Chapter 3

A family of triaxial mass models with cusps: inclusion of fourth order spherical harmonics

Summary A family of triaxial mass models is described which is nearly ellipsoidal, and for which the projected surface density can be calculated analytically. The models show ellipticity variations, isophote twists and high-order residuals in projections. These can be compared with observations. These models are flattened versions of the $\gamma$-model with density $\rho(r) \propto r^{-\gamma}(b_0 + r)^{-4}$ and are constructed by addition of five spherical harmonic terms to the spherical models. The potential of the models can be expressed in a simple form.
3.1 Introduction

A family of mass models, which is triaxial generalisation of γ-model of Dehnen is studied. This triaxial mass model can produce variations in ellipticity and position angle in its projection. A spherical model \( \rho(r) \), with density \( \rho \) as a function of radius \( r \), can be made triaxial by considering the expression for \( \rho(r) \) and replacing \( r^2 \) by \( m^2 = x^2 + y^2/p^2 + z^2/q^2 \), where \( (x, y, z) \) are the usual Cartesian coordinates. The density is stratified on coaxial ellipsoids. In case the axial ratios \( p \) and \( q \) are constants, projected density is stratified on similar and aligned ellipses (Stark, 1977, Binney, 1985). It was demonstrated by Madejsky & Mollenhoff (1990) that models with equal density coaxial ellipsoidal shells with radially varying axial ratios can generate a variety of ellipticity and position angle profiles.

Triaxial mass models of an alternative form are \( \rho(r, \theta, \phi) = f(r) - g(r) Y^2_2(\theta) + h(r) Y^3_2(\theta, \phi) \), where \( \rho \) is density in the usual spherical polar coordinates \( (r, \theta, \phi) \), \( f(r) \) is a spherical mass distribution (which is usually a widely studied model), \( g(r) \) and \( h(r) \) are two suitably chosen radial functions, \( Y^2_0 = \frac{3}{2} \cos^2 \theta - \frac{1}{2} \) and \( Y^3_2 = 3 \sin^2 \theta \cos 2\phi \) are the usual spherical harmonics. These models also exhibit ellipticity variations and isophote twists in their projections (de Zeeuw & Carollo, 1996, hereafter ZC96, Chakraborty & Thakur, 2000, hereafter CTO0). Such triaxial models with \( f(r) \) as the spherical γ-models of Dehnen (1993), are presented in ZC96. Schwarzchild (1979) studied the triaxial mass models, wherein \( f(r) \) is considered as the spherical modified Hubble mass model, in a numerical form. Later it was casted into an analytical form by de Zeeuw & Merritt (1983). Projected properties of such triaxial Hubble mass model are described in CTO0. We shall refer to these models as \( fgh \) models.

CCD photometry of elliptical galaxies shows that the isophotes are approximately elliptical, and ellipticity and position angle vary with radius. Further, high-order residuals are found to be common in isophotes of elliptical galaxies. Although, models with ellipticity variations and isophote twists have been discussed, high-order residuals in these models have not been adequately treated. A small value of \( \sim -0.4\% \) of the shape parameter has been reported in ZC96. Likewise, the isophotes at large radii are shown to be slightly boxy by CTO0.

Rix & White (1990) have discussed a photometric model, in which an exponential disk is embedded in the equatorial plane of an oblate spheroid of constant ellipticity, and have found that the isophotes are pointy. In a sense, the model is a two component model. Contopoulos & Grosbol (1989) have found an orbital family which can lead to pointy isophotes. Thus, the model of Rix & White is by no means an exclusive interpretation of ellipticals with pointy isophotes. Further, Binney & Petrou (1985) have found that an overpopulation of a particular two dimensional subset of tube orbits can yield boxy isophotes.

Therefore, it will be worthwhile to examine density forms in a triaxial mass model which
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3.2.1 A potential-density pair

We (Chakraborty & Das, 2002) consider a potential of the form

\[ V_2(r, \theta, \phi) = u_1(r) + u_2(r)Y_l^m(\theta, \phi), \quad (3.1) \]

where \( u_1(r) \) and \( u_2(r) \) are two radial functions, and \( Y_l^m(\theta, \phi) = P_l^m(\cos \theta) \cos m\phi \). Here, \( P_l^m(\cos \theta) \) are the usual associated Legendre polynomials, and we have considered only the real part of the spherical harmonics \( Y_l^m \). Then, Poisson's equation

\[ 4\pi G\rho_2 = \frac{1}{r^2\sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial V_2}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V_2}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 V_2}{\partial \phi^2} \right] \quad (3.2) \]
give the mass density \( \rho_2 \) of the form

\[
4\pi G \rho_2 = \frac{d^2 u_2}{dr^2} + \frac{2}{r} \frac{du_2}{dr} + g_2(r) Y_l^m,
\]

where \( g_2(r) \) is the contribution to density corresponding to \( u_2(r) \) in potential, and is given by

\[
g_2(r) = \frac{d^2 u_2}{dr^2} + \frac{2}{r} \frac{du_2}{dr} - \frac{l(l+1)}{r^2} u_2.
\]

In above, we have used the differential equation

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial P_m^l}{\partial \theta} \right) + \left[ (l+1) - \frac{m^2}{\sin^2 \theta} \right] P_m^l = 0
\]

satisfied by \( P_m^l \). For \( l = 2 \) expression (3) reduces to the form used by Schwarzschild (1979), which is given by

\[
g_2(r) = \frac{1}{r} \frac{d^2}{dr^2} (ru_2) - \frac{6u_2}{r^2},
\]

whereas for \( l = 4 \), we obtain

\[
g_2(r) = \frac{1}{r} \frac{d^2}{dr^2} (ru_2) - \frac{20u_2}{r^2}.
\]

### 3.2.2 A family of triaxial mass models

We now consider a potential \( V(r, \theta, \phi) \) of the form

\[
V(r, \theta, \phi) = V_1(r, \theta, \phi) + v_1(r)[Y_2^0(\theta) - \alpha Y_4^0(\theta, \phi)] + w_1 Y_4^2(\theta, \phi),
\]

where

\[
V_1(r, \theta, \phi) = u(r) - v(r) Y_2^0(\theta) + w(r) Y_4^2(\theta, \phi).
\]

In above, \( u(r), v(r), w(r), v_1(r) \) and \( w_1(r) \) are five radial functions, \( Y_2^0 = \frac{3}{8} \cos^2 \theta - \frac{1}{8} \), \( Y_4^2 = 3 \sin^2 \theta \cos 2\phi \), \( Y_4^0 = \frac{1}{8}(35 \cos^4 \theta - 30 \cos^2 \theta + 3) \), \( Y_4^2 = \frac{15}{2}(7 \cos^2 \theta - 1) \sin^2 \theta \cos 2\phi \), \( Y_4^4 = 105 \sin^4 \theta \cos 4\phi \) and \( \alpha \) is a parameter. Potential \( V_1(r, \theta, \phi) \) defines the \( gh \) models.

In above, we have chosen \( Y_2^0, Y_4^2, Y_4^0 \) and \( Y_4^4 \) terms of the spherical harmonics \( Y_l^m(\theta, \phi) \). The choice of even values of \( l \) and \( m \), makes the model symmetric with respect to reflection on the coordinate planes \( x = 0, y = 0 \) and \( z = 0 \).

