CHAPTER 7

TOLERANCE ALLOCATION ON SHAFTS
Shafts are used in a variety of ways in all kinds of mechanical equipments such as power shafts, line shafts etc. They are usually designed to produce desired strength and stiffness. No scientific method of specifying the tolerances on shaft dimensions has been cited in the literature. Stiffness is important from the point of view of critical speeds and vibration, therefore, stiffness deviation is important criterion to assign the tolerances on shaft diameter and length. With this criterion a scientific method for tolerance allocation is presented considering the random nature of dimensions during manufacturing and follow normal law of distribution. Chance constrained programming technique is used to develop equations and numerical examples are solved to illustrate the method.

INTRODUCTION

A shaft is a rotating or stationary member, usually of circular cross section, having mounted upon it such elements as gears, pulleys, flywheels and other power transmission elements. Depending on the loading shafts are subjected to constant bending or torsional stress or a combined of these stresses. Shafts must be strong enough to resist bending stresses and shear stresses. The approach used for springs in second chapter may be extended to the shafts to allocate the tolerances. The approach consists
of zero stiffness deviation together with specified stress variation due to tolerances. Probability must be associated to consider certainty with which the stress does not exceed the specified value. Equations are developed to allocate tolerances for two cases -

(1) transmission shaft subject to torsion,

(2) transmission shaft subject to torsion and bending.

THEORY

CASE 1 SHAFT SUBJECT TO TORSION

Consider the shaft shown in Fig.16, subjected to torque \( T \) only. The following equations (38) may be used.

\[ S = \text{torsional stiffness} = \frac{T}{\Theta} = \frac{G \Theta d^4}{L} \quad \ldots \quad (7.1) \]

where, \( \Theta \) = twist angle in radians,

\( G \) = shear modulus, Newton/cm²

\( L \) = shaft length, cm

\( d \) = shaft diameter, cm.

\[ f = \frac{G \Theta d}{2 L} \quad \ldots \quad \ldots \quad \ldots \quad (7.2) \]

in which \( f \) = maximum shear stress, N/cm²

Taking \( d \) and \( L \) as design parameters. There will be an error in specified \( S \) as practically it is not possible to manufacture \( d \) and \( L \) exact to the specifications.

\( \Delta d \) and \( \Delta L \) are the tolerances on \( d \) and \( L \) respectively. The deviation \( \Delta S \) can be expressed as follows -
FIG. 16. SHAFT SUBJECT TO TORSION.

FIG. 17(a). SHAFT SUBJECT TO TORSION AND BENDING.

FIG. 17(b). BENDING MOMENT DIAGRAM (SHAFT TREATED AS SIMPLY SUPPORTED BEAM).
\[
\Delta S = \frac{\delta S}{\delta d} \Delta d + \frac{\delta S}{\delta L} \Delta L \ldots \ldots (7.3)
\]

Use of equation (7.1) for partial derivatives of \( S \),
\[
\frac{\delta S}{\delta d} = \frac{4 G \pi d^3}{L} \quad \text{and} \quad \frac{\delta S}{\delta L} = -\frac{G \pi d^4}{L^2}
\]

Use of these partial derivatives in equation (7.3) gives
\[
\Delta S = \frac{G \pi d^3}{32 L} (4 \Delta d - \frac{d}{L}) \Delta L \ldots \ldots (7.4)
\]

For \( \Delta S \) to be zero,
\[
\Delta d = \frac{d}{4 L} \Delta L \ldots \ldots (7.4)
\]

Equation (7.2) can be written in the equality form,
\[
(C - \bar{f}) = 0 \ldots \ldots (7.5)
\]
where \( C = \frac{G \rho \sigma d}{2 L} \ldots \ldots (7.6) \)

In practice it is not possible to get \( f \) exact to the specification and hence equation (7.5) is violated as it involves manufacturing tolerances, the random variables.

The maximum permissible variation in the stress should be specified and tolerances allocated in such a way that this is not exceeded with a specified probability (high) say 99.7 percent. This concept leads to the following relation
\[(C - f) \leq g.f \text{ (with 99.7 percent probability)}\]

OR \(C - (1 + g)f \leq 0 \text{ (with 99.7 percent probability)}\)

\(f\) is the specified stress which in a deterministic sense is given by equation (7.2) and \(g\) is a fraction representing deviation of \(f\) due to tolerances and of the order of 0.01 corresponding to one percent deviation.

