Chapter 2

Preliminaries

In this chapter, we state some basic definitions, properties and classical results of ring theory which are mandatory for the forthcoming chapters. Initially we briefly give basic definitions of ring theory and lattice theory [15, 31].


The concept of intuitionistic fuzzy sets (IFS) introduced by K. T. Atanassov [16] in 1983 is a generalization of fuzzy sets. Among various branches of pure and applied Mathematics, Modern Algebra was one of the first few subjects where the notion of IFS was applied [23, 27, 44]. This led to the introduction of intuitionistic fuzzy groups, intuitionistic fuzzy subrings, intuitionistic fuzzy ideals etc. Some basic results of fuzzy sets and intuitionistic fuzzy sets from the books and papers of K. T. Atanassov [18, 19] and H. Bustince and P. Burillo [28] are given here. The notion of interval-valued intuitionistic fuzzy sets, introduced by K. T. Atanassov and G. Gargov [20]
Chapter 2. Preliminaries

is also stated in this chapter.

2.1 Rings, Lattices and \(L\)-fuzzy sets

A ring is a non-empty set \(R\) together with two binary operations denoted as addition ‘+’ and multiplication ‘·’ such that:

(i) \((R, +)\) is an abelian group

(ii) \((ab)c = a(bc)\) for all \(a, b, c \in R\). i.e., multiplication is associative.

\(a(b + c) = ab + ac\) and \((a + b)c = ab + bc\) for all \(a, b, c \in R\). i.e.,

\(\text{distributive law holds.}\)

(iii) A ring \(R\) is said to be a \textbf{commutative ring} if \(ab = ba\) for all \(a, b \in R\).

If a ring \(R\) contains an element 1, then it is said to be a \textbf{ring with identity}
if \(1a = a1 = a\) for all \(a \in R\). A non-empty subset \(S\) of a ring \(R\) is a \textbf{subring} of \(R\) if \(S\) is itself a ring under the operations of addition and multiplication in \(R\). An ideal \(I\) of a ring \(R\) is a subset of \(R\) which is an additive subgroup of \(R\) such that for \(x \in I\) and \(r \in R\), \(xr \in I\) and \(rx \in I\).

A \textbf{ring homomorphism} is a map \(\varphi : R \rightarrow R'\) from one ring to another which is compatible with the laws of composition and which carries 1 to 1. i.e.

(i) \(\varphi(a + b) = \varphi(a) + \varphi(b)\) and \(\varphi(ab) = \varphi(a) \varphi(b)\) for all \(a, b \in R\)

(ii) \(\varphi(1) = 1'\), where \(1'\) is the identity of \(R'\). A ring homomorphism which is surjective is an \textbf{epimorphism}. A ring homomorphism which is bijective is an \textbf{isomorphism}. An epimorphism \(f : R \rightarrow R/I, I\) an ideal of ring \(R\) is a natural homomorphism.

The \textbf{cosets} of an ideal \(I\) of \(R\) are the subsets \(\{a + I/a \in R\}\). The set of cosets \(R/I = \{(a + I)/a \in R\}\) is a \textbf{quotient ring}.

A poset \((A, \leq)\) is a \textbf{lattice} if for all \(a, b \in A\) the set \(\{a, b\}\) has both the greatest lower bound (glb) and least upper bound (lub). In this case we denote lub \(\{a, b\} = a \vee b\) and glb \(\{a, b\} = a \wedge b\). Throughout this thesis we denote a lattice \(L\) with join ‘\(\vee\)’ and meet ‘\(\wedge\)’. A \textbf{minimal element} \(a\) of \(L\)
is an element that satisfies \( a \leq x \) for all \( x \in L \) and a **maximal element** \( a \) of \( L \) is an element that satisfies \( a \geq x \) for all \( x \in L \).

A lattice \((L, \leq)\) is said to be **complete** if every non-empty subset \( A \) of \( L \) has both a lub and a glb existing in \( L \). A lattice \((L, \leq)\) is said to be **modular**, if

\[
(a \lor (c \land b)) = (a \lor c) \land b \quad \text{for all } a, b, c \in L \text{ with } a \leq b
\]

and \( L \) is said to be **distributive**, if

\[
(a \lor (b \land c)) = (a \lor b) \land (a \lor c) \quad \text{for all } a, b, c \in L.
\]

A subset \( M \) of a lattice \( L \) is a **sublattice** of \( L \) if \( a, b \in M, a \lor b \in M \) and \( a \land b \in M \).