We take \( u(r) \) to be the potential of the \( \gamma \)-model of Dehnen, defined by the choice

\[
u(r) = \begin{cases} 
\frac{MG}{\ln b_0} \frac{r}{b_0 + r}, & \text{for } \gamma = 2, \\
\frac{MG}{(2-\gamma)b_0} \left[ \left( \frac{r}{b_0 + r} \right)^{2-\gamma} - 1 \right], & \text{for } \gamma \neq 2,
\end{cases}
\]
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with 0 < \gamma < 3. In above, \( M \) is the mass of the model and \( b_0 \) is the scale length. The radial functions \( u(r) \) and \( w(r) \), chosen in ZC96, have a form \( r^{2-\gamma}/(1+r)^{4-\gamma} \). Then the corresponding radial functions in the expression of density, as obtained by using equation (3.4), turn out to be positive at all \( r \). Following this, we consider a form \( r^{n-\gamma}/(1+r)^{m-\gamma} \), both for \( u_1(r) \) and \( w_1(r) \). The corresponding radial functions in the expression of density, as obtained by using (3.4), is positive at all \( r \), if \( n = 4 \) and \( m \leq 9 - \gamma \). These considerations lead us to the following choices. We choose

\[
u(r) = - MG \frac{b_1 r^{2-\gamma}}{(b_2 + r)^{4-\gamma}}, \tag{3.11}
\]

\[
w(r) = - MG \frac{b_3 r^{2-\gamma}}{(b_4 + r)^{4-\gamma}}, \tag{3.12}
\]

\[
v_1(r) = - MG \frac{c_1 b_5}{b_0} \frac{r^{4-\gamma}}{(c_2 + r)^{8-\gamma}}, \tag{3.13}
\]

and

\[
w_1(r) = - MG \frac{c_6 b_5}{b_0} \frac{r^{4-\gamma}}{(c_4 + r)^{8-\gamma}}. \tag{3.14}
\]

We obtain the associated density \( \rho(r, \theta, \phi) \) by applying equations (3.1)-(3.4), and is of the form

\[
\rho(r, \theta, \phi) = \rho_1(r, \theta, \phi) + s(r) \left[ Y_0^0(\theta) - \alpha Y_4^0(\theta, \phi) \right] + t(r) Y_2^2(\theta, \phi), \tag{3.15}
\]

where

\[
\rho_1(r, \theta, \phi) = f(r) - g(r) Y_2^0(\theta) + h(r) Y_2^2(\theta, \phi) \tag{3.16}
\]

The density \( \rho_1(r, \theta, \phi) \) defines the \( fg \)h models. The radial functions \( f(r) \), \( g(r) \), \( h(r) \), \( s(r) \) and \( t(r) \) are obtained from \( u(r), v(r), w(r), v_1(r) \) and \( w_1(r) \), respectively, by using relations (3.1)-(3.4). We refer to these as model A. The radial functions \( f(r), g(r), h(r), s(r) \) and \( t(r) \) are as follows

\[
f(r) = \frac{M b_0}{4\pi} \frac{(3-\gamma)}{r^7(b_0 + r)^{4-\gamma}}, \tag{3.17}
\]

\[
g(r) = \frac{M b_1}{4\pi} \frac{[b_1 r^2 + 4(6-\gamma) r b_2 + \gamma (5-\gamma) b_2^2]}{r^7(b_2 + r)^{6-\gamma}}, \tag{3.18}
\]

\[
h(r) = \frac{M b_2}{4\pi} \frac{[b_2 r^2 + 4(6-\gamma) r b_4 + \gamma (5-\gamma) b_4^2]}{r^7(b_4 + r)^{6-\gamma}}, \tag{3.19}
\]

\[
s(r) = \frac{M c_1 b_5}{4\pi b_0^3} \frac{r^7[c^2 r^2 + 8(10-\gamma) r c_2 + \gamma (9-\gamma) c_2^2]}{r^7(c_2 + r)^{10-\gamma}} \tag{3.20}
\]
Figure 3.1: Axial ratios of constant-$\rho$ surface (solid line) and of constant-$\rho_1$ surfaces (dotted line), as a function of $r$ in units of $b_0$. Model with $p = 0.90, q = 0.70, \gamma = 1.5$.

and

$$
t(r) = \frac{Mc^6}{4\pi b_0^2} \frac{r^2[8r^2 + 8(10 - \gamma)r c_4 + \gamma(9 - \gamma)c_3^2]}{r^\gamma c_4 + r}^{10-\gamma}.
$$

where $b_1, b_2, b_3, b_4, c_1, c_2, c_3$ and $c_4$ are constant. Note that we have avoided the choice $m = 9 - \gamma$. In case, we intend to choose $m = 9 - \gamma$, the resultant radial term in density has the form (corresponding to $u_1(r)$)

$$
\frac{r^2 \left[ 2rc_4(8r - 45) - c_3^2 \gamma(9 - \gamma) \right]}{r^\gamma (r + c_2)^{11-\gamma}}
$$

which is slightly different from the form of $g(r)$ and $h(r)$. The numerator of $g(r)$ and $h(r)$ has three terms in powers of $r$, whereas above has just two terms.

In above equations the four ratios $\frac{b_1}{b_0}, \ldots, \frac{b_4}{b_0}$, can be expressed in terms of the axial ratios $(p_\infty, q_\infty)$ and $(p_0, q_0)$ of the density distribution $\rho$ at (asymptotically) large and at (asymptotically) small radii (see equations 2.12-2.16), where the surfaces of constant density are approximately ellipsoidal.

We find that even when $p_0 = p_\infty = p$ and $q_0 = q_\infty = q$ are considered, the axial ratios of constant-$\rho_1$ surfaces fall short of $p$ and $q$, at intermediate radii (see Figure 3.1-3.4.).
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Figure 3.2: Same as in Figure 3.1 with $p = 0.90, q = 0.70, \gamma = 0.5$.

Figure 3.3: Same as in Figure 3.1 with $p = 0.75, q = 0.6, \gamma = 1.5$. 
axial ratios of constant-$p_1$ surfaces deviates more for the flatter model ($p = 0.75, q = 0.60$) (figure 3.3-3.4) compared to that of the rounder model ($p = 0.90, q = 0.70$) (Figure 3.1-3.2). It also depends on the steepness of the cusp parameter of the model. From Figures 3.1-3.4 we find that the axial ratios of constant-$p_1$ surfaces with steep cusp ($\gamma = 1.5$) deviates less compare to the less steep cusp models, for given $p, q$. Confining ourselves to the case where $p_0 = p_{\infty} = p$ and $q_0 = q_{\infty} = q$, we set the values of $c_1, ..., c_4$ such that at a large (finite) and at a small (finite) major axis lengths, the axial ratios of constant-$p$ surfaces become close to $p$ and $q$.

We expand $s(r)$ and $t(r)$ at large and at small $r$ and retain the terms of highest orders only, in $p$. We note that at large radii, $s(r)$ and $t(r)$ decline as $1/r^6$, whereas $g(r)$ and $h(r)$ go as $1/r^4$. Likewise at small radii, $s(r)$ and $t(r)$ increases as $r^{2-\gamma}$, whereas $g(r)$ and $h(r)$ rise as $r^{-7}$. Therefore, in asymptotic large and small radii, only the original terms $f(r), g(r)$ and $h(r)$ of the $fgh$ model will survive, and so, the ratios $\frac{a_1}{a_0}, ..., \frac{a_4}{a_0}$ will be the same, as in $fgh$ model. Requiring that for a large and a small major axis lengths, denoted $a_1$ and $a_3$, respectively, axial ratios of constant-$p$ surfaces are same as $p$ and $q$, we obtain

$$c_1^\delta = a_1^\delta q^\delta \frac{\beta \{ \rho_{1z_l} - \rho_{1z_r} + p^\delta (\rho_{1y_l} - \rho_{1z_l}) \}}{\{1 + p^\delta - 2a_0 q^\delta\}}, \quad (3.22)$$
\[ c_3^6 = \frac{a_0^6 \beta \{ p_{iz} (p^6 - a_0q^6) - p_{iz} p^6 (1 - a_0q^6) + \rho_{iz} q^6 a_0 (1 - p^6) \}}{(1 + p^6 - 2a_0q^6)}, \]  
(3.23)

\[ \frac{c_3^6}{c_2^{\gamma-\gamma}} = \frac{\beta}{(2 - \gamma) a_2^{\gamma-\gamma}} \frac{\{ p_{iz} (p^2 - a_0p^2 - a_0^2) - p_{iy} (p^2 - a_0) - \rho_{iz} a_0 (1 - p^2) \}}{(q^2 - a_0 p^2)}, \]  
(3.24)

\[ \frac{c_3^6}{c_4^{\gamma-\gamma}} = \frac{\beta}{\gamma (9 - \gamma) a_2^{\gamma-\gamma}} \frac{\{ p_{iz} (p^2 - a_0p^2 - a_0^2) - p_{iy} (q^2 - a_0) - \rho_{iz} a_0 (1 - p^2) \}}{(q^2 - a_0 p^2)}, \]  
(3.25)

where \( p_{iz} = a_1 (a_1, 0, 0), p_{iy} = a_1 (0, a_i, 0), p_{iv} = a_1 (0, 0, a), p_{ix} = a_1 (0, a_i, 0), p_{ix} = a_1 (0, 0, a), \rho_{iz} = a_1 (a_1, 0, 0), \rho_{iy} = a_1 (0, a_i, 0), \rho_{iv} = a_1 (0, 0, a), \rho_{ix} = a_1 (0, a_i, 0). \) We take a = 20b and a_i = 0.5b, and find that the terms, which are dropped while obtaining (3.22)-(3.25), are small. Equations (3.22)-(3.25) are valid for \( \gamma \neq 0. \) For \( \gamma = 0, \) equations (3.22)-(3.25) are replaced by

