$, we have $R(A) \leq R_x(A)$. Hence $R(A) = R_x(A)$ implies $x \in A_c$.

Conversely, let $x \in A_c$. Then $R_x(A) = R(A)$. Now, for every $y \in A$ $d(x, y) \leq R_x(A) = R(A) \leq R(A) + \frac{1}{n}$ for every $n$. This shows that $x \in A_n(y)$ for every $y \in A$ and for every $n$.

This completes the proof. $\blacksquare$

**Proposition 3.3.9:** Let $M$ be a nonempty compact subset of a weak convex ultra metric space $(X, d, W)$ and $K$ be the least closed weak convex set containing $M$. If the diameter $\delta(M)$ is positive, then there exists an element $u \in K$ such that $\sup\{d(x, u) : x \in M\} \leq \delta(M)$. 

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**Proof:** Since M is compact, we may find \( x_1, x_2 \in M \) such that \( d(x_1, x_2) = \delta(M) \). Let \( M_0 \subseteq M \) be maximal so that \( M_0 \supseteq \{x_1, x_2\} \) and \( d(x, y) = 0 \) or \( \delta(M) \) for all \( x, y \in M_0 \). Since M is compact, \( M_0 \) is finite. Let us assume that \( M_0 = \{x_1, x_2, \ldots, x_n\} \).

Let \( \begin{align*} y_1 &= W(x_1, x_2, \frac{1}{2}); \\ y_2 &= W(x_2, y_1, \frac{1}{3}) \\ &\quad \ddots \\ y_{n-2} &= W(x_{n-1}, y_{n-3}, \frac{1}{n-1}) \\ y_{n-1} &= W(x_n, y_{n-2}, \frac{1}{n}) = u. \end{align*} \)

As \( K \) is a weak convex set, \( u \in K \). Since M is compact, we can find \( y_0 \in M \) such that \( d(y_0, u) = \sup \{d(x, u) : x \in M\} \).

Now by Definition 3.3.1, we have
\[
\begin{align*}
d(y_0, u) &= d(y_0, W(x_n, y_{n-2}, \frac{1}{n})) \\
&\leq \max \{d(y_0, x_n), d(y_0, y_{n-2})\} \\
&\leq \max \{d(y_0, x_n), \max \{d(y_0, W(x_{n-1}, y_{n-3}, \frac{1}{n-1}))\}\} \\
&\leq \max \{d(y_0, x_n), \max \{d(y_0, x_{n-1}), d(y_0, y_{n-3})\}\} \\
&\leq \max \{\delta(M), \delta(M)\} = \delta(M). \end{align*}
\]

Therefore, \( \sup \{d(x, u) : x \in M\} \leq \delta(M) \).

**Definition 3.3.10:** A weak convex ultra metric space \((X, d, W)\) is said to have the normal structure if for each closed bounded weak convex subset \( A \) of \((X, d, W)\) which contains at least two points, there exists \( x \in A \) which is not a diametral point of \( A \).
**Definition 3.3.11:** Let \((X, d, W)\) be a weak convex ultra metric space. Then \((X, d, W)\) is said to be strictly weak convex if for any \(x, y \in X\), \(\lambda d(x, y) = d(W(x, y, \lambda), y)\) and \((1-\lambda)d(x, y) = d(x, W(x, y, \lambda))\) hold.

**Theorem 3.3.12:** Let \((X, d, W)\) be a weak convex ultra metric space. Suppose that \((X, d, W)\) has the Property(WCU). Let \(K\) be a nonempty bounded closed weak convex ultra subset of \((X, d, W)\) with ultra normal structure. If \(T\) is a nonexpansive mapping of \(K\) into itself, then \(T\) has a fixed point in \(K\).

**Proof:** Let \(\Phi\) be a family of all nonempty closed and weak convex subsets of \(K\), such that each of which is mapped into itself by \(T\). By the Property(WCU) and Zorn’s Lemma, \(\Phi\) has a minimal element \(A\). We show that \(A\) consists of a single point.

Let \(x \in A_c\). Since \(T\) is nonexpansive map, \(d(T(x), T(y)) \leq d(x, y)\) for all \(y \in A\). Again for every \(y \in A\), \(d(x, y) \leq \rho_x(A)\), which implies that \(d(T(x), T(y)) \leq \rho_x(A) = \rho(A)\) for all \(y \in A\) and hence \(T(A)\) is contained in the closed ball \(S[T(x), \rho(A)]\). Since \(T(A \cap S[T(x), \rho(A)]) \subseteq A \cap S[T(x), \rho(A)]\), the minimality of \(A\) implies \(A \subseteq S[T(x), \rho(A)]\).