A lattice \((L, \leq)\) is said to be **regular** if for all \( a, b \in L \) such that \( a \neq 0, b \neq 0, a \land b \neq 0 \).

We recall some basic results from [15].

**Lemma 2.1.1.** The non-empty intersection of subrings (ideals) of a ring \( R \) is a subring (ideal) of \( R \).

**Theorem 2.1.2.** If \( f : R \to R' \) is a ring epimorphism and \( A, A' \) are ideals of \( R \) and \( R' \) respectively then \( f(A) \) and \( f^{-1}(A') \) are ideals of \( R' \) and \( R \), respectively.

**Theorem 2.1.3.** If \( f : R \to R' \) is a ring homomorphism then

- (i) \( f(0) = 0' \)
- (ii) \( f(-a) = -f(a) \), for all \( a \in R \)

where \( 0, 0' \) are the identity elements of the rings \( R \) and \( R' \), respectively.

**Theorem 2.1.4** (First isomorphism theorem). If \( f : R \to S \) is a homomorphism of rings, then \( f \) induces an isomorphism of rings \( R/\ker f \cong \text{img } f \).

**Theorem 2.1.5** (Third isomorphism theorem). Let \( I, J \) be ideals in a ring \( R \). If \( I \subseteq J \), then \( J/I \) is an ideal in \( R/I \) and there is an isomorphism of rings \( (R/I)/(J/I) \cong R/J \).
Chapter 2. Preliminaries

**Theorem 2.1.6** (Correspondence theorem). If $I$ is an ideal in a ring $R$, then there is a 1-1 correspondence between the set of ideals of $R$ which contain $I$ and the set of all ideals of $R/I$ given by $J \mapsto J/I$, where $J$ is an ideal of $R$ which contains $I$.

**Theorem 2.1.7.** Let $R$ be a ring and $I$ an ideal of $R$. Then the additive quotient group $R/I$ is a ring with multiplication given by $(a + I)(b + I) = ab + I$.

**Theorem 2.1.8.** If $A$ and $B$ are two ideals of a ring $R$. Then $A + B = \langle A \cup B \rangle$, where $\langle A \cup B \rangle$ is the ideal generated by $A \cup B$.

The set of all ideals of a ring $R$, denoted by $\mathcal{I}(R)$ forms a complete modular lattice under ordinary set inclusion with join ‘$\lor$’ and meet ‘$\land$’ of two ideals defined by:

for $P, Q \in \mathcal{I}(R)$, $P \lor Q = \langle P \cup Q \rangle$ and $P \land Q = P \cap Q$ respectively where $\langle P \cup Q \rangle$ is the ideal generated by $P \cup Q$.

An equivalence relation $C$ on $R$ is called congruence if for $(a, b), (c, d) \in C$,

(i) $(a + c, b + d) \in C$ and
(ii) $(ac, bd) \in C$

The set of all congruence relations on a ring $R$ is denoted by $\mathcal{C}(R)$. Let $P, Q \in \mathcal{C}(R)$. Then their composition denoted as $P \circ Q$ is defined by $(p, q) \in P \circ Q$ iff $\exists$ a $x \in R$ such that $(p, x) \in P$ and $(x, q) \in Q$.

The notion of congruence relation on a ring is closely related to that of ideals of a ring. We recall some results pertaining to these concepts.

**Theorem 2.1.9.** Let $E$ be an ideal of a commutative ring $R$. Define a relation $C(E)$ as

$$ C(E) = \{(x, y) \in R \times R/ x - y \in E\}. $$

Then $C(E)$ is a congruence relation on $R$.

**Theorem 2.1.10.** Let $C_1$ be a congruence relation on a commutative ring $R$ with zero element, denoted by ‘0’. Define a set $I(C_1)$ by

$$ I(C_1) = \{x \in R/(x, 0) \in C_1\}. $$
Then $I(C_1)$ is an ideal of $R$.