As \(d\) and \(L\) are random in nature, above inequality is probabilistic, changing the inequality to the equivalent deterministic form by applying chance constrained programming technique (18) as follows

\[
(P - (1 + g)f) = \sum_{j} \frac{\partial C}{\partial d} \sigma_d^2 + \frac{\partial C}{\partial L} \sigma_L^2 \quad \frac{1}{2} < 0
\]

\[
\sigma_d^2 = \text{Var} (d) \quad \text{and} \quad \sigma_L^2 = \text{Var} (L) \quad \ldots \quad \ldots \quad (7.7)
\]

Since maximum error is \(\pm 3\sigma_1\) with a probability of 99.7 percent for normal distribution, therefore, \(\Delta d = 3 \sigma_d\) and \(\Delta L = 3 \sigma_L\)

Since in the deterministic sense \((C - f) = 0\), and substituting the values of \(\sigma_d\) and \(\sigma_L\) in equation (7.7),

\[
(3gf)^2 = \frac{\partial C}{\partial d} (\Delta d)^2 + \frac{\partial C}{\partial L} (\Delta L)^2 \quad \ldots \quad (7.8)
\]

Consider equation (7.6) for partial derivatives of \(c\), we have

\[
\frac{\partial C}{\partial d} = \frac{g \Theta}{2L} \quad \text{and} \quad \frac{\partial C}{\partial L} = \frac{g \Theta d}{2L^2}
\]
Further substituting the value of $f$ (equation 7.2) and partial derivatives of $C$ in the relation (7.8) and on simplification, we get

\[
\left( \frac{3g}{P_j} \right)^2 = \left( \frac{\Delta d}{d} \right)^2 + \left( \frac{\Delta L}{L} \right)^2 \quad \ldots \quad (7.9)
\]

Use of equations (7.4) and (7.9) directly gives $\Delta d$ and $\Delta L$.

A numerical example will serve to illustrate the approach.

EXAMPLE 1

Allocate tolerances on shafts with the following specifications.

(i) $d = 3$ cm, $L = 80$ cm, permissible stress variation $0.5, 1.0, 1.5$ percent i.e. $g = 0.005, 0.010$ and $0.015$

(ii) $d = 5$ cm, $L = 90$ cm; $g = 0.5, 1.0, 1.5$ percent

(iii) $d = 6$ cm, $L = 90$ cm; $g = 0.01$

(iv) $d = 7$ cm, $L = 110$ cm; $g = 0.01$

(v) $d = 8$ cm, $L = 150$ cm; $g = 0.01$

Solution

$P_j = 2.75$ for 99.7 percent probability.

(i) $d = 3$ cm, $L = 80$ cm, $g = 0.005$

Equation (7.4) gives $\Delta d = 0.009375 \Delta L$

Substituting the values in equation (7.9) gives

$2.9752 \times 10^{-5} = 0.11111 (\Delta d)^2 + 1.5625 \times 10^{-4} (\Delta L)^2$

Simplifying the equations gives
\[ \Delta d = \pm 0.00396 \text{ cm} \text{ and } \Delta L = \pm 0.42333 \text{ cm}. \]

Similarly, one can solve equations (7.4) and (7.9) for various values of \( d \), \( L \) and \( g \) and results are given in Table 4.

**Table 4: Tolerances on Shaft Dimensions.**

<table>
<thead>
<tr>
<th>S.No.</th>
<th>( d )</th>
<th>( L )</th>
<th>( g )</th>
<th>( \Delta d )</th>
<th>( \Delta L )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>cm</td>
<td>cm</td>
<td>cm</td>
<td>cm</td>
<td>cm</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>80</td>
<td>0.005</td>
<td>( \pm 0.00396 )</td>
<td>( \pm 0.42333 )</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>80</td>
<td>0.010</td>
<td>( \pm 0.00793 )</td>
<td>( \pm 0.84666 )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>80</td>
<td>0.015</td>
<td>( \pm 0.01190 )</td>
<td>( \pm 1.27000 )</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>90</td>
<td>0.005</td>
<td>( \pm 0.00661 )</td>
<td>( \pm 0.47625 )</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>90</td>
<td>0.010</td>
<td>( \pm 0.01322 )</td>
<td>( \pm 0.95250 )</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>90</td>
<td>0.015</td>
<td>( \pm 0.01984 )</td>
<td>( \pm 1.42875 )</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>90</td>
<td>0.010</td>
<td>( \pm 0.01587 )</td>
<td>( \pm 0.95250 )</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>110</td>
<td>0.010</td>
<td>( \pm 0.01852 )</td>
<td>( \pm 1.16417 )</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>150</td>
<td>0.010</td>
<td>( \pm 0.02116 )</td>
<td>( \pm 1.58750 )</td>
</tr>
</tbody>
</table>
CASE 2  SHAFT SUBJECT TO TORSION AND BENDING

