Chapter 2

Basic Mathematical Techniques

Summary Basic mathematical formulations used for constructing the triaxial mass models with higher order spherical harmonic terms, and projecting these along the line-of-sight are presented. Numerical method, for orbit calculations and for constructing the self consistent triaxial dynamical models are also presented.
2.1 Introduction

In this thesis our main aim is to construct the triaxial mass models including fourth order spherical harmonic terms, representing more realistic elliptical galaxies, and to apply these models to the following studies, as listed below.

- Study of the properties of the 3-dimensional mass models.
- Study of the projected properties including the higher order residuals.
- In addition to above we have also studied the effect of additional radial density terms with second order spherical harmonics.
- Finally, we have studied the orbital structure of triaxial models with weak cusp, which can be utilised for the construction of numerical distribution function.

In this chapter we have presented the basic mathematical formulations required for the above listed studies.

In section 2.2 we present the basic mathematical methods used for the construction of models, and for the study of projected properties of various triaxial mass models. Section 2.3 give some details of isophotal shape analysis. Section 2.4 covers the orbital structure, needed for the construction of numerical distribution function. Section 2.5 is devoted to summary and discussion.

2.2 Different triaxial mass models and their projection formulas

2.2.1 Triaxial models without isophotal twists

Stark (1977) and Binney (1985) worked on a triaxial mass models wherein the density is stratified on coaxial and coaligned ellipsoids with constant intrinsic axial ratios \( p \) & \( q \). Binney (1985) re-derived the Stark's results in a slightly modified notation. Binney (1985) assumed that the luminosity density \( l \) in an elliptical galaxy is a function

\[
 l = l(a_v) \tag{2.1}
\]

of the elliptical radius variable

\[
a_v = \sqrt{\left( \frac{x^2}{p^2} + \frac{y^2}{q^2} \right)} \tag{2.2}
\]
Hence the two axial ratios \( p \) and \( q \) satisfy

\[ 1 \geq p \geq q. \tag{2.3} \]

Two angles \((\phi', \theta')\) are required to specify the direction of the line-of-sight with respect to the principle axes of a triaxial galaxy. Using the rotational matrix \( R \) the coordinate system \((x, y, z)\) of ellipsoids are transformed to observer coordinate system \((x', y', z')\) where \(x'\) axis runs along the line-of-sight and the \(z'\) axis runs in the \((x, y)\) plane.

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= R
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix}, \tag{2.4}
\]

where \( R \) is the rotation matrix

\[
R = \begin{pmatrix}
  -\sin \phi' & -\cos \phi' \cos \theta' & \cos \phi' \sin \theta' \\
  \cos \phi' & -\sin \phi' \cos \theta' & \sin \phi' \sin \theta' \\
  0 & \sin \theta' & \cos \theta'
\end{pmatrix}. \tag{2.5}
\]

The surface brightness along the line-of-sight \((x', y')\) is

\[
\Sigma(x', y') = \int_{-\infty}^{\infty} l(a_0^2) dz' \\
= \frac{2}{\sqrt{f}} \int_{0}^{\infty} l(z''^2 + a_2^2) dz''. \tag{2.6}
\]

where

\[
a_0^2 = z''^2 + a_2^2, \\
z'' = \sqrt{f} \left( x + \frac{g}{2f} \right), \\
a_2^2 = \left( h - \frac{g^2}{4f} \right), \\
f = \sin^2 \theta' \left( \cos^2 \phi' + \frac{\sin^2 \phi'}{p^2} \right) + \frac{\cos^2 \phi'}{q^2}, \\
g = \sin \theta' \sin 2\phi' \left( \frac{1}{p^2} - 1 \right) x' + \\
+ \sin 2\theta' \left( \frac{1}{q^2} - \cos^2 \phi' - \frac{\sin^2 \phi'}{p^2} \right) y', \\
h = \left( \sin^2 \phi' + \frac{\cos^2 \phi'}{p^2} \right) x'' + \sin 2\phi' \cos \theta' \left( 1 - \frac{1}{p^2} \right) x' y' + \\
+ \left[ \cos^2 \theta' \left( \cos^2 \phi' + \frac{\sin^2 \phi'}{p^2} \right) + \frac{\sin^2 \theta'}{q^2} \right] y''^2. \tag{2.7}
\]
The triaxial mass models of Stark (1977) and Binney (1985) show that the projected surface densities of these models are stratified on coaxial and coaligned ellipses with constant observed axial ratio $b/a$. But, many real galaxies show $b/a$ variations and isophote twists. In the next section, we will discuss the triaxial mass models which exhibit these features.