Hence \(\rho_{T(x)}(A) \leq \rho(A)\). We know that \(\rho(A) \leq \rho_{T(x)}(A)\) for all \(x \in A\). Thus \(\rho(A) = \rho_{T(x)}(A)\).

Hence \(T(x) \in A_c\) and \(T(A_c) \subseteq A_c\). By Proposition 3.3.8, \(A_c \in \Phi\).

If \(z, w \in A_c\), then \(d(z, w) \leq \rho_d(A) = \rho(A)\) that implies \(\delta(A_c) \leq \rho(A)\). By the definition of normal structure \(\rho(A) < \delta(A)\). This proves that \(\delta(A_c) \leq \rho(A) < \delta(A)\). This is a contradiction to the minimality of \(A\). Hence, \(\delta(A) = 0\) and \(A\) consists of a single point. ■

**Theorem 3.3.13:** Let \((X, d, W)\) be a weak convex ultra metric space. Suppose \((X, d, W)\) is strictly weak convex with the Property(WCU). Let \(K\) be a nonempty bounded and closed weak convex subset of \((X, d, W)\) with normal structure. If \(F\) is a
commuting family of nonexpansive mappings of K into itself, then the family has a common fixed point in K.

**Proof:** If T is a nonexpansive mapping of K into itself in a strictly weak convex space, then by Theorem 3.3.12 the set F of fixed points of T is nonempty. Since T is continuous, F is closed. We show that F is weak convex. Let x, y \in F and \lambda \in I. Since K is a weak convex subset of (X, d, W), W(x, y, \lambda) \subseteq K.

\[
d(T(x), T(y)) \leq \max \{d(T(x), T(W(x, y, \lambda))), d(T(W(x, y, \lambda), T(y)))\} \\
\leq \max \{d(x, W(x, y, \lambda)), d(W(x, y, \lambda), y)\} = d(x, y),
\]

and hence by the strict weak convex of the space T(W(x, y, \lambda)) = W(x, y, \lambda). This implies F is weak convex. Let \( F_\alpha \) be the fixed point sets of \( T_\alpha \in \mathcal{F} \). If \( u \in F_\alpha \), then for any \( \beta \), \( T_\alpha T_\beta u = T_\beta T_\alpha u = T_\beta u \) that is \( T_\beta u \) lies in \( F_\alpha \) and each \( T_\beta \) maps \( F_\alpha \) into itself. Suppose there are finite sequence \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m \) and consider \( T_{\alpha_m} \) as a nonexpansive mapping of \( \bigcap_{k=1}^{m} F_{\alpha_k} \) into itself. By using Theorem 3.3.12, \( \bigcap_{k=1}^{m} F_{\alpha_k} \neq \emptyset \). Hence by Property(WCU), the family \( \{F_\alpha\} \) has nonempty intersection. This consists of the common fixed point.

**Definition 3.3.14:** Let K be a compact weak convex ultra metric space. Then a family \( \mathcal{F} \) of nonexpansive mappings T of K into itself is said to have an invariant property in K if for any compact and weak convex subset A of K such that \( TA \subseteq A \) for each \( T \in \mathcal{F} \), there exists a compact subset M \( \subseteq A \) such that \( TM = M \) for each \( T \in \mathcal{F} \).

**Theorem 3.3.15:** Let (X, d, W) be a weak convex ultra metric space. Let K be a compact weak convex ultra metric space. If \( \mathcal{F} \) is a family of nonexpansive mappings with invariant property in K, then the family \( \mathcal{F} \) has a common fixed point.
Proof: By applying Zorn’s Lemma, we can find a minimal nonempty weak convex compact set \( A \subseteq K \) such that \( A \) is invariant under each \( T \in \mathcal{F} \). If \( A \) contains a single point, then the theorem is proved. If \( A \) contains more than one point, then by the hypothesis, there exists a compact subset \( M \) of \( A \) such that \( M = \{ T(x) : x \in M \} \) for each \( T \in \mathcal{F} \).

If \( M \) contains more than one point, by Proposition 3.3.9, there exists an element \( u \) in the least weak convex ultra set containing \( M \) such that \( \rho = \sup \{ d(u, x) : x \in M \} \) \( \leq \delta(M) \), where \( \delta(M) \) is the diameter of \( M \). Let us define \( A_0 = \bigcap_{x \in M} \{ y \in A : d(x, y) \leq \rho \} \). Clearly \( A_0 \) is a nonempty and closed set. By Proposition 3.3.6(i), \( A_0 \) is a weak convex set. Since \( u \) is not in \( A_0 \), \( A_0 \) is a proper subset of \( A \) invariant under each \( T \) in \( \mathcal{F} \). This is a contradiction to the minimality of \( A \).