**Theorem 2.1.11.** Let $R$ be a commutative ring and $E$ is an ideal of $R$. Then $I(C(E)) = E$.

**Theorem 2.1.12.** Let $C_1$ be a congruence relation on a commutative ring $R$. Then $C(I(C_1)) = C_1$.

**Theorem 2.1.13.** The set of all ideals of a ring $I(R)$ forms a complete lattice under set inclusion. Similarly, the set of all congruence relations of a ring $C(R)$ forms a complete lattice under a set inclusion.

**Theorem 2.1.14.** Let $R$ be a commutative ring. Then the lattices $C(R)$ and $I(R)$ are isomorphic.

**Corollary 2.1.15.** The lattice of ideals of a ring is modular.

The concept of fuzzy set was formulated by L. A. Zadeh [98] in his pioneering paper (1965), wherein a fuzzy set on a set $X$ is defined to be a mapping $\mu$ from $X$ to the closed unit interval $[0,1]$, i.e., $\mu : X \rightarrow [0,1]$. Later J. A. Goguen provided a more general definition, where $[0,1]$ was replaced by a lattice $L$ with a least element and a greatest element. Such type of functions $\mu : X \rightarrow L$, are called $L$-fuzzy sets.

In this thesis, we take $L$ as a complete distributive lattice with maximal element and minimal element denoted by 1 and 0, respectively.

For $x \in X$, $\mu(x)$ is interpreted as the **degree of membership** of $x$ in the $L$-fuzzy set $\mu$. For an ordinary set $A \subseteq X$, its characteristic function $\psi_A$ is the $L$-fuzzy set representing $A$.

J. N. Mordeson and D. S. Malik in [67] made remarkable studies in the area of $L$-fuzzy sets. They enriched the area of $L$-fuzzy algebraic structures by extending various concepts and results of classical algebra to $L$-fuzzy setting.

**Definition 2.1.16.** Let $\mu$ be an $L$-fuzzy set on a set $X$. For $a \in L$ the **level subset** or a **level set** of $\mu$ is defined as $\mu_a = \{x \in X / \mu(x) \geq a\}$. 

Chapter 2. Preliminaries

Definition 2.1.17. Let $Y \subseteq X$ and $a \in L$. A $L$-fuzzy set $a_Y$ is defined as

$$a_Y(x) = \begin{cases} a & \text{for } x \in Y \\ 0 & \text{for } x \in X \setminus Y \end{cases}$$

If $Y = \{y\}$ then $a_{\{y\}}$ is a $L$-fuzzy point denoted by $y_a$. $1_Y$ is called the characteristic function of $Y$.

From now onwards, let $R$ be a commutative ring.

Definition 2.1.18. Let $\mu : R \to L$. Then $\mu$ is called an $L$-fuzzy subring of $R$, if for all $x, y \in R$

(i) $\mu(x - y) \geq \mu(x) \land \mu(y)$

(ii) $\mu(xy) \geq \mu(x) \land \mu(y)$

Definition 2.1.19. A $L$-fuzzy set $\mu$ of $R$ is an $L$-fuzzy ideal of $R$ if, for all $x, y \in R$

(i) $\mu(x - y) \geq \mu(x) \land \mu(y)$

(ii) $\mu(xy) \geq \mu(x)$.

Theorem 2.1.20. Let $A \subseteq R$. Then $A$ is an ideal of $R$ if and only if $1_A$ is an $L$-fuzzy ideal of $R$.

Theorem 2.1.21. A $L$-fuzzy subset $\mu$ of $R$ is a $L$-fuzzy subring ($L$-fuzzy ideal) of $R$ if and only if each non-empty level subset $\mu_a$ is a subring (ideal) of $R$, $a \in \mu(R) \cup \{b \in L | b \leq \mu(0)\}$.

2.2 Intuitionistic $L$-fuzzy set

The theory of intuitionistic fuzzy sets (IFS) is an extension of the fuzzy set theory. In 1983, K. T. Atanassov [16] introduced IFS which is characterized by a degree of membership and a degree of non-membership function such that their sum be less than or equal to one. Later K. T. Atanassov and S. Stoeva [21] introduced intuitionistic $L$-fuzzy sets which is a generalization of $L$-fuzzy sets.
2.2. Intuitionistic L-fuzzy set

**Definition 2.2.1.** Let \((L, \leq)\) be the lattice with an involutive order reversing operation \(N : L \rightarrow L\). Let \(X\) be a non-empty set. An **intuitionistic L-fuzzy set** (ILFS) \(A\) in \(X\) is defined as an object of the form

\[
A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X\}
\]

where \(\mu_A : X \rightarrow L\) and \(\nu_A : X \rightarrow L\) define the degree of membership and the degree of non-membership for every \(x \in X\) satisfying \(\mu_A(x) \leq N(\nu_A(x))\).