2.2.2 Triaxial models with isophotal twists

Triaxial mass distribution can be produced by two different methods. First method is discussed in section 2.2.1 which gives coaxial and coaligned ellipses on their projections with no ellipticity and position angle variations. However, ellipticity variations and isophote twists can be incorporated by letting $p$ and $q$ to vary with radius. Second method of producing a triaxial mass distribution was discussed by Schwarzschild (1979). In this method a spherical mass model is made triaxial by adding two low order spherical harmonics, each multiplied by a radial function. Schwarzschild (1979) took the modified Hubble mass model, and showed that the numerical distribution function exists.

Adopting the same procedure, de Zeeuw & Carollo (1996, hereafter ZC96) made a triaxial generalisation of $\gamma$-models of Dehnen (1993) and studied the projected properties. Following the similar scheme, Chakrabarty & Thakur (2000, hereafter CT00) studied the projected properties of triaxial modified Hubble mass models. In this thesis we have studied these models with inclusion of higher order harmonic terms. Here, first we reproduce the results of ZC96 with slightly different symbols.

2.2.2.1 Triaxial models with cusps

Mass models

de Zeeuw & Carollo considered triaxial potentials of the form

\[
V(r, \theta, \phi) = u(r) - v(r)Y_2^0(\theta) + w(r)Y_2^2(\theta, \phi), \\
V(x, y, z) = u(r) - v(r)(2x^2 - y^2)/2R^2 \\
+ w(r)3(x^2 - y^2)/R^2,
\]

where $(r, \theta, \phi)$ are the spherical coordinates defined such that $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$. The functions $Y_2^0(\theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2}$ and $Y_2^2(\theta, \phi) = 3 \sin^2 \theta \cos 2\phi$ are the usual spherical harmonics, and $u(r)$, $v(r)$ and $w(r)$ three radial functions. They take $u(r)$ to be the
potential of the spherical $\gamma$-models, defined by the choice

$$
u(r) = \begin{cases} 
\frac{GM b_1 r^{2-\gamma}}{(r + b_0)^{4-\gamma}}, & \text{for } \gamma = 2, \\
\frac{GM}{(2-\gamma)b_0^2 [\frac{r}{r+b_0}]^{2-\gamma} - 1}, & \text{for } \gamma \neq 2,
\end{cases}$$

where $M$ is the total mass of the model, $b_0$ is the scale-length and take $0 \leq \gamma < 3$. They choose the functions $v(r)$ and $w(r)$ as follows

$$
v(r) = -\frac{GM b_1 r^{2-\gamma}}{(r + b_2)^{4-\gamma}}, \\
w(r) = -\frac{GM \gamma r^{2-\gamma}}{(r + b_3)^{4-\gamma}},$$

(2.9)

where $b_1, ..., b_4$ are constants.

The associated density distribution $\rho(r, \theta, \phi)$ follows from Poisson's equation

$$
\rho(r, \theta, \phi) = f(r) - g(r)Y_2^0(\theta) + h(r)Y_2^2(\theta, \phi),
$$

(2.10)

where

$$
f(r) = \frac{Mb_3(3-\gamma)}{4\pi} \frac{1}{r^4(r + b_0)^{4-\gamma}},
\quad
\frac{1}{r^4(r + b_2)^{4-\gamma}},
\quad
\frac{1}{r^4(r + b_4)^{4-\gamma}},$$

(2.13)

The four ratios $b_1/b_0, ..., b_4/b_0$ can be expressed in terms of the axial ratios of the density distribution at small and at large radii, where the surfaces of constant density are approximately ellipsoidal, i.e. $\rho \sim \rho(m^2)$ with $m^2 = x^2 + y^2/p^2 + z^2/q^2$. When $\gamma > 0$ the density diverges and is proportional to $r^{-\gamma}$ at small $r$, and they find,

$$
b_1 = Ab_0; \quad b_2^{4-\gamma} = (A/C)b_0^{4-\gamma},
\quad
b_3 = Bb_0; \quad b_4^{4-\gamma} = (B/D)b_0^{4-\gamma},$$

(2.12)

where

$$
A = \frac{(3-\gamma)(p_\infty^4 - 2q_\infty^4 + 1)}{4(1 + p_\infty^4 + q_\infty^4)},
\quad
B = \frac{(3-\gamma)(1 - p_\infty^4)}{8(1 + p_\infty^4 + q_\infty^4)},
\quad
C = \frac{(3-\gamma)(p_\infty^4 - 2q_\infty^4 + 1)}{\gamma(5-\gamma)(1 + p_\infty^4 + q_\infty^4)},
\quad
D = \frac{(3-\gamma)(1 - p_\infty^4)}{2\gamma(5-\gamma)(1 + p_\infty^4 + q_\infty^4)},$$