3.4. Quasi convex ultra metric spaces

In this section, we introduce the Quasi convex ultra structure in an ultra metric space by restricting \( \lambda = \frac{1}{2} \) in the definition given by Wataru Takahashi and establish some basic properties of quasi convex ultra metric spaces.

Definition 3.4.1: Let \((X, d)\) be an ultra metric space. A mapping \( W : X \times X \times [0, 1] \to X \) is said to be a quasi convex ultra structure on \((X, d)\), if for each \((x, y, \lambda) \in X \times X \times I \) and \( u \in X \), the condition \( d(u, W(x, y; \lambda)) \leq \max \{ \frac{1}{2} d(u, x), \frac{1}{2} d(u, y) \} \) holds.

If \( W \) is a quasi convex ultra structure on \((X, d)\), then the triplet \((X, d, W)\) is called a quasi convex ultra metric space.
Example 3.4.2: Consider a linear space $L$ which is also an ultra metric space with the following properties:

(i) For $x, y \in L$, $d(x, y) = d(x - y, 0)$.

(ii) For $x, y \in L$ and $\lambda (0 \leq \lambda \leq 1)$, $d(\lambda x + (1 - \lambda)y, 0) \leq \max \{\frac{1}{2} d(x, 0), \frac{1}{2} d(y, 0)\}$.

Let $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for all $x, y \in X$ and $\lambda \in I$.

Now $d(u, W(x, y, \lambda)) = d(u - \lambda x - (1 - \lambda)y, 0)$

$$= d(\lambda u + (1 - \lambda)u - \lambda x - (1 - \lambda)y, 0)$$

$$= d(\lambda (u - x) + (1 - \lambda)(u - y), 0)$$

$$\leq \max \{\frac{1}{2} d(u - x, 0), \frac{1}{2} d(u - y, 0)\}$$

$$= \max \{\frac{1}{2} d(u, x), \frac{1}{2} d(u, y)\}.$$

Hence, $(L, d, W)$ is a quasi convex ultra metric space.

Proposition 3.4.3: Let $(X, d, W)$ be a quasi convex ultra metric space. Let $u, x, y \in X$ and $0 \leq \lambda \leq 1$. Then

(i) $d(x, W(x, y, \lambda)) \leq \frac{1}{2} d(x, y)$.

(ii) $d(y, W(x, y, \lambda)) \leq \frac{1}{2} d(x, y)$.

(iii) $d(x, W(x, x, \lambda)) = 0$ for all $\lambda$ with $0 \leq \lambda \leq 1$.

(iv) $x = W(x, x, \lambda)$ for all $\lambda$ with $0 \leq \lambda \leq 1$.

(v) $d(x, y) = d(x, W(x, y, \lambda)) + d((W(x, y, \lambda), y)$.

Proof: In Definition 3.4.1, by taking $u = x$ we get

$$d(x, W(x, y, \lambda)) \leq \max \{\frac{1}{2} d(x, x), \frac{1}{2} d(x, y)\} = d(x, y).$$

Similarly by taking $u = y$ in Definition 3.4.1, we get

$$d(y, W(x, y, \lambda)) \leq \max \{\frac{1}{2} d(y, x), \frac{1}{2} d(y, y)\} = d(y, x).$$
Either in (i) or (ii) take $y=x$ we get $d(x, W(x, x, \lambda)) \leq \max \{\frac{1}{2}d(x, x), \frac{1}{2}d(x, x)\} = d(x, x) = 0$. This implies $d(x, W(x, x, \lambda)) = 0$. From this we conclude that $x = W(x, x, \lambda)$.

Since $(X, d, W)$ is a quasi convex ultra metric space, for $x, y \in X$ and $\lambda \in [0, 1]$, we obtain

$$d(x, y) \leq d(x, W(x, y, \lambda)) + d((W(x, y, \lambda), y)$$

$$\leq \max \{\frac{1}{2}d(x, x), \frac{1}{2}d(x, y)\} + \max \{\frac{1}{2}d(x, y), \frac{1}{2}d(y, y)\}$$

$$= \frac{1}{2}d(x, y) + \frac{1}{2}d(x, y) = d(x, y).$$

This shows that $d(x, y) = d(x, W(x, y, \lambda)) + d((W(x, y, \lambda), y)$.

**Proposition 3.4.4:**

(i) If $(X, d, W)$ is a quasi convex ultra metric space then it is a weak convex ultra metric space.