**Definition 2.2.2.** Let \(A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X\}\) and \(B = \{\langle x, \mu_B(x), \nu_B(x) \rangle / x \in X\}\) be two intuitionistic L-fuzzy sets of \(X\). Then we define

(i) \(A \subseteq B\) iff for all \(x \in X\), \(\mu_A(x) \leq \mu_B(x)\) and \(\nu_A(x) \geq \nu_B(x)\).

(ii) \(A = B\) iff for all \(x \in X\), \(\mu_A(x) = \mu_B(x)\) and \(\nu_A(x) = \nu_B(x)\).

(iii) \(\bar{A} = \{\langle x, \nu_A(x), \mu_A(x) \rangle / x \in X\}\).

(iv) \(A \cup B = \{\langle x, (\mu_A \lor \mu_B)(x), (\nu_A \land \nu_B)(x) \rangle / x \in X\}\).

(v) \(A \cap B = \{\langle x, (\mu_A \land \mu_B)(x), (\nu_A \lor \nu_B)(x) \rangle / x \in X\}\).

We denote the collection of all intuitionistic L-fuzzy sets on \(X\) by \(\text{ILFS}(X)\).

**Definition 2.2.3.** Let \(A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X\} \in \text{ILFS}(X)\). Let \(Y \subseteq X\). Then

\[
(\mu_A)_Y(x) = \begin{cases} 
\mu_A(x), & x \in Y \\
0, & x \in X \setminus Y
\end{cases}
\]

and

\[
(\nu_A)_Y(x) = \begin{cases} 
\nu_A(x), & x \in Y \\
0, & x \in X \setminus Y
\end{cases}
\]

If \(Y\) is a singleton then the above functions are the **intuitionistic L-fuzzy point** (ILFP).

**Definition 2.2.4.** Let \(A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X\} \in \text{ILFS}(X)\) and \(a, b \in L\). Then the set

\[
A_{[a,b]} = \{x \in X / (\mu_A)(x) \geq a \text{ and } (\nu_A)(x) \leq b\}
\]
is called **level subset** of \( A \) and

\[
A_{(a,b)} = \{ x \in X / (\mu_A)(x) > a \text{ and } (\nu_A)(x) < b \}
\]

is called **strong level subset** of \( A \).

For \( A \in \text{ILFS}(X) \), the set \( A^* = \{ x \in X / \mu_A(x) > 0, \nu_A(x) = 0 \} \) is called the **support** of \( A \).

**Definition 2.2.5.** Let

\[
A = \{ (x,y), \mu_A(x,y), \nu_A(x,y) \} / (x,y) \in (X \times X) \} \in \text{ILFS}(X).
\]

Then \( A \) is called an **intuitionistic L-fuzzy relation** (ILFR) on \( X \) if for all \( (x,y) \in X \times X, \mu_A(x,y) \leq N(\nu_A(x,y)) \), where \( N : L \to L, \mu_A : X \times X \to L \) and \( \nu_A : X \times X \to L \).

The set of all ILFR on \( X \) is denoted by \( \text{ILFR}(X) \).

**Definition 2.2.6.** Let \( A \in \text{ILFR}(X) \). Then the **inverse** of \( A \) denoted by \( A^{-1} \) is defined as \( A^{-1}(x,y) = A(y,x) \) for all \( (x,y) \in X \times X \).

**Definition 2.2.7.** Let \( A = \{ (x,y), \mu_A(x,y), \nu_A(x,y) \} / (x,y) \in (X \times X) \} \) and \( B = \{ (x,y), \mu_B(x,y), \nu_B(x,y) \} / (x,y) \in (X \times X) \} \in \text{ILFR}(X) \). Then the composition \( A \circ B \) of \( A \) and \( B \) is defined as

\[
A \circ B = \{ (x,y), \mu_{A \circ B}(x,y), \nu_{A \circ B}(x,y) \} / (x,y) \in X \times X \}
\]

where \( \mu_{A \circ B}(x,y) = \bigvee_{z \in X} \{ \mu_A(x,z) \land \mu_B(z,y) \} \)

and \( \nu_{A \circ B}(x,y) = \bigwedge_{z \in X} \{ \nu_A(x,z) \lor \nu_B(z,y) \} \).