(2.14)
and \((p\omega, q\omega)\) and \((p_\alpha, q_\alpha)\) are the axial ratios of the surfaces of constant density at small and at large radii, respectively. When \(\gamma = 0\), the central density is finite and equal to \(3M\omega \delta/4\pi\), but the central density gradient does not vanish, as it does in model with, e.g., a modified Hubble profile. In this case they find

\[ b_0^2 = (A/C)b_0^2 \quad ; \quad b_0^2 = (B/D)b_0^2, \tag{2.15} \]

with \(A\) and \(B\) as in equation (2.14), but now

\[ C = \frac{2p_\omega - q_\omega - p_\omega q_\omega}{2(p_\omega + q_\omega + p_\omega q_\omega)}; \quad D = \frac{q_\omega(1 - p_\omega)}{4(p_\omega + q_\omega + p_\omega q_\omega)}. \tag{2.16} \]

In all these models the density fall off as \(1/r^4\) at large radii.

Projected surface density

The projected surface density \(\Sigma\), can be calculated in a straight forward way for the density form (2.15), for any viewing direction. Following the convention of de Zeeuw & Franx (1989, hereafter ZF89), they choose a new coordinates \((x', y', z')\) with the \(z'\)-axis along the line-of-sight, and with \(z'\)-axis in the \((x, y)\)-plane. The model is projected onto the \((x', y')\)-plane. Let \(\theta'\) and \(\phi'\) be the standard spherical polar angles of the line-of-sight, i.e., \(z'\)-axis, in the \((x, y, z)\) coordinates. The coordinate transformation is given by (2.4) and (2.5). The inverse transformation is

\[ \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -\sin\phi' & -\cos\phi' & 0 \\ \cos\phi' \cos\theta' & -\sin\phi' \cos\theta' & \sin\theta' \\ \cos\phi' \sin\theta' & \sin\phi' \sin\theta' & \cos\theta' \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.17} \]

As a result, the \(x\)-axis project onto the \(y'\)-axis. They define \((R, \Theta)\) as the polar coordinates in the \((x', y')\)-plane. It follows that \(\Theta\) is the position angle measured on the sky counted counterclockwise from positive \(y'\) (see figure 2.1). The position angle \(\Theta_\alpha\) of the projected \(z\)-axis is therefore equal to zero. The position angles \(\Theta_x\) and \(\Theta_y\) of the projected \(x\)-axis and \(y\)-axis, respectively, are given by

\[ \tan \Theta_x = -\frac{\sin\phi' \cos\theta'}{\cos\phi' \cos\theta'}; \quad \tan \Theta_y = -\frac{\cos\phi' \sin\theta'}{\sin\phi' \cos\theta'}. \tag{2.18} \]

An observer will not know the direction of one of the symmetry axes a priori, and measures position angles with respect to the north. These they denoted by \(\hat{\Theta}\). They introduce an extra angle of projection \(\chi\), which is the position angle \(\Theta_\alpha\) of the projected \(x\)-axis (i.e., the \(y'\)-axis) with respect to north. The two set of angles are related by

\[ \hat{\Theta} = \Theta + \chi. \tag{2.19} \]
Figure 2.1: Coordinate systems used for projection. The \( \theta' \) and \( \phi' \) define the direction of the line of sight, which is taken to be the \( z' \)-axis. (b) Angles in the plane of the sky [i.e., the \((x',y')\)-plane]. This Figure has been taken from ZF89.
2.2. PROJECTION FORMULAS

Figure 2.1 illustrates the two coordinate systems \( (x, y, z) \) and \( (x', y', z') \) as well as convention for the various angles.

Then the projected surface density \( \Sigma(x', y') \) or \( \Sigma(R, \Theta) \) is obtained by integrating the model density (2.10) along the line-of-sight, i.e., \( z' \)-axis,

\[
\Sigma(R, \Theta) = \int_{-\infty}^{\infty} \rho \, dz'.
\]  

(2.20)

After transformation to \( (x', y', z') \) by means of equations (2.4) and (2.5), and integrating over \( z' \), ZC96 have obtained the simple form of projected surface density as

\[
\Sigma(R, \Theta) = \Sigma_\varphi(R) + \Sigma_\theta(R) \cos 2(\Theta - \Theta_\star),
\]  

(2.21)

where the two radial functions are given by

\[
\Sigma_\varphi(R) = 2F_1 + \left(1 - 3\cos^2 \theta'\right) \left[ G_1 - \frac{3}{2}G_2 \right] + \left[6H_1 - 9H_2\right] \sin^2 \theta' \cos 2\varphi',
\]  

(2.22)

\[
\Sigma_\theta(R) = \left[6H_2 \cos \theta' \sin 2\varphi' \right]^2 + \left[3G_2 \sin^2 \theta' - 3H_2 \left(1 + \cos^2 \theta'\right) \cos 2\varphi' \right]^2,
\]  