The shaft under study is shown in Fig. 17(a) and subjected to torsion and bending. When a shaft is subjected to combined loading the design is usually based on the maximum shear stress \( f \)

\[
f = \frac{16}{\pi d^3} \left( M^2 + T^2 \right)^{1/2} \quad \ldots \quad \ldots \quad (7.10)
\]

where \( M \) is the maximum bending moment in Nm. Equations (7.1), (7.3) and (7.4) are valid in second case also. In case of short bearings the shaft may be treated as a simply supported beam and corresponding bending moment diagram is shown in Fig. 17(b). Maximum bending moment is given by

\[
M = \frac{P a b}{L} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.11)
\]

\( P \) is the gear force or belt force or weight of component in Newtons, \( a \) and \( b \) are the distances of \( P \) from left and right support respectively.

Referring to Fig. 17(a),

\[a + b = L \quad \text{and let} \quad \frac{a}{b} = j\]

Solving these relations for \( a \) and \( b \), we get

\[
a = \frac{j L}{1 + j} \quad \text{and} \quad b = \frac{L}{1 + j}
\]

putting the values of \( a \) and \( b \) in equation (7.11)

\[
M = \frac{P j L}{(1 + j)^2}
\]
\[ M = P k L \ldots \ldots \ldots \ldots \ldots \quad (7.12) \]

where \( k = \frac{j}{(1+j)^2} \)

\( k \) is a constant and depends upon \( a/b \) ratio.

For \( a = b \), \( k = 0.25 \)

Substituting the value of \( M \) in equation (7.10), we get

\[ f = \frac{16}{\pi d^3} \left[ \left( P k L \right)^2 + T^2 \right]^{1/2} = D \text{ (say)} \ldots \quad (7.13) \]

Following the same concept as in case 1, specifying the maximum permissible variation in stress with a specified high probability, 99.7 percent, one can write

\[ (D - f) \lesssim g.f \quad (\text{with } 99.7\% \text{ probability}) \]

Following the line of equations (7.5 to 7.9),

\[ \left( \frac{3gf}{\pi j} \right)^2 = \left( \frac{\partial D}{\partial d} \right)^2 \left( \Delta d \right)^2 + \left( \frac{\partial D}{\partial L} \right)^2 \left( \Delta L \right)^2 \ldots \quad (7.14) \]

Consider equation (7.13) for partial derivatives of \( D \),

\[ \frac{\partial d}{\partial D} = - \frac{48}{\pi d^4} \left[ \left( P k L \right)^2 + T^2 \right]^{1/2} \]

\[ \frac{\partial D}{\partial L} = \frac{16 L (P k)^2}{\pi d^3 \left[ \left( P k L \right)^2 + T^2 \right]^{1/2}} \]

Substituting the value of \( f \) from equation (7.13) and partial derivatives of \( D \) in equation (7.14) and simplifying, we get
Equation (7.4) is valid in this case also and solution of equations (7.4) and (7.15) yields \( \Delta d \) and \( \Delta L \). The following numerical example illustrates the procedure.

**Example 2**

Allocate tolerances on the shaft with the following data:

- \( L = 200 \text{ cm} \), \( d = 8 \text{ cm} \), \( P = 8 \text{ kN} \), \( a = 150 \text{ cm} \), \( b = 50 \text{ cm} \), \( T = 900 \text{ kN cm} \) and maximum stress variation 0.5 percent.

Solution

\[
\begin{align*}
\frac{3}{P_j} &= 9 \left( \frac{\Delta d}{d} \right)^2 + \frac{(P \times L)^4}{((P \times L)^2 + T^2)^2} \left( \frac{\Delta L}{L} \right) \\
\ldots \ldots \quad (7.15)
\end{align*}
\]

Use of given data in equations (7.4) and (7.15) gives

\[
\Delta d = 0.01 \quad \Delta L
\]

\[
2.9752 \times 10^{-5} = 9 \left( \frac{\Delta d}{d} \right)^2 + 0.01 \left( \frac{\Delta L}{L} \right)^2
\]

Simplifying the equations, we get

\[
\Delta d = \pm 0.01441 \text{ cm}
\]

\[
\Delta L = \pm 1.44178 \text{ cm}
\]
OBSERVATIONS

The method is precise, versatile and useful as it provides the dimensional tolerances on the shaft directly. The method can be extended to multiple loading system. Also, procedure does not change even when fatigue, notch sensitivity etc. are considered in designing the shaft as they affect only the nominal dimensions 'd' and 'L' which must be known to us prior to the application of equations (7.4) and (7.9) or relations (7.4) and (7.15). The method may be applied to shafts supported in long bearings, they are assumed to have fixed ends. The maximum tolerances can be specified in order to control the stress variation below a specified level. This does not need any special mathematical technique.