\[ c_1^6 = a_0^6 \beta \{ p_{iz} (p^6 - a_0q^6) - \rho_{ij} p^6 (1 - a_0q^6) + \rho_{iz} q^6 a_0 (1 - p^6) \}, \]  
(3.26)

\[ c_3^6 = a_0^6 \beta \{ p_{iz} (p^6 - a_0q^6) - p_{iz} p^6 (1 - a_0q^6) + \rho_{iz} q^6 a_0 (1 - p^6) \}, \]  
(3.27)

\[ \frac{c_3^6}{c_2^{\gamma-\gamma}} = \frac{\beta}{8(10 - \gamma) a_2^{3-\gamma}} \frac{\{ p_{iz} (p^3 - a_0q^3 - a_0^3) + p_{iy} (p^3 - a_0) - \rho_{iz} a_0 (1 - p^3) \}}{(q^3 - a_0 p^3)}, \]  
(3.28)

\[ \frac{c_3^6}{c_4^{\gamma-\gamma}} = \frac{\beta}{60(10 - \gamma) a_2^{3-\gamma}} \frac{\{ p_{iz} (q^3 - a_0q^3 - a_0^3) + p_{iy} (q^3 - a_0) - \rho_{iz} a_0 (1 - p^3) \}}{(q^3 - a_0 p^3)}. \]  
(3.29)

We have adopted a form (3.15) of density distribution and a procedure to determine \( c_1, \ldots, c_4. \) The resulting density figure is nearly ellipsoidal: serious dimples of the \( f_{gh} \) models have almost disappeared (see Figure 3.5). The method adopted here is different from that adopted by Schwarzschild (1993), who used a form equivalent to a 10 term development in spherical harmonics.

It was realized by Schwarzschild (1979) that while the first term in (3.16) is spherical, second term containing \( Y_2^0 \) shortens the z-axis and lengthens the \( x \) and \( y \) axes equally. The third term containing \( Y_2^2 \) lengthens the x-axis and shortens the y-axis. This can be seen easily by examining the values of \( Y_2^0 \) and \( Y_2^2 \) along Cartesian coordinate axes. The values of \( Y_2^0 (x, y, z) \) and \( Y_2^2 (x, y, z) \) are shown in Table 3.1. We note that term with \( Y_2^0 \) has an overall negative sign (see 3.16). Therefore, the above mentioned realization of Schwarzschild (1979) follows.
This gives rise to a triaxial figure with the major axis along the $x$-coordinate, the median one along the $y$-coordinate and the minor one along the $z$-coordinate. The additional, second and third terms in (3.15) just reverse the above effects, provided $\alpha > 3/(8 \times 105)$. This also follows from the inspection of Table 3.1 indicating the values $Y_4^0$, $Y_4^2$ and $Y_4^4$ on coordinate axes, and arguing in the same manner as Schwarzschild (1979) did. This also explains our motivation for taking a particular combination of $Y_4^0$ and $Y_4^4$ terms in (3.16) and the value of $\alpha$.

A further motivation of adopting a form (3.15) is that we have just two more radial functions $s$ and $t$, involving just four constants $c_1, \ldots, c_4$, which may be fixed by adopting procedure, close to that of ZC96. In case more number of different radial functions are chosen, we shall have larger number of constants and a more complex procedure has to be adopted for determining these. This may be a program of future investigation. We take $\alpha = 0.01$ for the present study.

The radial profiles of $f(r)$, $g(r)$, $h(r)$, $s(r)$ and $t(r)$ for the rounder model with $p = 0.90, q = \ldots$
Table 3.1: Values of the $Y_2^0$, $Y_2^2$, $Y_4^0$, $Y_2^2$ and $Y_4^4$ along Cartesian coordinate axes.

<table>
<thead>
<tr>
<th>Coordinate axis</th>
<th>$Y_2^0$</th>
<th>$Y_2^2$</th>
<th>$Y_4^0$</th>
<th>$Y_2^2$</th>
<th>$Y_4^4$</th>
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</thead>
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<td>3</td>
<td>3/8</td>
<td>$-15/2$</td>
<td>105</td>
</tr>
<tr>
<td>$y = 0, z = 0$</td>
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<td>-3</td>
<td>3/8</td>
<td>15/2</td>
<td>105</td>
</tr>
<tr>
<td>$y = 0, x = 0$</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

0.70 and for the flatter model with $p = 0.75$, $q = 0.60$ are presented in Table 3.2 and 3.3. The average value $<s + t/g + h>$ for these models are also presented in these tables.

3.3 Projected properties

The form (3.15) of the density of the models allows a straightforward calculation of the projected surface density $\Sigma$, for any viewing direction. We choose coordinates $(x', y', z')$ with the $z'$-axis along the line of sight and with the $z'$-axis in the $(x, y)$ plane. Let $(\theta', \phi')$ be the standard spherical coordinates of the line of sight (de Zeeuw & Franx, 1989) and $(R, \theta)$ be the polar coordinates in $(x', y')$ plane. It follows that $\theta$ is positive when measured from the $y'$ in the counter clockwise direction. The projected surface density $\Sigma(x', y')$ or $\Sigma(R, \theta)$ is obtained by integrating the model density (3.15) along the line-of-sight. We have the relations (2.4) and (2.5) relating the body coordinate $(x, y, z)$ and the observed coordinate $(x', y', z')$, and

$$R^2 = x'^2 + y'^2, \quad x' = R \sin \theta$$
$$\theta = \tan^{-1} \frac{x'}{y'}, \quad y' = R \cos \theta$$

and

$$r^2 = x'^2 + y'^2 + z'^2 = x'^2 + y'^2 + z^2 = R^2 + z'^2$$

Therefore, it follows that the projected surface density $\Sigma(R, \theta)$ is given by

$$\Sigma(R, \theta) = 2 \int_0^\infty \rho \, dz' = 2 \int_R^\infty \frac{\rho \, r}{\sqrt{r^2 - R^2}} \, dr.$$  (3.31)

Using above, we integrate each term in the expression of $\rho$. Terms odd in $z'$ give zero contributions to $\Sigma$.  


Table 3.2: Values of the radial profiles of $f(r)$, $g(r)$, $h(r)$, $s(r)$ and $t(r)$ for the rounder model with $p = 0.90, q = 0.70, \gamma = 1.5$.

<table>
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<th>$r/b_0$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
<th>$s$</th>
<th>$t$</th>
<th>Average</th>
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<td>7.06E+00</td>
<td>8.09E-01</td>
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<td>3.40E-03</td>
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<td>4.75E-04</td>
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</tr>
<tr>
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<td>1.02E-02</td>
<td>1.48E-03</td>
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<td>1.45E-03</td>
<td>9.99E-05</td>
<td></td>
</tr>
</tbody>
</table>
3.3. PROJECTED PROPERTIES

Table 3.3: Same as in table 3.2 with \( p = 0.75, q = 0.60, \gamma = 1.5 \).

<table>
<thead>
<tr>
<th>( r/h_o )</th>
<th>( f )</th>
<th>( g )</th>
<th>( h )</th>
<th>( s )</th>
<th>( t )</th>
<th>Average</th>
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<tr>
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<td>1.73E-02</td>
<td>2.39E-02</td>
<td>2.72E-03</td>
<td>(&lt;s + t/g + h&gt;)</td>
</tr>
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<td>3.18E-02</td>
<td>9.94E-03</td>
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<td>1.23E-03</td>
<td>1.47E-03</td>
<td>2.13E-04</td>
<td></td>
</tr>
</tbody>
</table>
After some long, but straightforward, algebra we obtain the projected surface density \( \Sigma(R, \Theta) \), given by

\[
\Sigma(R, \Theta) = \Sigma_1(R, \Theta) + P_1(R) \cos 4\Theta + P_2(R) \sin 4\Theta,
\]

where

\[
\Sigma_1(R, \Theta) = P_0(R) - P_2(R) \cos 2\Theta - P_3(R) \sin 2\Theta.
\]

\( P_0, \ldots, P_5 \) are expressed in terms of the integrals \( S_1(R), S_2(R), S_3(R) \), given by

\[
S_1(R) = \int_R^\infty \frac{r s(r) \, dr}{\sqrt{r^2 - R^2}},
\]

\[
S_2(R) = R^2 \int_R^\infty \frac{s(r) \, dr}{\sqrt{r^2 - R^2}},
\]

\[
S_3(R) = R^4 \int_R^\infty \frac{s(r) \, dr}{r^3 \sqrt{r^2 - R^2}},
\]

and similar integrals \( T_1, T_2, T_3, G_1, G_2, H_1, H_2 \) and \( F_1 \) in terms of functions \( t(r), g(r), h(r) \) and \( f(r) \), respectively. The actual forms of these functions \( P_0, \ldots, P_5 \) are presented below.

