3-Equitable Labeling for Some Star and Bistar Related Graphs

S.K. Vaidya and N.H. Shah

Abstract—In this paper we prove that the splitting graphs of $K_{1,n}$ and $B_{n,n}$ are 3-equitable graphs. We also show that the shadow graph of $B_{n,n}$ is a 3-equitable graph. Further we prove that square graph of $B_{n,n}$ is 3-equitable for $n \equiv 0 \pmod{3}$ and $n \equiv 1 \pmod{3}$ and not 3-equitable for $n \equiv 2 \pmod{3}$.

Index Terms—3-equitable labeling, star, bistar, splitting graph, shadow graph, square graph.

MSC 2010 Codes - 05C78.

I. INTRODUCTION

In this paper we consider simple, finite, connected and undirected graph $G = (V(G),E(G))$ with order $p$ and size $q$. For all standard terminology and notations we follow West [9]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 : If the vertices of the graph are assigned values subject to certain condition(s) then is known as graph labeling.

A detailed study on applications of graph labeling is reported in Bloom and Golomb [3]. According to Beineke and Hegde [2] graph labeling serves as a frontier between number theory and structure of graphs. For an extensive survey on graph labeling and bibliographic references we refer to Gallian [4].

Definition 1.2 : Let $G = (V(G),E(G))$ be a graph. A mapping $f : V(G) \rightarrow \{0,1,2\}$ is called ternary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0,1,2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ and $v_f(2)$ be the number of vertices of $G$ having labels 0, 1 and 2 respectively under $f$ and let $e_f(0), e_f(1)$ and $e_f(2)$ be the number of edges having labels 0, 1 and 2 respectively under $f^*$.

Definition 1.3 : A ternary vertex labeling of a graph $G$ is called a 3-equitable labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph $G$ is 3-equitable if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit [1] and he proved that an Eulerian graph with number of edges congruent to $3(\text{mod } 6)$ is not 3-equitable. In the same paper he proved that $C_n$ is 3-equitable if and only if $n \not\equiv 3 \pmod{6}$ and all caterpillars are 3-equitable. Cahit [1] claimed to prove that $W_n$ is 3-equitable if and only if $n \not\equiv 3 \pmod{6}$ but Youssef [8] proved that $W_n$ is 3-equitable for all $n \geq 4$. The 3-equitable labeling in the context of vertex duplication is discussed by Vaidya et al. [5] while same authors in [6] have investigated 3-equitable labeling for some shell related graphs. Vaidya et al. [7] have discussed 3-equitability of graphs in the context of some graph operations and proved that the shadow and middle graph of cycle $C_n$, path $P_n$ are 3-equitable.

Generally there are three types of problems that can be considered in this area.

1) How 3-equitability is affected under various graph operations?
2) Construct new families of 3-equitable graph by investigating suitable labeling.
3) Given a graph theoretic property $P$, characterize the class of graphs with property $P$ that are 3-equitable.

The problems of second type are largely discussed while the problems of first and third types are rarely discussed and they are of great importance also. The present work is aimed to discuss the problems of first kind.

Definition 1.4 : The splitting graph of a graph $G$ is obtained by adding to each vertex $v$ a new vertex $v'$ such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$, i.e. $N(v) = N(v')$. The resultant graph is denoted by $S'(G)$.

Definition 1.5: The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $u''$ in $G''$.

Definition 1.6: For a simple connected graph $G$ the square of graph $G$ is denoted by $G^2$ and defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$. 

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II. MAIN RESULTS

Theorem 2.1 : $S'(K_{1,n})$ is 3-equitable graph.

Proof : Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $v$ be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3, \ldots, u_n$ are added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ to obtain $S'(K_{1,n})$. Let $G$ be the graph $S'(K_{1,n})$ then $|V(G)| = 2n + 2$ and $|E(G)| = 3n$. To define $f : V(G) \rightarrow \{0, 1, 2\}$ we consider following three cases.