(2.23)

and they have defined the integrals

\[
G_1(R) = \int_R^\infty \frac{r \, g(r) \, dr}{\sqrt{r^2 - R^2}},
\]

(2.24)

\[
G_2(R) = R^2 \int_R^\infty \frac{g(r) \, dr}{r \sqrt{r^2 - R^2}},
\]

and similarly for \( F_1, H_1 \) and \( H_2 \) in terms of functions \( f(r) \) and \( h(r) \) respectively. The function \( \Sigma_\varphi(R) \) given in equation (2.22) is the azimuthally averaged surface density profile. The expressions above show that only one-dimensional integrals which are independent of the viewing angles \( \theta' \) and \( \phi' \) are required to find the projected surface density of a model with a density of the general form (2.10) (cf. Palmer 1994). For the specific choice (2.11) and general \( \gamma \), the radial integrals (2.24) need to be evaluated numerically.

The projected surface density (2.21) has its major axis at position angle \( \Theta_\star \), given by

\[
\tan 2\Theta_\star = \frac{T \, h_3}{h_1 + \left(1 - T\right) \, h_2},
\]  

(2.25)

where

\[
h_1 = \sin^2 \phi' - \cos^2 \phi' \cos^2 \theta',
\]

\[
h_2 = \cos^2 \phi' - \sin^2 \phi' \cos^2 \theta',
\]

\[
h_3 = \sin 2\phi' \cos \theta',
\]  

(2.26)
which depend only on two viewing angles \((\theta', \phi')\). The quantity \(T \equiv T(R)\) in equation (2.25) is the triaxiality parameter, which is a function of radius, is independent of the viewing angles, and is given by

\[
T \equiv T(R) = \frac{4 H_2(R)}{G_2(R) + 2 H_3(R)} .
\]  

(2.27)

For a given model and viewing angles, equation (2.25) has two solution \(\Theta_*\) and \(\Theta_* - \frac{\pi}{2}\). The former is the position angle of the major axis and the latter of the minor axis when \(\Theta_*\) satisfies the condition

\[
H_2 h_3 \sin 2\Theta_* < 0 .
\]

(2.28)

When \(T\), i.e., \(H_2/G_2\), is independent of radius, then \(\Theta_*\) is constant, so that the projected surface density does not show twisting isophotes.

In the case of specific set of models (2.10), the triaxiality parameters \(T_0\) and \(T_{\infty}\) at small and, respectively, large radii are related to axial ratios of the density at these radii by

\[
T_0 = \frac{1 - p_0^7}{1 - q_0^7} , \quad T_{\infty} = \frac{1 - p_{\infty}^4}{1 - q_{\infty}^4} .
\]

(2.29)

The position angle of the apparent major axis of isophotes is the same at large and small radii if \(T_0 = T_{\infty}\).

The observed axial ratio \(b/a\) should be calculated from

\[
\Sigma \left( b, \Theta_* - \frac{\pi}{2} \right) = \Sigma \left( a, \Theta_* \right) .
\]

(2.30)

It is clear from equations (2.25) and (2.30) that the position angle \(\Theta_*\) and axial ratio \(b/a\) are functions of \(R\). Hence the model exhibits isophote twist and variations in observed axial ratios. Figure 2.2 exhibits such variations. Besides, the isophotes are not elliptical, as shown by plot of \(a_0\).

2.2.2.2 Triaxial models with central core

CT00, studied the projected properties of triaxial modified Hubble mass models, following the scheme of ZC96. This model has a core at the center and density goes as \(1/r^3\). We present the main results of CT00, below.

CT00 considered the density distribution of the form (2.10), with \(f(r), g(r)\) and \(h(r)\) as

\[
f(r) = \frac{M}{4\pi} \frac{1}{(b_0^2 + r^2)^{3/2}}
\]

\[
g(r) = \frac{3M}{4\pi} \frac{b_1^3}{b_0^3} \frac{2r^4 + 7b_3^2r^2}{(b_0^2 + r^2)^{3/2}}
\]

\[
h(r) = \frac{3M}{4\pi} \frac{b_3^3}{b_0^3} \frac{2r^4 + 7b_3^2r^2}{(b_0^2 + r^2)^{3/2}} .
\]

(2.31)
Figure 2.2: The projected surface density distribution $\Sigma$ of a triaxial $\gamma$-model. The left panel shows $\Sigma$ for a model with $\gamma = 1.5$ and $p_0 = 0.9$, $q_0 = 0.7$, $p_{\infty} = 0.88$, $q_{\infty} = 0.72$, seen from the direction $\theta' = 45^\circ$, $\phi' = 30^\circ$. The right panel shows the variation of ellipticity $e$, position angle (PA) of the major axis, and the isophotal shape parameter $a_4$ of the same model, all as a function of radius. The small negative value of $a_4$ shows that the isophotes are slightly boxy. This Figure has been taken from ZC96.