Integrating \( \rho \) along the line of sight, we obtain

\[
\Sigma = E_1 + E_2 \sin^2 \Theta + E_3 \cos^2 \Theta + E_4 \sin \Theta \cos \Theta + E_5 \sin^4 \Theta
\]

\[
+ E_6 \cos^4 \Theta + E_7 \sin^3 \Theta \cos^2 \Theta + E_8 \cos \Theta \sin^3 \Theta + E_9 \sin \Theta \cos^3 \Theta,
\]

where

\[
E_1 = 2F_1 + G_1 + \frac{3}{4} S_1 + 3 \cos^2 \theta' \left( G_2 - G_1 + \frac{5}{4} (S_2 - S_1) \right)
\]

\[
+ \sin^2 \theta' \cos 2\phi' \left( 6(H_1 - H_2) + 15(T_2 - T_1) \right)
\]

\[
+ 105 \sin^2 \theta' \cos^2 \theta' \cos 2\phi' \left( T_2 - 2T_2 + T_1 \right)
\]

\[
- (210 \alpha \sin^4 \theta' \cos 4\phi' - \frac{35}{4} \cos^4 \theta') \left( S_3 - 2S_2 + S_1 \right);
\]

\[
E_2 = \frac{3}{2} \sin^2 \theta' \left( 2G_2 + 5S_2 \right) + 3 \cos^2 \theta' \cos 2\phi' \left( 2H_2 - 5T_2 \right)
\]

\[
+ \frac{105}{2} \sin^2 \theta' \cos^2 \theta' \left( 24 \alpha \cos 4\phi' - 1 \right) \left( S_3 - S_2 \right)
\]

\[
- 105 \left( \cos^4 \theta' + \sin^4 \theta' - 4 \sin^2 \theta' \cos^2 \theta' \right) \cos 2\phi' \left( T_3 - T_2 \right);
\]

\[
E_3 = -3 \cos 2\phi' \left( 2H_2 - 5T_2 \right) - 1260 \alpha \sin^2 \theta' \cos 4\phi' \left( S_3 - S_2 \right)
\]

\[
+ 105 \cos^2 \theta' \cos 2\phi' \left( T_3 - T_2 \right),
\]

\[
E_4 = 6 \sin 2\phi' \cos \theta' \left( 2H_2 - 5T_2 \right) + 5040 \alpha \sin 2\phi' \cos 2\phi' \sin^2 \theta' \cos \theta' \left( S_3 - S_2 \right)
\]

\[
- 210 \sin 2\phi' \cos \theta' \left( \cos^2 \theta' - 2 \sin^2 \theta' \right) \left( T_3 - T_2 \right),
\]

\[
E_5 = \left( G_1 + \frac{5}{4} S_1 \right) \left( 2H_1 - H_2 \right) + 3 \cos^2 \theta' \cos 2\phi' \left( H_2 - T_2 \right)
\]

\[
+ 105 \sin^2 \theta' \cos^2 \theta' \left( 24 \alpha \cos 4\phi' - 1 \right) \left( S_3 - S_2 \right)
\]

\[
- 105 \left( \cos^4 \theta' + \sin^4 \theta' - 4 \sin^2 \theta' \cos^2 \theta' \right) \cos 2\phi' \left( T_3 - T_2 \right);
\]

\[
E_6 = \left( G_1 + \frac{5}{4} S_1 \right) \left( 2H_1 - H_2 \right) + 3 \cos^2 \theta' \cos 2\phi' \left( H_2 - T_2 \right)
\]

\[
+ 105 \sin^2 \theta' \cos^2 \theta' \left( 24 \alpha \cos 4\phi' - 1 \right) \left( S_3 - S_2 \right)
\]

\[
- 105 \left( \cos^4 \theta' + \sin^4 \theta' - 4 \sin^2 \theta' \cos^2 \theta' \right) \cos 2\phi' \left( T_3 - T_2 \right);
\]

\[
E_7 = \left( G_1 + \frac{5}{4} S_1 \right) \left( 2H_1 - H_2 \right) + 3 \cos^2 \theta' \cos 2\phi' \left( H_2 - T_2 \right)
\]

\[
+ 105 \sin^2 \theta' \cos^2 \theta' \left( 24 \alpha \cos 4\phi' - 1 \right) \left( S_3 - S_2 \right)
\]

\[
- 105 \left( \cos^4 \theta' + \sin^4 \theta' - 4 \sin^2 \theta' \cos^2 \theta' \right) \cos 2\phi' \left( T_3 - T_2 \right);
\]

\[
E_8 = \left( G_1 + \frac{5}{4} S_1 \right) \left( 2H_1 - H_2 \right) + 3 \cos^2 \theta' \cos 2\phi' \left( H_2 - T_2 \right)
\]

\[
+ 105 \sin^2 \theta' \cos^2 \theta' \left( 24 \alpha \cos 4\phi' - 1 \right) \left( S_3 - S_2 \right)
\]

\[
- 105 \left( \cos^4 \theta' + \sin^4 \theta' - 4 \sin^2 \theta' \cos^2 \theta' \right) \cos 2\phi' \left( T_3 - T_2 \right).
\]
$E_5 = \frac{35}{4} \sin^4 \theta' S_3 - 210 \alpha \cos^4 \theta' \cos 4\phi' S_3 + 105 \sin^2 \theta' \cos^2 \theta' \cos 2\phi' T_3,$

$E_6 = -210 \alpha \cos 4\phi' S_3,$

$E_7 = 1260 \alpha \cos 4\phi' \cos^2 \theta' S_3 - 105 \cos 2\phi' \sin^2 \theta' T_3,$

$E_8 = -1680 \alpha \cos 2\phi' \sin 2\phi' \cos^3 \theta' S_3 + 210 \sin 2\phi' \sin^2 \theta' \cos \theta' T_3,$

$E_9 = 1680 \alpha \sin 2\phi' \cos 2\phi' \cos \theta' S_3,$

(3.38)

which can be written in the form (3.32). We have

$$P_0 = E_1 + \frac{1}{2}(E_2 + E_3) + \frac{1}{8}(3E_5 + 3E_6 + E_7),$$

$$P_2 = \frac{1}{2}(-E_2 + E_3 - E_5 + E_6),$$

$$P_3 = \frac{1}{4}(2E_4 + E_5 + E_9),$$

$$P_4 = \frac{1}{8}(E_6 + E_7 - E_9),$$

$$P_5 = \frac{1}{8}(E_9 - E_8).$$

(3.39)

The integrals needed in $P_0, \ldots, P_5$ are calculated by numerical quadrature of the formulae given below. These can also be expressed in terms of elementary functions for integer $\gamma$.

The formulae given in (3.34)-(3.36) are not very useful for numerical computations because of the infinite integration intervals and singularities of the integrands. We follow the procedure of Dehne (1993) and use substitution

$$\tau = \sqrt{\frac{r}{r + b_0}} (\sigma + 1) - \sigma,$$

(3.40)

where

$$\sigma = \frac{R}{b_0},$$

Then $F_1(R)$ reduces to

$$F_1(R) = \frac{M(3 - \gamma)}{2\pi b_0^2 (1 + \sigma)^{7/3 - \gamma}} \int_0^1 \frac{(\sigma + \tau^2)^{1 - \gamma} (1 - \tau^2) d\tau}{\sqrt{2\sigma + \tau^2 (1 - \sigma)}} .$$

(3.41)

Redefining $\tau$ as

$$\tau = \sqrt{\frac{r}{r + b_2}} (\sigma + 1) - \sigma,$$

(3.42)

where

$$\sigma = \frac{R}{b_2},$$
we obtain

\[ G_1(R) = \frac{Mb_1}{2\pi b_2^3 (1 + \sigma)^{11/2 - \gamma}} \int_0^1 f_2(\tau) \left(1 - \tau^2\right)^2 (\sigma + \tau^2)^{1 - \gamma} d\tau, \]  

(3.43)

and

\[ G_2(R) = \frac{Mb_1 R^2}{2\pi b_2^3 (1 + \sigma)^{11/2 - \gamma}} \int_0^1 f_2(\tau) \left(1 - \tau^2\right)^4 d\tau \left(\sigma + \tau^2\right)^{1 + \gamma}, \]

(3.44)

where

\[ f_2(\tau) = \frac{4(\sigma + \tau^2)^2 + 4(6 - \gamma)(\sigma + \tau^2)(1 - \tau^2) + \gamma(5 - \gamma)(1 - \tau^2)^2}{\sqrt{2\sigma + \tau^2 (1 - \sigma)}}. \]

\(H_1(R)\) and \(H_2(R)\) are obtained from (3.43)-(3.44) by replacing \(b_2\) by \(b_4\) and \(b_1\) by \(b_3\).