(ii) If $(X, d, W)$ is a quasi convex ultra metric space then it is a quasi convex metric space.

(iii) If $(X, d, W)$ is a quasi convex ultra metric space then it is a weak convex metric space.

**Proof:** Let $(X, d, W)$ be a quasi convex ultra metric space. Let $x, y \in X$ and $\lambda \in (0, 1]$.

Then $d(u, W(x, y, \lambda)) \leq \max \{\frac{1}{2}d(u, x), \frac{1}{2}d(u, y)\}$ for all $u \in X$

$$\leq \max \{d(u, x), d(u, y)\}.$$  

This shows that $(X, d, W)$ is a weak convex ultra convex metric space.

Let $(X, d, W)$ be a quasi convex ultra metric space. Let $x, y \in X$ and $\lambda \in (0, 1]$.

Then $d(u, W(x, y, \lambda)) \leq \max \{\frac{1}{2}d(u, x), \frac{1}{2}d(u, y)\}$ for all $u \in X$

$$\leq \frac{1}{2}d(u, x) + \frac{1}{2}d(u, y).$$
Let \((X, d, W)\) be a quasi convex ultra metric space. Let \(x, y \in X\) and \(\lambda (0 \leq \lambda \leq 1)\). Then
\[
d(u, W(x, y, \lambda)) \leq \max\{\frac{1}{2} d(u, x), \frac{1}{2} d(u, y)\} \quad \text{for all } u \in X.
\]
\[
\leq d(u, x) + d(u, y).
\]
This shows that \((X, d, W)\) is a weak convex metric space. ■

**Definition 3.4.5:** A subset \(M\) of quasi convex ultra metric space \((X, d, W)\) is said to be a quasi convex set if \(W(x, y, \lambda) \in M\) for all \(x, y \in M\) and for all \(\lambda \in [0,1]\).

**Proposition 3.4.6:** Let \(\{K_\alpha : \alpha \in \Delta\}\) be a family of quasi convex subsets of the quasi convex ultra metric space \((X, d, W)\), then \(\bigcap_{\alpha \in \Delta} K_\alpha\) is also a quasi convex subset of \((X, d, W)\).

**Proof:** Let \(M = \bigcap_{\alpha \in \Delta} K_\alpha\). If \(M = \emptyset\), then \(M\) is a quasi convex subset of \((X, d, W)\). Suppose \(M \neq \emptyset\). Let \(x, y \in M\) and \(\lambda \in I\). Since each \(K_\alpha\) is a quasi convex subsets, \(W(x, y, \lambda) \in K_\alpha\) for every \(\alpha \in \Delta\). Hence \(W(x, y, \lambda) \in M\). This proves that \(M\) is a quasi convex subset of \((X, d, W)\). ■

**Proposition 3.4.7:** Let \((X, d, W)\) be a quasi convex ultra metric space. Then the balls \(S(z, r)\) and \(S[z, r]\) in \((X, d, W)\) are quasi convex subsets of \((X, d, W)\).

**Proof:** Let \(x, y \in S(z, r)\) and \(\lambda (0 \leq \lambda \leq 1)\). Then \(d(x, z) < r\) and \(d(y, z) < r\). By using Definition 3.4.1, \(d(z, W(x, y, \lambda)) \leq \max\{\frac{1}{2} d(z, x), \frac{1}{2} d(z, y)\} < r\).

This shows that \(W(x, y, \lambda) \in S(z, r)\) that implies \(S(z, r)\) is quasi convex. Analogously it can be proved that \(S[z, r]\) is quasi convex. ■

**Proposition 3.4.8:** Let \((X, d, W)\) be a quasi convex ultra metric space and \(z \in X, r >0\). Let \(x, y \in S(z, r)\) and \(\lambda (0 \leq \lambda \leq 1)\). Then \(S(x, r) = S(y, r) = S(W(x, y; \lambda), r) = S(z, r)\).
Proof: Let \( x, y \in S(z, r) \) and \( \lambda \in I \). In the quasi convex ultra metric space \( S(x, r) = S(y, r) = S(z, r) \). Since \( S(z, r) \) is quasi convex, \( W(x, y, \lambda) \subseteq S(z, r) \). Hence \( S(x, r) = S(y, r) = S(z, r) = S(W(x, y, \lambda), r) \).

Definition 3.4.9: A quasi convex ultra metric space \((X, d, W)\) is said to have the Property(QCU) if every bounded decreasing sequence of nonempty closed quasi convex subsets of \((X, d, W)\) has nonempty intersection.