The other symbols are having the same meaning as described for ZC96. The four ratios $b_1$, ..., $b_4$ can be expressed in terms of axis ratios of the density distribution at large and at small radii, where the constant $\rho$ surfaces are approximately ellipsoidal. For the prescribed values of $(p_{\infty}, q_{\infty})$ and $(p_0, q_0)$ CT00 have

$$\left(\frac{b_1}{b_0}\right)^3 = \frac{1 + p_{\infty}^3 - 2q_{\infty}^3}{6(1 + p_{\infty}^3 + q_{\infty}^3)},$$

$$\left(\frac{b_3}{b_0}\right)^3 = \frac{1 - p_{\infty}^3}{12(1 + p_{\infty}^3 + q_{\infty}^3)},$$

$$\left(\frac{b_1}{b_2}\right)^3 \left(\frac{b_0}{b_2}\right)^2 = \frac{p_{\infty}^3(1 - q_{\infty}^2) + p_0^3 - q_0^2}{14(p_0^2 + q_0^2 + p_{\infty}^2q_{\infty}^2)},$$

$$\left(\frac{b_3}{b_4}\right)^3 \left(\frac{b_0}{b_4}\right)^2 = \frac{q_{\infty}^6(1 - p_{\infty}^6)}{28(p_0^2 + q_0^2 + p_{\infty}^2q_{\infty}^2)}. \quad (2.32)$$

The projected surface density is of the form (2.21) where $\Sigma_0(R)$ and $\Sigma_2(R)$ are of form (2.22) and (2.23) respectively. They calculated the integrals $F_1(R)$, $G_2(R)$, $G_2(R)$, $H_1(R)$ and $H_2(R)$ analytically. These integrals are given by

$$F_1(R) = \frac{M}{4\pi b_0^3} \frac{b_0^3}{b_0^3 + R^2}.$$
\[ G_1(R) = \frac{M}{4\pi b_3^2} \frac{2b_1^2}{(b_2^2 + R^2)^3} [2b_4^2 + 9b_2^2R^2 + 3R^4] \]
\[ G_2(R) = \frac{M}{4\pi b_3^2} \frac{4b_1^2}{(b_2^2 + R^2)^3} R^2 [3b_2^2 + R^2] \]

and similarly for \( H_1(R) \) and \( H_2(R) \) in terms of \( b_3 \) and \( b_4 \), in place of \( b_1 \) and \( b_2 \) respectively. The analytical expressions of \( F_1, G_1(R), G_2(R), H_1(R) \) and \( H_2(R) \) can be regarded as an added advantage of using modified Hubble model. Similar integrals have to be calculated numerically, for the \( \gamma \)-models of Dehnen. The axis ratio \( \frac{b_4}{b_3} \) can be calculated by using

\[ \Sigma_0(a) + \Sigma_2(a) = \Sigma_0(b) - \Sigma_2(b) \] (2.34)

The analytical expressions (2.33) allows one to write \( \frac{G_2}{H_2} \) and therefore \( T \) and \( \Theta \), analytically. In particular, at large \( R \), the ratio \( \left( \frac{G_2}{H_2} \right)_\infty \) and the triaxiality parameter \( T_\infty \) are given by

\[ \left( \frac{G_2}{H_2} \right)_\infty = \frac{b_4^3}{b_3^3} \] (2.35)

and

\[ T_\infty = \frac{4 b_3^2}{b_4^2 + 2 b_3^2} = \frac{1 - p_\infty^3}{1 - q_\infty^3} \] (2.36)

Likewise, at small \( R \), the ratio \( \left( \frac{G_2}{H_2} \right)_0 \) and the triaxiality \( T_0 \) are

\[ \left( \frac{G_2}{H_2} \right)_0 = \frac{b_4^3}{b_3^3} \frac{b_3^4}{b_2^4} \] (2.37)

and

\[ T_0 = \frac{4 b_3^2 b_4^2}{b_3^4 b_4^2 + 2 b_3^2 b_2^4} \] (2.38)