**Definition 2.2.8.** Let \( A = \{ (x,y), \mu_A(x,y), \nu_A(x,y) \} / (x,y) \in (X \times X) \} \in \text{ILFR}(X) \). Then \( A \) is called an **intuitionistic L-fuzzy equivalence relation** (ILFER) on \( X \) if it satisfies the following conditions:

(i) \( A \) is **intuitionistic L-fuzzy reflexive** i.e. \( A(x,x) = (1,0) \) for all \( x \in X \)

(ii) \( A \) is **intuitionistic L-fuzzy symmetric** i.e., \( A = A^{-1} \)

(iii) \( A \) is **intuitionistic L-fuzzy transitive** i.e., \( A \circ A \subseteq A \)

The set of all intuitionistic L-fuzzy equivalence relations on \( X \) is denoted by \( \text{ILFER}(X) \).

**Definition 2.2.9.** Let \( P \in \text{ILFR}(X) \). Then the **intuitionistic L-fuzzy transitive closure** of \( P \) denoted by \( P^\infty \) is defined as \( P^\infty = \bigcup_{n \in N} P^n \) where
2.3. Interval-Valued Intuitionistic L-fuzzy Set

\[ P^n = P \circ P \circ \cdots \circ P, \] where \( P \) occurs \( n \) times. Here

\[
P^\infty = \{(x, y), \mu_{P^\infty}(x, y), \nu_{P^\infty}(x, y)\} / (x, y) \in X \times X\]

where

\[
\mu_{P^\infty}(x, y) = \vee_{n \in \mathbb{N}} \mu_{P^n}(x, y) \quad \text{and} \quad \nu_{P^\infty}(x, y) = \wedge_{n \in \mathbb{N}} \nu_{P^n}(x, y).
\]

2.3 Interval-Valued Intuitionistic L-fuzzy Set

With the demand for knowledge-handling systems capable of dealing with and distinguishing between various facets of imprecision ever increasing, clear and formal characterization of the mathematical models implementing such services is quintessential. This task was undertaken in 1975, when L. A. Zadeh [100] made an extension of the concept of a fuzzy subset by an interval-valued fuzzy subset. Interval-valued fuzzy sets serve to capture a feature of uncertainty with respect to the assignment of membership degrees. The central notion is to replace crisp \([0, 1]\)-valued membership degrees by intervals in \([0, 1]\), understood to contain the true, incompletely known membership degree.

The notion of interval-valued intuitionistic fuzzy set was introduced by K. T. Atanassov and G. Gargov [20]. Interval-valued intuitionistic fuzzy set theory has been successful in modelling uncertainty to a great extent and has found applications in various fields. Here we give some preliminary notions related to interval-valued intuitionistic fuzzy sets which is applicable in chapter 7.

We list some basic concepts and define different operators over interval-valued intuitionistic fuzzy sets [20] which are applied in chapter 7. Let \( D(I) \) be the set of all closed subintervals of the unit interval \( I = [0, 1] \). The elements of \( D[0, 1] \) called interval numbers which are generally denoted by \( \hat{\alpha} \) where \( \hat{\alpha} = [\alpha^L, \alpha^U] \) where \( 0 \leq \alpha^L \leq \alpha^U \leq 1 \) where \( \alpha^L \) and \( \alpha^U \) are the lower and upper end points, respectively. The interval \([\alpha, \alpha]\) is
identified with the number $\alpha \in [0, 1]$. For interval numbers $\hat{\alpha} = [\alpha^L, \alpha^U]$, $\hat{\beta} = [\beta^L, \beta^U] \in D[0, 1]$, we define

\[
\hat{\alpha} \lor \hat{\beta} = [\alpha^L \lor \beta^L, \alpha^U \lor \beta^U],
\]
\[
\hat{\alpha} \land \hat{\beta} = [\alpha^L \land \beta^L, \alpha^U \land \beta^U],
\]
\[
\forall i \in I \hat{\alpha}_i = [\forall i \in I \alpha_i^L, \forall i \in I \alpha_i^U],
\]
\[
\land i \in I \hat{\alpha}_i = [\land i \in I \alpha_i^L, \land i \in I \alpha_i^U]
\]
where $\hat{\alpha}_i = [\alpha_i^L, \alpha_i^U]$.