Case 1: $n \equiv 0 \pmod{3}$

$$f(v) = 2, \quad f(u) = 0,$$

$$f(v_i) = 0; \quad 1 \leq i \leq \frac{n}{3} + 1$$

$$f\left(\frac{n}{3} + i\right) = 1; \quad 1 \leq i \leq \frac{n}{3} - 1$$

$$f\left(\frac{2n}{3} + i\right) = 2; \quad 1 \leq i \leq \frac{n}{3}$$

$$f(u_i) = 0; \quad 1 \leq i \leq \frac{n}{3} - 1$$

$$f\left(\frac{n}{3} - 1 + i\right) = 1; \quad 1 \leq i \leq \frac{n}{3} + 2$$

$$f\left(\frac{2n}{3} - 1 + i\right) = 2; \quad 1 \leq i \leq \frac{n}{3} - 1$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(1) = \frac{2n}{3} + 1 = v_f(2) + 1$$

$$e_f(0) = e_f(1) = e_f(2) = n$$

Case 2: $n \equiv 1 \pmod{3}$

Since $n \equiv 1 \pmod{3}$, $n = 3k + 1$ some $k \in \mathbb{N}$.

$$f(v) = 2,$$

$$f(u) = 0,$$

$$f(v_i) = 0; \quad 1 \leq i \leq k + 1$$

$$f\left(k + 1 + i\right) = 1; \quad 1 \leq i \leq k - 1$$

$$f\left(2k + 1 + i\right) = 2; \quad 1 \leq i \leq k + 1$$

$$f(u_i) = 0; \quad 1 \leq i \leq k - 1$$

$$f\left(k - 1 + i\right) = 1; \quad 1 \leq i \leq k + 3$$

$$f\left(2k + 1 + i\right) = 2; \quad 1 \leq i \leq k - 1$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(2) = \frac{2n + 1}{3} = v_f(1) - 1$$

$$e_f(0) = e_f(1) = e_f(2) = n$$

Case 3: $n \equiv 2 \pmod{3}$

Since $n \equiv 2 \pmod{3}$, $n = 3k + 2$ some $k \in \mathbb{N}$.

$$f(v) = 2,$$

$$f(u) = 0,$$

$$f(v_i) = 0; \quad 1 \leq i \leq k + 1$$

$$f\left(k + 1 + i\right) = 1; \quad 1 \leq i \leq k$$

$$f\left(2k + 1 + i\right) = 2; \quad 1 \leq i \leq k + 1$$

$$f(u_i) = 0; \quad 1 \leq i \leq k$$

$$f\left(k - 1 + i\right) = 1; \quad 1 \leq i \leq k + 2$$

$$f\left(2k + 2 + i\right) = 2; \quad 1 \leq i \leq k$$

In view of the above labeling pattern we have

$$v_f(0) = v_f(2) = v_f(1) = \left\lfloor \frac{2(n + 1)}{3} \right\rfloor$$

$$e_f(0) = e_f(1) = e_f(2) = n$$

Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, for all $0 \leq i, j \leq 2$.

Hence $S'(K_{1,n})$ is a 3-equitable graph.

Illustration 2.2 : 3-equitable labeling of the graph $S'(K_{1,7})$ is shown in Fig. 1.

$$\text{Figure 1}$$

Theorem 2.3 : $S'(B_{n,n})$ is 3-equitable graph.

Proof : Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices. In order to obtain $S'(B_{n,n})$, add $u', v', u'_i, v'_i$ vertices corresponding to $u, v, u_i, v_i$ where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n + 1)$ and $|E(G)| = 6n + 3$. To define $f : V(G) \rightarrow \{0, 1, 2\}$ we consider following four cases.

Case 1: $n = 2, 5$

The graphs $S'(B_{2,2})$ and $S'(B_{5,5})$ are to be dealt separately and their 3-equitable labeling is shown is Fig. 2. and Fig. 3.

$$\text{Figure 2}$$

$$\text{Figure 3}$$
Case 2: \( n \equiv 0(\text{mod} \ 3) \)
Since \( n \equiv 0(\text{mod} \ 3) \), \( n = 3k \) some \( k \in \mathbb{N} \).

\[
\begin{align*}
  f(u) &= 0, \\
  f(u') &= 0, \\
  f(v) &= 1, \\
  f(v') &= 1, \\
  f(u_i) &= 2; \quad 1 \leq i \leq n \\
  f(u'_i) &= 2; \\
  f(u_{1+i}) &= 1; \quad 1 \leq i \leq k \\
  f(u'_{k+1+i}) &= 0; \quad 1 \leq i \leq n - k - 1 \\
  f(v_i) &= 0; \quad 1 \leq i \leq 2(k - 1) \\
  f(v_{2k-2+i}) &= 1; \quad 1 \leq i \leq n - 2k + 2 \\
  f(v'_i) &= 2; \quad 1 \leq i \leq k \\
  f(v'_{k+1+i}) &= 1; \quad 1 \leq i \leq 2k - 2 \\
  f(v'_i) &= 0; \quad i = n, n - 1
\end{align*}
\]