Further, they defined few more functions

\[ h_4 = (6 h_3)^2 + 9 (h_1 - h_2)^2 \]
\[ h_5 = \frac{9}{4} (h_1 + h_2)^2 \]
\[ h_6 = 9 (h_1^2 - h_2^2) \]
\[ h_7 = 1 - 3 \cos^2 \psi' \]
\[ h_8 = \sin^2 \psi' \cos 2\phi' \] (2.39)

of the viewing angles \( \psi' \) and \( \phi' \) and defined

\[ Z = \left[ h_4 + h_5 \left( \frac{G_2}{H_2} \right)^2 + h_6 \left( \frac{G_2}{H_2} \right)^2 \right]^\frac{1}{2} \] (2.40)
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Figure 2.3: The projected surface density distribution $\Sigma$ of a triaxial modified Hubble mass model. The left panel shows $\Sigma$ for a model with $p_o = p_{oo} = 0.65$ and $q_o = q_{oo} = 0.60$, seen from the direction $\theta' = 60$, $\phi' = 10$. The right panel shows the profiles of $\Theta_*$, $b/a$ and $\Sigma$ as a function of $R$. $\Theta_*$ are in degrees while $R = \sqrt{a^2 b^2}$ and $\Sigma$ are in the units of $b_o$ and $M/\Omega b_o^2$, respectively. The viewing angles $(\theta', \phi')$ of each frame are mentioned therein. The parameters $(p, q)$ for the frames on first four rows are (0.65, 0.55) for prolate and (0.95, 0.55) for oblate and parameters $(p_0, q_0, p_{oo}, q_{oo})$ for the last row are (0.75, 0.65, 0.65, 0.55) for prolate and (0.95, 0.75, 0.95, 0.55) for oblate. This figure has been taken from Thakur 2002 and CT00.

and

$$A = \frac{2}{b_o} + \frac{12}{b_o} \frac{b_0^3}{b_2^3} h_7 + \frac{72}{b_o} \frac{b_0^3}{b_4^3} h_8,$$  (2.41)

In terms of the above, quantifies, the axial ratios at asymptotic limits are

$$\left(\frac{b}{a}\right)_\infty = \frac{b_0^3 - 2 b_2^3 Z_{oo}}{b_0^3 + 2 b_2^3 Z_{oo}}$$  (2.42)

and

$$\left(\frac{b}{a}\right)_o = \frac{A - \frac{12}{b_4^3} Z_o}{A + \frac{12}{b_4^3} Z_o}$$  (2.43)
where $Z_{\infty}$ and $Z_0$ are the values of $Z$ at very large and at very small radii, respectively.

At large $R$, the surface density $\Sigma$ goes as $1/R^2$ and writing $(\Theta - \Theta_*)$ as $\lambda$, the equation (2.21) can be written as

$$\Sigma(R, \lambda) = \frac{P}{R^2} + \frac{Q}{R^2} \cos^2 \lambda,$$

(2.44)

where

$$P = \frac{2M}{4\pi b_0^3} \left( b_0^2 - 2b_0^2 Z_{\infty} \right),$$

$$Q = \frac{2M}{4\pi b_0^3} 4b_0^2 Z_{\infty}.$$

(2.45)

As $Q << P$, the equation (2.44) can be written as

$$\frac{P}{\Sigma R^2} = 1 - \frac{Q}{P} \cos^2 \lambda + \left( \frac{Q}{P} \right)^2 \cos^4 \lambda + ... ,$$

(2.46)

which, again, can be re-casted in the form

$$\frac{P}{\Sigma R^2} \left( 1 + \frac{Q^2}{8P^2} \right) = 1 - \frac{Q}{P} \left( 1 - \frac{Q}{P} \right) \cos^2 \lambda + \frac{1}{8} \left( \frac{Q}{P} \right)^2 \cos 4\lambda ,$$

(2.47)

retaining terms up to $Q/P)^2$ only. Comparing equation (2.47) with the equation of an ellipse

$$\frac{b^2}{R_{\text{ellip}}^2} = 1 - \varepsilon^2 \cos^2 \theta ,$$

(2.48)

where $b$ is semilatus rectum and $\varepsilon$ is eccentricity, we find that a constant $\Sigma$ - contour is approximately elliptical with

$$\varepsilon^2 = \frac{Q}{P} \left( 1 - \frac{Q}{P} \right), \quad (a)$$

$$b = \frac{P}{\Sigma} \left( 1 + \frac{Q^2}{8P^2} \right). \quad (b)$$

(2.49)

The numerical evaluation of equation (2.49a) agrees closely with values, as obtained by using equation (2.42).

Further, isophotal deviation from true ellipse can be determined by departure between $R$ values of equation (2.47) and $R_{\text{ellip}}$ of equation (2.48) for any given value of $\lambda$. This departure can be represented by

$$\left( \frac{1}{R^2} - \frac{1}{R_{\text{ellip}}^2} \right) = \frac{1}{8b^2} \left( \frac{Q}{P} \right)^2 \cos 4\lambda.$$

(2.50)

The plot of equation (2.47) shows that constant $\Sigma$ contour is slightly boxy with $0.4 \leq q \leq p \leq 1$. Figure 2.4 shows the boxy shape.
Figure 2.4: Constant $\Sigma$ contour for modified Hubble mass model at very large $R$ for fixed value of viewing angles $(\theta', \phi') = (60^\circ, 10^\circ)$, model with $p_0 = p_{\infty} = 0.65$ and $q_0 = q_{\infty} = 0.60$. This figure has been taken from Thakur 2002.