Redefining \(\tau\) and \(\sigma\) further by

\[ \tau = \sqrt{\frac{\tau}{\tau + c_2}} (\sigma + 1) - \sigma, \]  

(3.45)

where

\[ \sigma = \frac{R}{c_2}, \]

we find that

\[ S_1(R) = \frac{M c_1^5}{2\pi b_2^3 c_2^3 (1 + \sigma)^{11/2 - \gamma}} \int_0^1 f_3(\tau) (\sigma + \tau^2)^{2 - \gamma} (1 - \tau^2)^4 d\tau, \]

(3.46)

\[ S_2(R) = \frac{M c_1^5 R^2}{2\pi b_2^3 c_2^3 (1 + \sigma)^{11/2 - \gamma}} \int_0^1 f_3(\tau) (\sigma + \tau^2)^{1 - \gamma} (1 - \tau^2)^6 d\tau, \]

(3.47)

and

\[ S_3(R) = \frac{M c_1^5 R^4}{2\pi b_2^3 c_2^3 (1 + \sigma)^{11/2 - \gamma}} \int_0^1 f_3(\tau) (1 - \tau^2)^4 (\sigma + \tau^2)^{1 + \gamma} d\tau, \]

(3.48)

where

\[ f_3(\tau) = \frac{8(\sigma + \tau^2)^2 + 8(10 - \gamma)(\sigma + \tau^2)(1 - \tau^2) + \gamma(9 - \gamma)(1 - \tau^2)^2}{\sqrt{2\sigma + \tau^2 (1 - \sigma)}}. \]

Integrals \(T_1, T_2, T_3\) can be calculated from (3.46)-(3.48) by replacing \((c_1, c_2)\) by \((c_3, c_4)\).

The integrals are free from the singularities and intervals of integrations are finite. Those can be computed easily by using standard routines of numerical quadrature.

For integer values of \(\gamma\), these integrals can be calculated analytically, following the procedure given in ZC96. The integrals are of the form

\[ I = \int \frac{dx}{(x + p)\sqrt{R_1}}, \]

(3.49)
where

\[ R_1 = a + bx + cx^2. \]

We obtain

\[ I = \int \frac{t^{k-1} \, dt}{\sqrt{c + (b - 2pc)t + (a - bp + cp^2)t^2}}, \tag{3.50} \]

using the substitution

\[ t = \frac{1}{x + p}. \tag{3.51} \]

(cf: Gradshteyn and Ryzhik, 1965). In form (3.50), the integration can be carried out analytically, for positive integer \( k \) with \( k - 1 \leq 3 \).

For integer \( k \), the integral \( F_t, G_t, H_t, S_t, S_2, T_1, T_2 \) and \( T_3 \) can be written in the form

\[ W_k(A) = \int_A^\infty \frac{dx}{(x + 1)^k \sqrt{x^2 - A^2}} \tag{3.52} \]

\[ = -\int_{1/A}^\infty \frac{t^{k-1}}{\sqrt{1 - 2t + (1 - A^2)t^2}} \tag{3.53} \]

For \( k = 1 \), (3.52) gives

\[ W_1(A) = \begin{cases} \frac{1}{\sqrt{1 - A^2}} \ln \frac{1 + \sqrt{1 - A^2}}{A} & (0 \leq A < 1) \\ 1 & (A = 1) \\ \frac{1}{\sqrt{A^2 - 1}} \arccos \frac{1}{A} & (A > 1) \end{cases} \tag{3.54} \]

and for \( k = 2 \), we obtain

\[ W_2(A) = \frac{W_1(A) - 1}{1 - A^2} \tag{3.55} \]

For \( k > 2 \), we use (cf: Gradshteyn and Ryzhik, 1965)

\[ \int \frac{x^m}{\sqrt{R_1}} = \frac{x^{m-1}}{cm} \sqrt{R_1} - \frac{(2m - 1)}{2cm} \int \frac{x^{m-1}}{\sqrt{R_1}} \, dx - \frac{(m - 1)a}{mc} \int \frac{x^{m-2}}{\sqrt{R_1}} \, dx. \tag{3.56} \]

We find

\[ W_n(A) = \frac{(2n - 3)W_{n-1}(A) - (n - 2)W_{n-2}(A)}{(n - 1)(1 - A^2)}. \tag{3.57} \]

Equations (3.54), (3.55), (3.57) are reported in ZC96. We re-derived the same, in a slightly different notation.
CHAPTER 3. TRIAXIAL MASS MODELS WITH CUSPS

For \( \gamma = 0, 1 \) and 2, the integrals \( F_1, G_1, G_2 \) are reported in ZC96, in terms \( W_k \)-functions of various order of \( n \) and their arguments. Here, we present \( S_1, S_2 \) and \( S_3 \) for \( \gamma = 0 \), we obtain

\[
S_1(R) = \frac{Mc_1^6}{4\pi c_2^5} \int_0^\infty \frac{8x^5 + 80x^4}{(x + 1)^{10}} \sqrt{x^2 - A^2} \, dx
\]  

(3.58)

where \( A = R/c_2 \)

Above is an illustration of the form of integral. We obtain similar forms for \( S_2 \) and \( S_3 \), and also for \( \gamma = 1 \) and \( \gamma = 2 \). Decomposition the integrand in partial fractions, we obtain integrals, each are of form (3.52) for various \( k \), which can be expressed in terms of \( W_k \)-functions. Table 3.4 presents the \( S_1, S_2, S_3 \) in terms of \( W_k \)-functions. Similar expressions for \( F_1, G_1, G_2 \) in terms of \( W_k \)-functions are given in ZC96.

In Table 3.5, we present the values of \( F_1, G_1, G_2, H_1, H_2, S_1, S_2 \) and \( S_3 \) as calculated from numerical integration and that from \( W_k \)-functions. The values from both the calculation matches.

Although the potential \( \Psi_1(r, \theta, \phi) \) and the associated density \( \rho_1(r, \theta, \phi) \) are the same as in the \( fgh \) models studied by earlier workers, we note that \( \Sigma_1(R, \Theta) \) is now different from the projected density \( \Sigma_{fgh}(R, \Theta) \) of the \( fgh \) models. The additional terms with radial functions \( s(r) \) and \( t(r) \), which are included in the present study, also contribute to \( \Sigma_1(R, \Theta) \). However, the contributions from these additional terms are small compared to those from the terms with radial functions \( f(r), g(r) \) and \( h(r) \) (see Table 3.5). It also compares the integrals as obtained by numerical integration and as obtain by using the \( w \)-functions, for case \( \gamma = 1 \). The functional form of \( \Sigma_1(R, \Theta) \) is same as that of \( \Sigma_{fgh}(R, \Theta) \). We follow the derivations made in ZC96 and define the position angle \( \Theta_{s\Sigma_1} \) of \( \Sigma_1 \) by

\[
\Theta_{s\Sigma_1} = \frac{1}{2} \tan^{-1} \frac{P_3}{P_2},
\]

(3.59)

and the axial ratio \( b/a \) of \( \Sigma_1 \) by

\[
P_0(a) - P_3(a) \cos 2\Theta_{s\Sigma_1} - P_3(a) \sin 2\Theta_{s\Sigma_1} = P_0(b) - P_2(b) \cos 2(\Theta_{s\Sigma_1} - \pi/2)
\]

\[
- \frac{P_3(b) \sin 2(\Theta_{s\Sigma_1} - \pi/2)}{\sin 2(\Theta_{s\Sigma_1} - \pi/2)}.
\]

(3.60)

Relation (3.59) gives the position angle of the major axis, provided

\[
P_3 \sin 2\Theta_{s\Sigma_1} < 0.
\]

(3.61)