In view of the above labeling pattern we have

\[
\begin{align*}
  v_f(0) &= v_f(2) = 4k + 1 = v_f(1) - 1 \\
  e_f(0) &= e_f(1) = e_f(2) = 2n + 1
\end{align*}
\]

Case 3: \( n \equiv 1(\text{mod} \ 3) \)
Since \( n \equiv 1(\text{mod} \ 3) \), \( n = 3k + 1 \) some \( k \in \mathbb{N} \).

\[
\begin{align*}
  f(u) &= 0, \\
  f(u') &= 0, \\
  f(v) &= 1, \\
  f(v') &= 1, \\
  f(u_i) &= 2; \quad 1 \leq i \leq n \\
  f(u'_i) &= 2; \\
  f(u_{1+i}) &= 1; \quad 1 \leq i \leq k \\
  f(u'_{k+1+i}) &= 0; \quad 1 \leq i \leq 2k \\
  f(v_i) &= 1; \quad 1 \leq i \leq k + 2 \\
  f(v_{2k+i}) &= 0; \quad 1 \leq i \leq 2k - 1 \\
  f(v'_i) &= 2; \quad 1 \leq i \leq k \\
  f(v'_{k+1+i}) &= 1; \quad 1 \leq i \leq 2k - 1 \\
  f(v'_i) &= 0; \quad i = n, n - 1
\end{align*}
\]

In view of the above labeling pattern we have

\[
\begin{align*}
  v_f(0) &= v_f(1) = 4k + 1 = v_f(2) + 1 \\
  e_f(0) &= e_f(1) = e_f(2) = 2n + 1
\end{align*}
\]

Case 4: \( n \equiv 2(\text{mod} \ 3) \)
Since \( n \equiv 2(\text{mod} \ 3) \), \( n = 3k + 2 \) some \( k \in \mathbb{N} - \{1\} \).

\[
\begin{align*}
  f(u) &= 0, \\
  f(u') &= 0, \\
  f(v) &= 1, \\
  f(v') &= 1, \\
  f(u_i) &= 2; \quad 1 \leq i \leq n \\
  f(u'_i) &= 2; \\
  f(u_{1+i}) &= 1; \quad 1 \leq i \leq k + 1 \\
  f(u'_{k+2+i}) &= 0; \quad 1 \leq i \leq 2k \\
  f(v_i) &= 1; \quad 1 \leq i \leq k + 4 \\
  f(v_{3k+i}) &= 0; \quad 1 \leq i \leq 2k - 2 \\
  f(v'_i) &= 2; \quad 1 \leq i \leq k + 1 \\
  f(v'_{k+1+i}) &= 1; \quad 1 \leq i \leq 2k - 3 \\
  f(v'_i) &= 0; \quad n - 3 \leq i \leq n
\end{align*}
\]

In view of the above labeling pattern we have

\[
\begin{align*}
  v_f(0) &= v_f(1) = 4k + 1 = v_f(2) \\
  e_f(0) &= e_f(1) = e_f(2) = 2n + 1
\end{align*}
\]

Thus in each case we have \(|v_f(i) - v_f(j)| \leq 1\) and \(|e_f(i) - e_f(j)| \leq 1\), for all \( 0 \leq i, j \leq 2 \).

Hence \( S'(B_{n,n}) \) is a 3-equitable graph.

Illustration 2.4 : 3-equitable labeling of the graph \( S'(B_{6,6}) \) is shown in Fig. 4.

![Figure 4](image-url)
Illustration 2.6: 3-equitable labeling of the graph $D_2(B_{5,5})$ is shown in Fig. 5.

Theorem 2.7: $B^2_{n,n}$ is 3-equitable graph for $n \equiv 0 \pmod{3}$ and $n \equiv 1 \pmod{3}$. 

Proof: Consider $B_{n,n}$ with vertex set \{u, v, u_i, v_i, 1 \leq i \leq n\} where $u_i,v_i$ are pendant vertices. Let $G$ be the graph $B^2_{n,n}$ then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. To define $f : V(G) \to \{0, 1, 2\}$, we consider following two cases.