For any two interval numbers $\hat{\alpha}, \hat{\beta}$ we define,

1. $\hat{\alpha} \leq \hat{\beta} \iff \alpha^L \leq \beta^L$ and $\alpha^U \leq \beta^U$.
2. $\hat{\alpha} = \hat{\beta} \iff \alpha^L = \beta^L$ and $\alpha^U = \beta^U$.
3. $\hat{\alpha} < \hat{\beta} \iff \alpha^L < \beta^L$ and $\alpha^U < \beta^U$.

**Definition 2.3.1.** An interval-valued intuitionistic fuzzy set (IVIFS) on $X$, is defined as an expression of the form $A = \{ \langle x, \hat{\mu}_A(x), \hat{\nu}_A(x) \rangle / x \in X \}$ where $\hat{\mu}_A : X \rightarrow D[0, 1]$, $\hat{\nu}_A : X \rightarrow D[0, 1]$ with $\hat{\mu}_A(x) = [\mu^L_A(x), \mu^U_A(x)]$ and $\hat{\nu}_A(x) = [\nu^L_A(x), \nu^U_A(x)]$ such that $0 \leq \mu^L_A(x) + \nu^L_A(x) \leq 1$ and $0 \leq \mu^U_A(x) + \nu^U_A(x) \leq 1$.

We denote interval-valued intuitionistic fuzzy by simply IVIF and the set of all interval-valued intuitionistic fuzzy sets on $R$ is denoted by IVIFS ($X$).

**Definition 2.3.2.** [17] Let $A = \{ \langle x, \hat{\mu}_A(x), \hat{\nu}_A(x) \rangle / x \in X \}$ and $B = \{ \langle x, \hat{\mu}_B(x), \hat{\nu}_B(x) \rangle / x \in X \} \in$ IVIFS ($X$).

Let $\{A_a\}_{a \in I} \in$ IVIFS ($X$). Then

(a) $A \subseteq B \iff \hat{\mu}_A(x) \leq \hat{\mu}_B(x)$ and $\hat{\nu}_A(x) \geq \hat{\nu}_B(x)$ for all $x \in X$.

(b) $A = B \iff \hat{\mu}_A(x) = \hat{\mu}_B(x)$ and $\hat{\nu}_A(x) = \hat{\nu}_B(x)$ for all $x \in X$.

(c) $A^c = \{ \langle x, \hat{\mu}_A^c(x), \hat{\nu}_A^c(x) \rangle / x \in X \}$ where

$\hat{\mu}^c_A(x) = [1 - \mu^U_A(x), 1 - \mu^L_A(x)]$, $\hat{\nu}^c_A(x) = [1 - \nu^U_A(x), 1 - \nu^L_A(x)]$.

(d) $A \cup B = \{ \langle x, (\hat{\mu}_A \lor \hat{\mu}_B)(x), (\hat{\nu}_A \lor \hat{\nu}_B)(x) \rangle / x \in X \}$ where

$(\hat{\mu}_A \lor \hat{\mu}_B)(x) = [\mu^L_A(x) \lor \mu^L_B(x), \mu^U_A(x) \lor \mu^U_B(x)]$

and

$(\hat{\nu}_A \lor \hat{\nu}_B)(x) = [\nu^L_A(x) \lor \nu^L_B(x), \nu^U_A(x) \lor \nu^U_B(x)]$. 

(e) \( A = \cap_{i \in I} A_i = \{ \langle x, \hat{\mu}_A(x), \hat{\nu}_A(x) \rangle/ x \in X \} \) where
\[
\hat{\mu}_A(x) = \left[ \land_{i \in I} \mu^L_{A_i}(x), \land_{i \in I} \mu^U_{A_i}(x) \right]
\]
and
\[
\hat{\nu}_A(x) = \left[ \lor_{i \in I} \nu^L_{A_i}(x), \lor_{i \in I} \nu^U_{A_i}(x) \right].
\]