The terms \( P_4 \) and \( P_5 \) are related to high-order residuals, on constant-\( \Sigma_1 \) contour, as can be seen by rewriting (3.32) as

\[
\Sigma(R, \Theta) = \Sigma_1(R, \Theta) + B_4\Sigma_1 \cos 4(\Theta - \Theta_{s\Sigma_1}) + A_4\Sigma_1 \sin 4(\Theta - \Theta_{s\Sigma_1}),
\]

(3.62)
3.3 PROJECTED PROPERTIES

Table 3.4: The functions $S_1$, $S_2$, $S_3$ for special values of $\gamma$, expressed in terms of the elementary functions $W_k(x)$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$S_1(R) = \frac{Mc^2}{\pi R^2} \cdot \left[ 2W_5(R/c_2) + 10W_6(R/c_2) - 60W_7(R/c_2) + 100W_8(R/c_2) - 70W_9(R/c_2) + 18W_{10}(R/c_2) \right]$</td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>$S_2(R) = \frac{Mc^2 R^2}{\pi R_0^2 c_2} \cdot \left[ 2W_7(R/c_2) + 14W_8(R/c_2) - 34W_9(R/c_2) + 18W_{10}(R/c_2) \right]$</td>
</tr>
<tr>
<td>$\frac{4}{3}$</td>
<td>$S_3(R) = \frac{Mc^2 R^4}{\pi R_0^4 c_2^2} \cdot \left[ 2W_9(R/c_2) + 18W_{10}(R/c_2) \right]$</td>
</tr>
<tr>
<td>1</td>
<td>$S_1(R) = \frac{Mc^2}{\pi R^2} \cdot \left[ 2W_5(R/c_2) + 10W_6(R/c_2) - 40W_7(R/c_2) + 42W_8(R/c_2) - 14W_9(R/c_2) \right]$</td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>$S_2(R) = \frac{Mc^2 R^2}{\pi R_0^2 c_2} \cdot \left[ 2W_7(R/c_2) + 14W_8(R/c_2) - 14W_9(R/c_2) \right]$</td>
</tr>
<tr>
<td>$\frac{4}{3}$</td>
<td>$S_3(R) = \frac{Mc^2 R^4}{\pi R_0^4 c_2^2} \cdot \left[ 2W_9(R/c_2) + 18W_{10}(R/c_2) \right]$</td>
</tr>
<tr>
<td>2</td>
<td>$S_1(R) = \frac{Mc^2}{2\pi R_0^2 c_2^2} \cdot \left[ 4W_5(R/c_2) + 20W_6(R/c_2) - 45W_7(R/c_2) + 21W_8(R/c_2) \right]$</td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>$S_2(R) = \frac{Mc^2 R^2}{2\pi R_0^2 c_2^2} \cdot \left[ \frac{75}{3R} - 7W_1(R/c_2) + 14W_2(R/c_2) + 7W_3(R/c_2) + 7W_4(R/c_2) + 7W_5(R/c_2) + 7W_6(R/c_2) - 3W_7(R/c_2) + 21W_8(R/c_2) \right]$</td>
</tr>
<tr>
<td>$\frac{4}{3}$</td>
<td>$S_3(R) = \frac{Mc^2 R^4}{2\pi R_0^4 c_2^4} \cdot \left[ 24W_2(R/c_2) + 41W_3(R/c_2) + 51W_4(R/c_2) + 54W_5(R/c_2) + 50W_6(R/c_2) - \frac{243}{4R^2} + \frac{7\pi c_2^2}{4} \right]$</td>
</tr>
</tbody>
</table>
Table 3.5: Values of the integrals $F_1, G_1, H_1, S_1, S_2, S_3$, all as a function of $R$. For $\gamma = 1.0$ case.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$F_1$</th>
<th>$G_1$</th>
<th>$H_1$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0000</td>
<td>.0279</td>
<td>.0207</td>
<td>.0148</td>
<td>.0075</td>
<td>.0056</td>
<td>.0046</td>
<td>using numerical integration</td>
</tr>
<tr>
<td>3.2000</td>
<td>.0240</td>
<td>.0178</td>
<td>.0127</td>
<td>.0063</td>
<td>.0048</td>
<td>.0039</td>
<td></td>
</tr>
<tr>
<td>3.4000</td>
<td>.0207</td>
<td>.0155</td>
<td>.0111</td>
<td>.0054</td>
<td>.0041</td>
<td>.0034</td>
<td></td>
</tr>
<tr>
<td>3.6000</td>
<td>.0180</td>
<td>.0135</td>
<td>.0097</td>
<td>.0046</td>
<td>.0035</td>
<td>.0029</td>
<td></td>
</tr>
<tr>
<td>3.8000</td>
<td>.0158</td>
<td>.0118</td>
<td>.0085</td>
<td>.0039</td>
<td>.0030</td>
<td>.0025</td>
<td></td>
</tr>
<tr>
<td>4.0000</td>
<td>.0139</td>
<td>.0104</td>
<td>.0075</td>
<td>.0034</td>
<td>.0026</td>
<td>.0022</td>
<td></td>
</tr>
<tr>
<td>4.2000</td>
<td>.0123</td>
<td>.0092</td>
<td>.0067</td>
<td>.0029</td>
<td>.0022</td>
<td>.0019</td>
<td></td>
</tr>
<tr>
<td>4.4000</td>
<td>.0109</td>
<td>.0082</td>
<td>.0059</td>
<td>.0025</td>
<td>.0020</td>
<td>.0016</td>
<td></td>
</tr>
<tr>
<td>4.6000</td>
<td>.0098</td>
<td>.0073</td>
<td>.0053</td>
<td>.0022</td>
<td>.0017</td>
<td>.0014</td>
<td></td>
</tr>
<tr>
<td>4.8000</td>
<td>.0088</td>
<td>.0066</td>
<td>.0048</td>
<td>.0019</td>
<td>.0016</td>
<td>.0013</td>
<td></td>
</tr>
</tbody>
</table>

| 3.0000 | .0279 | .0207 | .0148 | .0074 | .0055 | .0045 | using $W_k$-functions     |
| 3.2000 | .0240 | .0178 | .0127 | .0063 | .0048 | .0039 |                          |
| 3.4000 | .0207 | .0155 | .0111 | .0054 | .0041 | .0034 |                          |
| 3.6000 | .0180 | .0135 | .0097 | .0046 | .0035 | .0029 |                          |
| 3.8000 | .0158 | .0118 | .0085 | .0039 | .0030 | .0025 |                          |
| 4.0000 | .0139 | .0104 | .0075 | .0034 | .0026 | .0022 |                          |
| 4.2000 | .0123 | .0092 | .0067 | .0029 | .0022 | .0019 |                          |
| 4.4000 | .0109 | .0082 | .0059 | .0025 | .0020 | .0016 |                          |
| 4.6000 | .0098 | .0073 | .0053 | .0022 | .0017 | .0014 |                          |
| 4.8000 | .0088 | .0066 | .0048 | .0019 | .0015 | .0013 |                          |
3.4 Results and Discussion

The detailed analysis of triaxial generalization of \( \gamma \)-model of Dehnen with the inclusion of fourth order spherical harmonics shows that the present mass models are more realistic model, as these have a nearly ellipsoidal shape. The complexity of the problem led us to some long expressions, but the numerical evaluation of these to obtain projected parameters, in terms of different choices of model parameters and viewing angles, is straightforward.