Case 1: \(n \equiv 0 \pmod{3}\)

Since $n \equiv 0 \pmod{3}$, $n = 3k$ some $k \in \mathbb{N}$.

Case 2: \(n \equiv 1 \pmod{3}\)

Since $n \equiv 1 \pmod{3}$, $n = 3k + 1$ some $k \in \mathbb{N}$.

Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, for all $0 \leq i, j \leq 2$.

Hence $D_2(B_{n,n})$ is a 3-equitable graph.

Illustration 2.8: 3-equitable labeling of the graph $B^2_{7,7}$ is shown in Fig. 6.
Theorem 2.9: \( B_{2n,n}^2 \) is not a 3-equitable graph for \( n \equiv 2 \pmod{3} \).

Proof: Let \( G \) be the graph \( B_{2n,n}^2 \) then \( |V(G)| = 2n + 2 \) and \( |E(G)| = 4n + 1 \). Here \( n \equiv 2 \pmod{3} \) therefore \( n = 3k_1 + 2 \) for some \( k_1 \in \mathbb{N} \). Hence \( |V(G)| = 3k \) and \( |E(G)| = 6k - 3 \) where \( k = 2k_1 + 2 \). So if \( B_{2n,n}^2 \) is 3-equitable then we must have \( v_f(0) = v_f(1) = v_f(2) = k \) and \( e_f(0) = e_f(1) = e_f(2) = 2k - 1 \).

In \( B_{2n,n}^2 \), note that each \( u_i \) and \( v_i \) (\( 1 \leq i \leq n \)) are adjacent to \( u \) and \( v \) both moreover \( u \) and \( v \) are adjacent vertices. It is obvious that any edge will have label 1 if it is incident to one vertex with label 1. Following Table 1 shows all possible assignments of vertex label. From the Table 1 (column 6) we can observe that the edge condition violates in all the possible assignments. Hence \( B_{2n,n}^2 \) is not a 3-equitable graph for \( n \equiv 2 \pmod{3} \).

III. Concluding Remarks

The graphs \( K_{1,n} \) and \( B_{n,n} \) are 3-equitable being caterpillars while we show that the splitting graphs of \( K_{1,n} \) and \( B_{n,n} \) also admit 3-equitable labeling. Thus 3-equitability remains invariant for the splitting graphs of \( K_{1,n} \) and \( B_{n,n} \). It is also invariant for shadow graph of \( B_{n,n} \). Moreover we prove that the square graph of \( B_{n,n} \) is 3-equitable for \( n \equiv 0 \pmod{3} \) and \( n \equiv 1 \pmod{3} \) while not 3-equitable for \( n \equiv 2 \pmod{3} \). To investigate similar results for other graph families and for various graph operations is a potential area of research.

REFERENCES

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Where $v_f'(j) = \text{number of vertices having label } j \text{ for } u_i \text{ and } v_i \text{ where } 1 \leq i \leq n \text{ and } 0 \leq j \leq 2$. 
Graceful and odd graceful labeling of some graphs

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Abstract

In this paper, we prove that the square graph of bistar $B_{n,n}$, the splitting graph of $B_{n,n}$ and the splitting graph of star $K_{1,n}$ are graceful graphs. We also prove that the splitting graph and the shadow graph of bistar $B_{n,n}$ admit odd graceful labeling.

Keywords: Graceful labeling, odd graceful labeling, shadow graph, splitting graph, square graph.

AMS Subject Classification(2010): 05C78.

1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $|V(G)| = p$ and $|E(G)| = q$. For standard terminology and notation we follow Gross and Yellen [5]. We provide a brief summary of definitions and other information which serve as prerequisites for the present investigation.

Definition 1.1. If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.

Graph labelings is an active area of research in graph theory which has rigorous applications in coding theory, communication networks, optimal circuits layouts and graph decomposition problems. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and the structure of graphs. For a dynamic survey of various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

Definition 1.2. A function $f$ is called graceful labeling of a graph $G$ if $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which admits graceful labeling is called a graceful graph.