The model of CT00 provides a simple analytical representation of the observed surface brightness distributions of moderately triaxial elliptical galaxies. Here, the projected surface density can be calculated analytically while for the models of ZC96, one has to resort to numerical integrations. The model shows the variation in $\Sigma$, $b/a$ and $\Theta_\phi$ with $R$ for different choices of intrinsic parameters and viewing angles (see Figure 2.3) and the constant $\Sigma$ contours are boxy (Figure 2.4). We have refined this model further by introducing higher order spherical harmonic terms, which will produce more nearly elliptical isophote.

2.3 Fitting of Ellipses

Galaxies are three-dimensional objects and what we see in the sky is the 2D projections of these objects. Thus when a profile is obtained, the 2D distribution is being further averaged to a 1D distribution leading to loss of information. Due to the regularities on a large scale in galaxies it is often enough to consider the 1D profile to obtain a fair idea of the light distribution.

After going through the process of cleaning and combining the raw CCD image, one can
use this for the further analysis to get valuable information. Isophotes of galaxies are well approximated by ellipse. One can thus fit ellipses to the isophotes and obtain a mean intensity as a function of radius. Besides mean intensity, one also obtains the radial profiles of ellipticity, position angle etc., each with errors. All these parameters can be used to obtain some idea regarding the 3D shape of the underlying galaxy (see, e.g. Thakur & Chakrabarty 2001). The set of Fourier components $A_3$, $B_3$, $A_4$, $B_4$ are indicative of the deviations of the shapes of the fitted isophotes from true ellipses. These components are very important to draw the conclusion for dust, disk etc in the galaxies. Below, we are presenting the short discription of ellipse fitting and isophotal shape analysis.

Isophote of a galaxy is the contour of constant intensity. It is a function of semimajor axis. In order to obtain the surface brightness profiles of galaxies isophotal shape analysis technique is employed in which ellipse are fitted to each isophote using Fourier series expansion. Any ellipse can be fully described by five parameters, namely, semi-major axis length, position angle, ellipticity and the coordinates of the center, $x_c$, $y_c$. The aim here is to fit the isophotes a series of ellipses with different semi-major lengths. An ellipse with a given semi-major axis will then have an ellipticity ($e$), a position angle ($PA$) and an $x-$ and $y-$center. The ellipticity is given by $e = 1 - b/a$ where $a$ and $b$ are the major-axis and minor-axis lengths respectively. The position angle ($PA$) is the angle ($0^\circ < PA < 180^\circ$) made by the major-axis of the galaxy with the $+ve$ $x$-axis. The peak central intensity defines its $x-$ and $y-$center. For smooth and featureless elliptical galaxies the ellipses will be concentric with the same ellipticity, $PA$, $x-$ and $y$-center. However, in a real galaxy these parameters will be function of the radius. Image reduction and analysis facilities (IRAF) are used for analysing the clean and combined images. For ellipse fitting and profile generation particular package "STSDAS.analysis.isophote.ellipse" available within IRAF are used. The algorithm for ellipse fitting follows the procedure described by Jedrzejewski (1987; see also Young 1976; Kent 1984). For a given semi-major axis length, an initial guess for the fitting parameters, namely center, ellipticity and position angle of ellipse, are made and the galaxy image is sampled along a trial ellipse, at equal intervals in the eccentric anomaly $E$. The fitting of ellipses to the galaxy images are done in a three step procedure. First, a harmonic expansion along concentric circles is performed. Second, the residuals from this expansion are used to flag additional pixels. Third, the actual ellipse fit is performed, using another harmonics expansion to calculate an initial guess. In general, the sampling ellipse does not match with the true isophote and therefore, the sampled intensity has low order harmonics superimposed on it. Fourier expansion of the sampled intensity as a function of eccentric anomaly are taken in the following form

$$I = I_0 + A_1 \sin(E) + B_1 \cos(E) + A_2 \sin(2E) + B_2 \cos(2E) \quad (2.51)$$
where $I_o$ is the mean intensity along the ellipse and $A_1, B_1, A_2, B_2$ give a set of corrections to the initial parameters of the ellipse. Iterative method is adopted to find a fit which has minimum value of $A_1, B_1, A_2$ and $B_2$. When the best fit ellipse is found, the third and fourth order harmonics for the resultant intensity distribution are measured by a least square fit to

$$I = I_o + A_3 \sin (3E) + B_3 \cos (3E) + A_4 \sin (4E) + B_4 \cos (4E) \quad (2.52)$$