The axial ratios of constant density surfaces of the models as depicted in Figure 3.1-3.4 shows that the axial ratios of constant-\( \rho_1 \) at intermediate radii, vary considerably from their values at asymptotic radii. These variations are relatively small for the constant-\( \rho \) surfaces. For the flatter model with \( p = 0.75, q = 0.6, \gamma = 1.5 \), the maximum relative change in the axial ratios of constant-\( \rho_1 \) surfaces is \( \sim 25.0\% \), which drops to \( \sim 8.3\% \) for constant-\( \rho \) surfaces. For rounder models \( p = 0.90, q = 0.70, \gamma = 1.5 \) the variation in axial ratios of constant-\( \rho_1 \) at intermediate radii is relatively small (Figure 3.1-3.2) compared to the flatter model. These

\[
B_{4\Sigma_1} = P_4 \cos 4\Theta_{\Sigma_1} + P_5 \sin 4\Theta_{\Sigma_1}, \\
A_{4\Sigma_1} = P_6 \cos 4\Theta_{\Sigma_1} - P_4 \sin 4\Theta_{\Sigma_1},
\]

\( B_{4\Sigma_1} \) and \( A_{4\Sigma_1} \) are the higher order residuals on constant-\( \Sigma_1 \) contour. For real galaxies the higher order residuals are given by variation in best fitting ellipse, based on algorithm of Jedrzejewski (1987), on the surface photometric data of the galaxies using standard IRAF package. We have fitted the ellipse on constant-\( \Sigma \) surface. We first produce the image of the pseudo galaxies using our models and then use ‘ellipse’ task under ‘STSDAS package’ in IRAF, and fit the ellipse on constant-\( \Sigma \) surface. We find that the basic parameters i.e., ellipticity and position angle of constant-\( \Sigma \) contour matches well with that of the basic parameters of \( \Sigma_1 \) contour for all the models with specific values of \( p, q, \gamma, \theta' \) and \( \phi' \). We find the center of \( \Sigma \) contour same, as that of the best-fitting ellipse. \( B_4 \) and \( A_4 \) of the best fit ellipse is close to the \( B_{4\Sigma_1} \) and \( A_{4\Sigma_1} \) on constant-\( \Sigma_1 \) contour, for the rounder model with steep cusp (\( p = 0.90, q = 0.70, \gamma = 1.5 \)). For core model with \( \gamma = 0 \) the match between these parameters are not good. Tables 3.6, 3.7, 3.9 and 3.10 present the ellipticity and position angle of the best-fitting ellipse on constant-\( \Sigma_1 \) and constant-\( \Sigma \) contour for specific \( p, q, \gamma, \theta' \) and \( \phi' \). Tables 3.8 and 3.11 shows the values of \( B_4 \) of constant-\( \Sigma \) as obtain from IRAF and \( B_{4\Sigma_1} \) of constant-\( \Sigma_1 \) contour for specific \( p, q, \gamma, \theta' \) and \( \phi' \). We have also developed a FORTRAN code to find out the best fit ellipse on constant-\( \Sigma \) contour and we find the results very close that of the IRAF.
Figure 3.6: Profiles of the $B_4$, $A_4$, position angle $\Theta_4$ of the major axis and the ellipticity $\epsilon$ of the best-fitting ellipses, all as a function of semimajor axis length $a$, expressed in terms of scale length $b_0$. Model with $p = 0.90$, $q = 0.70$, $\gamma = 1.5$, $\theta' = 80^\circ$, $\phi' = 20^\circ$. 
Figure 3.7: Same as that of Figure 3.6 with $\theta' = 30^\circ$, $\phi' = 20^\circ$. 
Table 3.6: Values of the position angle $\Theta_*$ of the major axis and the ellipticity $\epsilon$ of the best-fitting ellipses, all as a function of semimajor axis length $a$, expressed in terms of scale length $b_0$. These can be compared with the corresponding parameters $\Theta_{\Sigma_1}$ and $\epsilon_{\Sigma_1}$ of the constant-$\Sigma_1$ contours. Model with $p = 0.90$, $q = 0.70$, $\gamma = 1.5$, $\theta = 80^\circ$, $\phi = 20^\circ$.

<table>
<thead>
<tr>
<th>$a/b_0$</th>
<th>$\Theta_*$</th>
<th>$\Theta_{\Sigma_1}$</th>
<th>$\epsilon$</th>
<th>$\epsilon_{\Sigma_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>88.9°</td>
<td>88.7°</td>
<td>0.23</td>
<td>0.23</td>
</tr>
<tr>
<td>2</td>
<td>88.4°</td>
<td>88.6°</td>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td>3</td>
<td>88.8°</td>
<td>89.3°</td>
<td>0.23</td>
<td>0.24</td>
</tr>
<tr>
<td>4</td>
<td>89.5°</td>
<td>89.9°</td>
<td>0.22</td>
<td>0.23</td>
</tr>
<tr>
<td>5</td>
<td>90.1°</td>
<td>89.3°</td>
<td>0.21</td>
<td>0.22</td>
</tr>
<tr>
<td>6</td>
<td>90.7°</td>
<td>90.9°</td>
<td>0.21</td>
<td>0.22</td>
</tr>
<tr>
<td>7</td>
<td>91.0°</td>
<td>91.2°</td>
<td>0.20</td>
<td>0.21</td>
</tr>
<tr>
<td>8</td>
<td>91.4°</td>
<td>91.5°</td>
<td>0.20</td>
<td>0.21</td>
</tr>
<tr>
<td>9</td>
<td>91.5°</td>
<td>91.6°</td>
<td>0.20</td>
<td>0.21</td>
</tr>
<tr>
<td>10</td>
<td>91.7°</td>
<td>91.8°</td>
<td>0.20</td>
<td>0.21</td>
</tr>
</tbody>
</table>

Table 3.7: Same as in Table 3.6 with $\theta = 30^\circ$, $\phi = 20^\circ$.

<table>
<thead>
<tr>
<th>$a/b_0$</th>
<th>$\Theta_*$</th>
<th>$\Theta_{\Sigma_1}$</th>
<th>$\epsilon$</th>
<th>$\epsilon_{\Sigma_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-22.47°</td>
<td>-22.53°</td>
<td>0.088</td>
<td>0.087</td>
</tr>
<tr>
<td>2</td>
<td>-22.05°</td>
<td>-22.38°</td>
<td>0.098</td>
<td>0.096</td>
</tr>
<tr>
<td>3</td>
<td>-24.35°</td>
<td>-24.90°</td>
<td>0.100</td>
<td>0.098</td>
</tr>
<tr>
<td>4</td>
<td>-26.36°</td>
<td>-27.08°</td>
<td>0.100</td>
<td>0.096</td>
</tr>
<tr>
<td>5</td>
<td>-27.89°</td>
<td>-28.66°</td>
<td>0.098</td>
<td>0.095</td>
</tr>
<tr>
<td>6</td>
<td>-29.02°</td>
<td>-29.77°</td>
<td>0.096</td>
<td>0.094</td>
</tr>
<tr>
<td>7</td>
<td>-29.85°</td>
<td>-30.55°</td>
<td>0.095</td>
<td>0.093</td>
</tr>
<tr>
<td>8</td>
<td>-30.47°</td>
<td>-31.11°</td>
<td>0.094</td>
<td>0.0915</td>
</tr>
<tr>
<td>9</td>
<td>-30.95°</td>
<td>-31.52°</td>
<td>0.092</td>
<td>0.090</td>
</tr>
<tr>
<td>10</td>
<td>-31.32°</td>
<td>-31.82°</td>
<td>0.091</td>
<td>0.090</td>
</tr>
</tbody>
</table>
3.4. RESULTS AND DISCUSSION

Table 3.8: Values of the $B_4$ of the best-fitting ellipses and $B_{4k}$ as a function of semi-major axis length $a$, expressed in terms of scale length $b_0$. Model with $p = 0.90, q = 0.70, \gamma = 1.5$.

<table>
<thead>
<tr>
<th>$a/b_0$</th>
<th>$B_4$</th>
<th>$B_{4k}$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.006</td>
<td>0.018</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.025</td>
<td>0.033</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.036</td>
<td>0.037</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.038</td>
<td>0.036</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.035</td>
<td>0.032</td>
<td>$\theta' = 80^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>0.031</td>
<td>0.028</td>
<td>$\phi' = 30^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>0.025</td>
<td>0.024</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.020</td>
<td>0.021</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.016</td>
<td>0.018</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.012</td>
<td>0.016</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.029</td>
<td>-0.023</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.051</td>
<td>-0.036</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.053</td>
<td>-0.037</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.049</td>
<td>-0.032</td>
<td>$\theta' = 30^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>-0.042</td>
<td>-0.026</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.035</td>
<td>-0.021</td>
<td>$\phi' = 20^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>-0.030</td>
<td>-0.018</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.025</td>
<td>-0.015</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.022</td>
<td>-0.012</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.019</td>
<td>-0.017</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.9: Values of the position angle $\Theta_*$ of the major axis and the ellipticity $\epsilon$ of the best-fitting ellipses, all as a function of semimajor axis length $a$, expressed in terms of scale length $b_0$. These can be compared with the corresponding parameters $\Theta_{\Sigma_1}$ and $\epsilon_{\Sigma_1}$ of the constant-$\Sigma_1$ contours. Model with $p = 0.75$, $q = 0.60$, $\gamma = 1.5$, $\theta = 80^\circ$, $\phi = 20^\circ$.

