Chapter 2

Basic Terminology and Preliminaries
Chapter 2. Basic Terminology and Preliminaries

2.1 Introduction

This chapter is intended to provide all the fundamental terminology and notations which are required for the subsequent chapters.

2.2 Basic definitions

**Definition 2.2.1.** A graph $G = (V(G), E(G))$ consists of two finite sets, the vertex set $V(G)$ which is a non-empty set of elements called vertices, and the edge set $E(G)$ which is a possibly empty set of elements called edges, such that each edge $e$ in $E(G)$ is assigned an unordered pair of vertices $(u, v)$ called the end vertices of $e$.

**Definition 2.2.2.** The number of vertices in graph $G$ is called the *order* of $G$.

**Definition 2.2.3.** The number of edges in graph $G$ is called the *size* of $G$.

**Definition 2.2.4.** Two adjacent vertices are called *neighbours*. The set of all neighbours of vertex $v$ is called the *neighbourhood set* of $v$. It is denoted by $N(v)$ or $N[v]$ and they are respectively known as open and closed neighbourhood sets.

$$N(v) = \{ u \in V(G) / u \text{ adjacent to } v \text{ and } u \neq v \}$$

$$N[v] = N(v) \cup \{ v \}$$

**Definition 2.2.5.** The *degree* of a vertex $v$ in a graph $G$, denoted as $d(v)$ or $d_G(v)$, is the number of edges incident on $v$, counting each loop twice.

**Definition 2.2.6.** A graph $G$ is *$k$-regular graph* if for some positive integer $k$, $d(v) = k$ for every vertex $v$ of the graph $G$.

**Definition 2.2.7.** A *complete graph* is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on $n$ vertices is denoted as $K_n$. 
Definition 2.2.8. A bipartite graph $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $U$ and $W$, such that each edge of $G$ has one endpoint in $U$ and one endpoint in $W$. The pair $U, W$ is called a (vertex) bipartition of $G$, and $U$ and $W$ are called the bipartition subsets.

Definition 2.2.9. A complete bipartite graph is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has $m$ vertices in one of its bipartition subsets and $n$ vertices in the other is denoted by $K_{m,n}$.

Definition 2.2.10. The graph $K_{1,n}$ is called a star. It is the graph with a vertex of degree $n$ called the apex and $n$ pendant vertices.

Definition 2.2.11. The bistar $B_{n,n}$ is a graph obtained by joining the center (apex) vertices of two copies of $K_{1,n}$ by an edge.

Definition 2.2.12. A vertex $u$ is said to be connected to a vertex $v$ in a graph $G$ if there is a path in $G$ from $u$ to $v$ and a graph $G$ is called connected if every two of its vertices are connected otherwise it is called disconnected.

Definition 2.2.13. A graph $G$ is called acyclic if it contains no cycle.

Definition 2.2.14. A graph $G$ is called a tree if it is a connected acyclic graph.

Definition 2.2.15. A caterpillar is a tree in which a single path (the spine) is incident to (or contains) every edge. In other words a caterpillar is a tree with the property that the removal of its endpoints leaves a path.

Definition 2.2.16. A lobster is a tree with the property that the removal its endpoints leaves a caterpillar.

Definition 2.2.17. The graph obtained by joining a single pendant edge to each vertex of a path is called a comb graph denoted by $P_n \odot K_1$.

Definition 2.2.18. The join of two graphs $G_1$ and $G_2$, denoted by $G_1 + G_2$, to be the graph with vertex set and edge set given as follows:
$V(G_1 + G_2) = V(G_1) \cup V(G_2)$,  
$E(G_1 + G_2) = E(G_1) \cup E(G_2) + J$,  
where $J = \{uv : u \in V(G_1), v \in V(G_2)\}$. Thus $J$ consists of edges which join every vertex of $G_1$ to every vertex of $G_2$.

**Definition 2.2.19.** The *fan* $f_n$ is the graph obtained by taking $n - 2$ concurrent chords in cycle $C_{n+1}$. The vertex at which all the chords are concurrent is called the apex vertex. In other words $f_n = P_n + K_1$.

**Definition 2.2.20.** The *double fan* $Df_n$ consists of two fan graph that have a common path. In other words $Df_n = P_n + K_2$.

**Definition 2.2.21.** A *one point union* of regular graph $G$ denoted by $G'$ is the graph obtained by taking $v$ as a common vertex such that any two copy of $G$ are edge disjoint and do not have any vertex in common except $v$.

**Definition 2.2.22.** Let $e = uv$ be an edge of graph $G$ and $w$ is not a vertex of $G$. The edge $e$ is *subdivided* when it is replaced by edges $e' = uw$ and $e'' = wv$.

**Definition 2.2.23.** The *wheel* $W_n$ is defined to be the join $K_1 + C_n$. The vertex corresponding to $K_1$ is known as apex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges. We continue to recognize apex of wheel as the apex of respective graphs introduced in Definitions 2.2.24 to 2.2.27.

**Definition 2.2.24.** The *gear graph* $G_n$ is obtained from the wheel by subdividing each of its rim edge.

**Definition 2.2.25.** The *helm* $H_n$ is the graph obtained from a wheel $W_n = K_1 + C_n$ by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree $n$, $n$ vertices of degree 4 and $n$ pendant vertices.

**Definition 2.2.26.** The *closed helm* $CH_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to form a cycle. It contains three types of vertices: an apex of degree $n$, $n$ vertices of degree 4 and $n$ vertices degree 3.
Definition 2.2.27. The flower $F_{ln}$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree $2n$, $n$ vertices of degree $4$ and $n$ vertices of degree $2$.