Isophotes are generally not exactly elliptical. Therefore, these higher order harmonics $A_3, A_4, B_3, B_4$ determine the deviations of the isophote from perfect ellipse. For perfectly elliptical isophote, these higher order harmonics will all be zero. When non-zero, the $A_3$ and $B_3$ components indicate that the isophotes deviate from ellipses and that the deviation has a 3-fold symmetry. Similarly, the presence of $A_4$ and $B_4$ denote 4-fold symmetric deviations. $B_4$ component is used as an indicator for disk and boxy structure in ellipticals. When $B_4$ is positive it represents disk structure and for negative $B_4$ it represents a boxy structure in ellipticals.

### 2.4 Orbital structure for the construction self-consistent model

#### 2.4.1 Method

It is difficult to construct self-consistent solution analytically for less symmetric models. Schwarzschild (1979) had given a method to construct self-consistent solution numerically for less symmetric models, using the families of orbits. The method employed the common procedure (Schwarzschild 1979, 1993; Richstone 1980; Levinson & Richstone 1987; Merritt & Fridman 1996) which works as follows (Figure 2.4.1).

- Specify a mass model and find the potential by solving Poisson's equation.

- Calculate a large number of orbits by numerical integration of the equations of motion and compute their individual density distributions on a grid of cells, by determining the time spent in each cell by each orbit.

- Find a combination of orbital densities that reproduces the model density distribution with all occupation numbers non-negative.

The model is divided into cells with the model mass in each cell designated by $D(j)$, where the cell index $j = 1, ..., N$. Then one can calculate the fraction of time spent $B(i, j)$ by each orbit $i$ in a cell $j$, where $i$ and $j$ are respectively orbital and cell index. To reproduce the known
Figure 2.5: Schwarzschild's method for the numerical construction of a distribution function for a galaxy model.
mass of the model in each cell linear combination of orbits are investigated with an appropriate number of stars known as non-negative occupation numbers. Or more precisely, the following set of equations,

$$\sum_{i=1}^{M} C(i) B(i, j) = D(j), \quad (j = 1, \ldots, N)$$  \hspace{1cm} (2.53)

have a physically acceptable solution with non-negative occupational number

$$C(i) \geq 0, \quad (i = 1, \ldots, M)$$  \hspace{1cm} (2.54)

The value of 'D' and 'B' in equation (2.53) are introduced and is solved for 'C'. For solving this one can adopt any one procedure out of the four methods available: (i) by linear programming (Schwarzschild 1979); (ii) by Lucy's (1974) iterations method (Newton & Binney 1984; Statler 1987; Schwarzschild 1993); (iii) by non-negative least squares (Pfenniger 1984), (iv) maximum entropy (Richstone & Tremaine 1988). One can consider any one of these methods to establish the self-consistent solution for triaxial mass models. Schwarzschild (1979) considered the linear programming method to construct the self-consistent solution for triaxial generalisation of modified Hubble mass models.

### 2.4.2 Orbit families

Following Schwarzschild (1993) we assign initial conditions from one of the two sets of starting points either on an equipotential with zero initial velocity (stationary start space), or in the $x-z$ plane with $v_1 = 0, v_2 = 0$. Orbits started in the $x-z$ plane are mostly tube orbits, while the orbits started in the equipotential surfaces are mostly boxes or boxlets, reaching center in the course of time. These two families of orbits include most of the orbits in the full phase space of a triaxial model.

The box orbits have zero time averaged angular momentum while tube orbits have definite value of angular momentum about the $z$-axis or about the $x$-axis.

As the potential is explicitly known for the models of ZC96, it is straightforward to calculate the orbits by solving differential equations of motion. We use the FORTRAN version of routine ‘MSTEP’ of H.M.Antia (1991) for this calculations.

### 2.5 Discussion

In this chapter, we have presented the basic mathematical formulations that have been used in the remaining chapters of this thesis. In the chapter 3 and chapter 4 we have studied the projected properties of $\gamma$-models of Dehnen (1993) and modified Hubble mass models...
respectively with the inclusion of fourth order spherical harmonics terms. For these studies we have used the model and methodology (ZC96 & CT00) described here in detail. In both these chapters we have used the isophotal shape analysis for calculating distortion in isophotes from pure ellipse. In chapter 5 we have studied the projected properties of CT00 model with the inclusion of extra radial functions. In chapter 6 we have calculated the shell parameters, and the $x - z$ and stationary start spaces for the de Zeeuw & Carollo's potential with logarithmic cusp using the method described here. We have studied the orbital structure, so that the self consistent solution, may be attempted.

In the relevant places of this thesis, we have also quoted the equations from this chapter.