<table>
<thead>
<tr>
<th>$a/b_0$</th>
<th>$\Theta_*$</th>
<th>$\Theta_{\Sigma_1}$</th>
<th>$\epsilon$</th>
<th>$\epsilon_{\Sigma_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90.28$^\circ$</td>
<td>89.90$^\circ$</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>2</td>
<td>89.84$^\circ$</td>
<td>89.85$^\circ$</td>
<td>0.18</td>
<td>0.19</td>
</tr>
<tr>
<td>3</td>
<td>91.37$^\circ$</td>
<td>91.77$^\circ$</td>
<td>0.17</td>
<td>0.18</td>
</tr>
<tr>
<td>4</td>
<td>93.23$^\circ$</td>
<td>93.36$^\circ$</td>
<td>0.16</td>
<td>0.17</td>
</tr>
<tr>
<td>5</td>
<td>94.81$^\circ$</td>
<td>94.59$^\circ$</td>
<td>0.16</td>
<td>0.17</td>
</tr>
<tr>
<td>6</td>
<td>96.02$^\circ$</td>
<td>95.50$^\circ$</td>
<td>0.16</td>
<td>0.17</td>
</tr>
<tr>
<td>7</td>
<td>96.78$^\circ$</td>
<td>96.16$^\circ$</td>
<td>0.16</td>
<td>0.17</td>
</tr>
<tr>
<td>8</td>
<td>97.34$^\circ$</td>
<td>96.65$^\circ$</td>
<td>0.15</td>
<td>0.16</td>
</tr>
<tr>
<td>9</td>
<td>97.73$^\circ$</td>
<td>97.02$^\circ$</td>
<td>0.15</td>
<td>0.16</td>
</tr>
<tr>
<td>10</td>
<td>98.00$^\circ$</td>
<td>97.30$^\circ$</td>
<td>0.15</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 3.10: Same as in Table 3.9 with $\theta = 30^\circ$, $\phi = 20^\circ$.

<table>
<thead>
<tr>
<th>$a/b_0$</th>
<th>$\Theta_*$</th>
<th>$\Theta_{\Sigma_1}$</th>
<th>$\epsilon$</th>
<th>$\epsilon_{\Sigma_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-21.1^\circ$</td>
<td>$-20.7^\circ$</td>
<td>0.26</td>
<td>0.27</td>
</tr>
<tr>
<td>2</td>
<td>$-20.6^\circ$</td>
<td>$-20.9^\circ$</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>3</td>
<td>$-21.4^\circ$</td>
<td>$-21.8^\circ$</td>
<td>0.30</td>
<td>0.29</td>
</tr>
<tr>
<td>4</td>
<td>$-22.2^\circ$</td>
<td>$-22.6^\circ$</td>
<td>0.30</td>
<td>0.29</td>
</tr>
<tr>
<td>5</td>
<td>$-22.8^\circ$</td>
<td>$-23.1^\circ$</td>
<td>0.30</td>
<td>0.29</td>
</tr>
<tr>
<td>6</td>
<td>$-23.3^\circ$</td>
<td>$-23.5^\circ$</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>7</td>
<td>$-23.6^\circ$</td>
<td>$-23.8^\circ$</td>
<td>0.29</td>
<td>0.28</td>
</tr>
<tr>
<td>8</td>
<td>$-23.9^\circ$</td>
<td>$-23.9^\circ$</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td>9</td>
<td>$-24.0^\circ$</td>
<td>$-24.1^\circ$</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td>10</td>
<td>$-24.2^\circ$</td>
<td>$-24.2^\circ$</td>
<td>0.27</td>
<td>0.27</td>
</tr>
</tbody>
</table>
3.4. RESULTS AND DISCUSSION

Table 3.11: Values of the $B_4$ of the best-fitting ellipses and $B_{4\Sigma_1}$ as a function of semimajor axis length $a$, expressed in terms of scale length $b_o$. Model with $p = 0.75$, $q = 0.6$, $\gamma = 1.5$

<table>
<thead>
<tr>
<th>$a/b_o$</th>
<th>$B_4$</th>
<th>$B_{4\Sigma_1}$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.029</td>
<td>0.029</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.062</td>
<td>0.054</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.080</td>
<td>0.060</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.076</td>
<td>0.055</td>
<td>for $\theta' = 80^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>0.066</td>
<td>0.048</td>
<td>$\theta' = 80^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>0.055</td>
<td>0.041</td>
<td>$\phi' = 20^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>0.046</td>
<td>0.034</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.037</td>
<td>0.029</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.031</td>
<td>0.025</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.025</td>
<td>0.021</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.047</td>
<td>-0.020</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.075</td>
<td>-0.028</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.085</td>
<td>-0.026</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.086</td>
<td>-0.022</td>
<td>for $\theta' = 30^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>-0.081</td>
<td>-0.018</td>
<td>$\theta' = 30^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>-0.076</td>
<td>-0.015</td>
<td>$\phi' = 20^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>-0.070</td>
<td>-0.012</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.065</td>
<td>-0.011</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.061</td>
<td>-0.010</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.057</td>
<td>-0.009</td>
<td></td>
</tr>
</tbody>
</table>
Figure 3.8: Profiles of the $B_4$, $A_4$, position angle $\theta_*$ of the major axis and the ellipticity $\epsilon$ of the best-fitting ellipses, all as a function of semimajor axis length $a$, expressed in terms of scale length $b_0$. Model with $p = 0.75$, $q = 0.60$, $\gamma = 1.5$, $\theta' = 80^\circ$, $\phi' = 20^\circ$. 
Figure 3.9: Same as in Figure 3.8 with $\theta' = 30^\circ$, $\phi' = 20^\circ$. 

3.4. RESULTS AND DISCUSSION
variation also depends upon the cusp parameter \( \gamma \) of the models. Steep cusp \( \gamma - 1.5 \) shows less variation in their axial ratios of constant-\( \rho_1 \) (Figure 3.3-3.4).

Figure 3.5 depicts the sections of constant density surfaces of the models and the section of an ellipsoid, with the same \( p \) and \( q \), in \((x - z)\) plane. We found that the constant-\( \rho_1 \) is dimpled (peanut-shaped), the constant-\( \rho \) is relatively smooth and closer to the ellipsoid. The dimplesness found in the constant-\( \rho_1 \) surface depends on the amount of variation in the axial ratios of the constant-\( \rho_1 \) surface.

The projected properties of the mass models shows ellipticity and position angle variations, for different viewing directions. Figure 3.6-3.9 shows the variations in position angle, ellipticity, \( A_4 \) and \( B_4 \) as function of radius, for different viewing directions with a given \( p, q \) and \( \gamma \). The basic parameters of the best-fitting ellipses are very close to those of the constant-\( \Sigma_1 \) contours [see Tables (3.6, 3.7, 3.9 and 3.10)]. Similarly Tables 3.8 and 3.11 compares the values of \( B_4 \) of the best-fitting ellipses with \( B_{4\Sigma_1} \) on the constant-\( \Sigma_1 \) contours.

We found the higher order residuals on constant-\( \Sigma_1 \) contours. These fourth order residuals \( B_{4\Sigma_1} \) and \( A_{4\Sigma_1} \) arise, only from the fourth order spherical harmonics terms in the density form (3.15), and no third order or higher order residuals are present in the analytical formulation.

The higher order residuals of elliptical galaxies are obtained by considering variations around the best-fitting ellipse on the surface photometric data of elliptical galaxies. In order to compare the model with data, we use the standard packages of ellipse-fitting based on the algorithm of Jedrzejewski (1987), and obtain the parameters of the best-fitting ellipses on \( \Sigma \) and the higher order residuals \( A_3, B_3, A_4, B_4 \) on the ellipses. We find that \( B_4 \) is positive when viewed almost along the \((x', y')\) plane and it is negative when viewed as narrow angles with respect to the \( \pm z' \) directions (Figure 3.6-3.9). At intermediate \( \delta' \), \( B_4 \) is partly positive and partly negative. \( A_4 \) is small compared to \( B_4 \), and the residuals \( A_3 \) and \( B_3 \) are negligible.

The boxy and the pointy appearance of the models depend upon the viewing direction. The region of these viewing direction are shown in Figure 3.10 which exhibits three distinct regions of viewing angles producing boxy (region 1), partly boxy and partly pointy (region 2) and pointy (region 3) isophotes. We present another model in Chapter 4 and we will make some more discussion of both the models in Chapter 4.
Figure 3.10: Regions of viewing angles for $B_4$ negative (shaded with vertical lines), positive (shaded with horizontal lines) and partly negative-partly positive (white).