Definition 2.2.28. A shell $S_n$ is the graph obtained by taking $n - 3$ concurrent chords in cycle $C_n$. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan $f_{n-1}$.

\[ i.e. \ S_n = f_{n-1} = P_{n-1} + K_1 \]

Definition 2.2.29. The Crown $(C_n \odot K_1)$ is obtained by joining a pendant edge to each vertex of $C_n$.

Definition 2.2.30. A Friendship graph $F_n$ is a one point union of $n$ copies of cycle $C_3$.

Definition 2.2.31. For a simple connected graph $G$ the square of graph $G$ is denoted by $G^2$ and defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$.

Definition 2.2.32. The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $u''$ in $G''$.

Definition 2.2.33. The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it.

Definition 2.2.34. The total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$.

Definition 2.2.35. Let $G = (V(G), E(G))$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \ldots S_t \cup T$ where each $S_i$ is a set of all vertices of the same degree with at least two elements and $T = V \setminus \bigcup_{i=1}^t S_i$. The degree splitting graph of $G$ denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of $S_i$ for $1 \leq i \leq t$. 
We introduce following concept.

**Definition 2.2.36.** Let \( G \) be a connected graph with a unique vertex \( v \) with \( \text{deg}(v) = \Delta(G) \) (called the apex vertex) then new graph \( G \ast G \) is so constructed from two copies of \( G \) with respectively apex vertices \( v' \) and \( v'' \) such that,

(i) \( v' \) and \( v'' \) are adjacent,

(ii) \( N(v') = N(v'') \).

**Definition 2.2.37.** The *arbitrary supersubdivision* of a graph \( G \) produces a new graph by replacing each edge of \( G \) by complete bipartite graph \( K_{2,m_i} \) (where \( m_i \) is any positive integer) in such a way that the ends of each \( e_i \) are merged with two vertices of 2-vertices part of \( K_{2,m_i} \) after removing the edge \( e_i \) from the graph \( G \).

**Definition 2.2.38.** Consider two copies of a graph \( G \) and define a new graph known as *joint sum* is the graph obtained by connecting a vertex of first copy with corresponding vertex of second copy.

**Definition 2.2.39.** \( H_{n,n} \) is the graph with vertex set \( V(H_{n,n}) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\} \) and the edge set \( E(H_{n,n}) = \{v_i u_j : 1 \leq i \leq n, n - i + 1 \leq j \leq n\} \).

**Definition 2.2.40.** A *vertex switching* \( G_v \) of a graph \( G \) is the graph obtained by taking a vertex \( v \) of \( G \), removing all the edges incident to \( v \) and adding edges joining \( v \) to every other vertex which are not adjacent to \( v \) in \( G \).

**Definition 2.2.41.** The *duplication of a vertex* \( v_k \) of graph \( G \) produces a new graph \( G' \) by adding a vertex \( v_k' \) with \( N(v_k) = N(v_k') \). In other words a vertex \( v_k' \) is said to be duplication of \( v_k \) if all the vertices which are adjacent to \( v_k \) are now adjacent to \( v_k' \) also.

**Definition 2.2.42.** For a graph \( G \) the *splitting graph* \( S'(G) \) of a graph \( G \) is obtained by adding a new vertex \( v' \) corresponding to each vertex \( v \) of \( G \) such that \( N(v) = N(v') \).

**Definition 2.2.43.** The *duplication of an edge* \( e = uv \) of graph \( G \) produces a new graph \( G' \) by adding an edge \( e' = u'v' \) such that \( N(u') = N(u) \cup \{v'\} - \{v\} \) and \( N(v') = N(v) \cup \{u'\} - \{u\} \).
**Definition 2.2.44.** Consider two copies of cycle $C_n$. Then the *mutual duplication of a pair of vertices* $v_k$ and $v'_k$ from each of two copies of cycle $C_n$ produces a new graph $G$ such that $N(v_k) = N(v'_k)$.

**Definition 2.2.45.** Consider two copies of cycle $C_n$ and let $e_k = v_kv_{k+1}$ be an edge in the first copy of $C_n$ with $e_{k-1} = v_{k-1}v_k$ and $e_{k+1} = v_{k+1}v_{k+2}$ be its incident edges. Similarly let $e'_k = u_ku_{k+1}$ be an edge in the second copy of $C_n$ with $e'_{k-1} = u_{k-1}u_k$ and $e'_{k+1} = u_{k+1}u_{k+2}$ be its incident edges. The *mutual duplication of a pair of edges* $e_k$ and $e'_k$ from each of two copies of cycle $C_n$ produces a new graph $G$ in such a way that $N(v_k) \cap N(u_k) = \{v_{k-1}, u_{k-1}\}$ and $N(v_{k+1}) \cap N(u_{k+1}) = \{v_{k+2}, u_{k+2}\}$.

**Definition 2.2.46.** The cartesian product of $G$ and $H$, written as $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting $(u, v)$ adjacent to $(u', v')$ if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

**Definition 2.2.47.** The *ladder graph* is obtained by $P_n \times P_2$ while *Circular ladder* is obtained by $C_n \times P_2$.

**Definition 2.2.48.** The *triangular snake* $T_n$ is obtained from path $P_n$ by joining vertices $v_i$ and $v_{i+1}$ by $v'_i$ where $1 \leq i \leq n - 1$.

### 2.3 Concluding Remarks

This chapter provides basic definitions and terminology required for the advancement of the topic. For any undefined term we refer to Harary [42], Clark and Holton [30], Wilson [98], Gross and Yellen [41], Chartrand and Lesniak [29], West [97], Bondy and Murty [21], Balakrishnan and Ranganathan [12].

For standard notations and terminology related to domination in graphs we refer to Haynes *et al.* [44] while the terms related to number theory are used in the sense of Burton [23].
The next chapter is focused on graceful and harmonious labelings as well as their variants.