Proposition 3.4.10: Let \((X, d, W)\) be a quasi convex ultra metric space. If \((X, d, W)\) has the Property(WCU) as weak convex ultra metric space, then for every subset \( A \) of \( X \), \( A_c = \{ x \in X : R_x(A) = R(A) \} \) is a nonempty set, closed and weak convex.

Proof: Analogue proof to Lemma 2.1.4.

Proposition 3.4.11: Let \( M \) be a nonempty compact subset of a quasi convex ultra metric space \((X, d, W)\) and \( K \) be the least closed quasi convex set containing \( M \). If the diameter \( \delta(M) \) is positive, then there exists an element \( u \in K \) such that \( \sup \{d(x, u) : x \in M\} < \delta(M) \).

Proof: Since \((X, d, W)\) is a quasi convex ultra metric space, by using Proposition 3.4.4(i) \((X, d, W)\) is a weak convex ultra metric space. Then \( K \) is also a weak convex set in the weak convex ultra metric space in \((X, d, W)\). By applying Proposition 3.3.9, there exists an element \( u \in K \) such that \( \sup \{d(x, u) : x \in M\} < \delta(M) \).

Definition 3.4.12: A quasi convex ultra metric space \((X, d, W)\) is said to have normal structure if for each closed bounded quasi convex subset \( A \) of \((X, d, W)\) which contains at least two points, there exists \( x \in A \) which is not a diametral point of \( A \).
**Definition 3.4.13:** Let \((X, d, W)\) be a quasi convex ultra metric space. Then \((X, d, W)\) is said to be strictly quasi convex if for any \(x,y \in X\) and \(0 \leq \lambda \leq 1\), 
\[
\lambda d(x,y) = d(W(x,y,\lambda), y) \\
(1-\lambda)d(x, y) = d(x, W(x, y, \lambda))
\]
hold.

**Theorem 3.4.14:** Let \((X, d, W)\) be a quasi convex ultra metric space. Suppose that \((X, d, W)\) has the Property(WCU). Let \(K\) be a nonempty bounded closed quasi convex subset of \((X, d, W)\) with normal structure. If \(T\) is a nonexpansive mapping of \(K\) into itself, then \(T\) has a fixed point in \(K\).

**Proof:** Since \((X, d, W)\) is a quasi convex ultra metric space, by using Proposition 3.4.4(i) \((X, d, W)\) is a weak convex ultra metric space. Then \(K\) is also a nonempty bounded and closed weak convex set of convex ultra metric space in \((X, d, W)\) with normal structure. If \(T\) is a nonexpansive mapping of \(K\) into itself, then by applying Theorem 3.3.12 the family \(\mathcal{F}\) has a common fixed point.

**Theorem 3.4.15:** Let \((X, d, W)\) be a quasi convex ultra metric space. Suppose \((X, d, W)\) is strictly quasi convex with the Property(QCU). Let \(K\) be a nonempty bounded closed quasi convex subset of \((X, d, W)\) with normal structure. If \(\mathcal{F}\) is a commuting family of nonexpansive mappings of \(K\) into itself, then the family has a common fixed point in \(K\).

**Proof:** Since \((X, d, W)\) is a quasi convex ultra metric space, by using Proposition 3.4.4(i) \((X, d, W)\) is a weak convex ultra metric space. Then \(K\) is also a nonempty bounded closed weak convex ultra metric space in \((X, d, W)\). If \(\mathcal{F}\) is a commuting family of nonexpansive mappings of \(K\) into itself with invariant property in \(K\), then by applying Theorem 3.3.13 the family \(\mathcal{F}\) has a common fixed point.
**Definition 3.4.16:** Let $K$ be a compact quasi convex ultra metric space. Then a family $\mathcal{F}$ of nonexpansive mappings $T$ of $K$ into itself is said to an invariant property in $K$ if for any compact and quasi convex subset $A$ of $K$ such that $TA \subseteq A$ for each $T \in \mathcal{F}$, there exists a compact subset $M \subseteq A$ such that $TM = M$ for each $T \in \mathcal{F}$.

**Theorem 3.4.17:** Let $(X, d, W)$ be a quasi convex ultra metric space. Let $K$ be a compact quasi convex ultra metric space. If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in $K$, then the family $\mathcal{F}$ has a common fixed point.

**Proof:** Since $(X, d, W)$ is a quasi convex ultra metric space, by using Proposition 3.4.4(i) $(X, d, W)$ is a weak convex ultra metric space. Then $K$ is also a weak ultra convex metric space in $(X, d, W)$. If $\mathcal{F}$ is a family of nonexpansive mappings of $K$ into itself with invariant property in $K$, then by applying Theorem 3.3.15 the family $\mathcal{F}$ has a common fixed point. ■