Rosa [9] initially called this labeling a $\beta$ - valuation and later Golomb [4] named it as graceful labeling which is now the popular term. Several infinite families of graceful and non-graceful graphs have been studied. The famous Ringel-Kotzig tree conjecture [8] and many illustrious work on graceful
graphs brought a tide of labeling scheme having graceful theme. Vaidya et al. [11] discussed gracefulness of union of two path graphs with grid graph and complete bipartite graph. Kaneria et al. [6] discussed gracefulness of some classes of disconnected graphs. Vaidya and Lekha [12] investigated graceful labeling of some cycle related graphs. Some variants of graceful labeling were also introduced in recent past, such as edge graceful labeling, Fibonacci graceful labeling, odd graceful labeling and the like.

**Definition 1.3.** A function \( f \) is called odd graceful labeling of a graph \( G \) if \( f : V(G) \to \{0, 1, 2, \ldots, 2q - 1\} \) is injective and the induced function \( f^*: E(G) \to \{1, 3, \ldots, 2q - 1\} \) defined as \( f^*(e = uv) = |f(u) - f(v)| \) is bijective. The graph which admits odd graceful labeling is called an odd graceful graph.

The concept of odd graceful labeling was introduced by Gnanajothi [3]. It is possible to decompose the graph \( K_{n,n} \) with suitable odd graceful labeling of tree \( T \) of order \( n \). The splitting graphs of path \( P_n \) and even cycle \( C_n \) are proved to be odd graceful by Sekar [10] while ladders and graphs obtained from them by subdividing each step exactly once are shown odd-graceful by Kathiresan [7]. Vaidya and Lekha [13–15] proved many results on odd graceful labeling.

The following three types of problems are considered generally in the area of graph labeling.
1. How particular labeling is affected under various graph operations;
2. Investigation of new families of graphs which admit particular graph labeling;
3. Given a graph theoretic property \( P \), characterizing the class/classes of graphs with property \( P \) that admit particular graph labeling.

From the literature survey, it is clear that the problems of second type are largely studied than the problems of first and third types. The present work is aimed to discuss some problems of the first kind in the context of graceful and odd graceful labeling.

**Definition 1.4.** For a graph \( G \) the splitting graph \( S' \) of \( G \) is obtained by adding a new vertex \( v' \) corresponding to each vertex \( v \) of \( G \) such that \( N(v) = N(v') \).

**Definition 1.5.** The shadow graph \( D_2(G) \) of a connected graph \( G \) is constructed by taking two copies of \( G \) say \( G' \) and \( G'' \). Join each vertex \( u' \) in \( G' \) to the neighbours of the corresponding vertex \( v' \) in \( G'' \).

**Definition 1.6.** For a simple connected graph \( G \) the square of graph \( G \) is denoted by \( G^2 \) and defined as the graph with the same vertex set as of \( G \) and two vertices are adjacent in \( G^2 \) if they are at a distance 1 or 2 apart in \( G \).

### 2 Results on graceful labeling

**Theorem 2.1.** \( B_{n,n}^2 \) is a graceful graph.

**Proof.** Consider \( B_{n,n} \) with the vertex set \( \{u, v, u_i, v_i, 1 \leq i \leq n\} \) where \( u_i, v_i \) are the pendant vertices. Let \( G \) be the graph \( B_{n,n}^2 \) then \( |V(G)| = 2n + 2 \) and \( |E(G)| = 4n + 1 \). We define the vertex labeling \( f : V(G) \to \{0, 1, 2, \ldots, 4n + 1\} \) as follows.


\[ f(v) = 0, \]
\[ f(u) = 4n + 1, \]
\[ f(v_i) = i; \quad 1 \leq i \leq n \]
\[ f(u_i) = f(v_n) + i; \quad 1 \leq i \leq n. \]

The vertex function \( f \) defined above induces a bijective edge function \( f^*: E(G) \to \{1, 2, 3, \ldots, 4n + 1\} \). Thus \( f \) is graceful labeling of \( G = B_{2,n,n}^2 \). Hence, \( B_{2,n,n}^2 \) is a graceful graph.

**Illustration 2.2.** Graceful labeling of the graph \( B_{7,7}^2 \) is shown in Figure 1.

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**Theorem 2.3.** \( S'(B_{n,n}) \) is a graceful graph.

**Proof.** Consider \( B_{n,n} \) with the vertex set \( \{u, v, u_i, v_i, 1 \leq i \leq n\} \) where \( u_i, v_i \) are the pendant vertices. In order to obtain \( S'(B_{n,n}) \), add \( u', v', u_i', v_i' \) vertices corresponding to \( u, v, u_i, v_i \) where, \( 1 \leq i \leq n \). If \( G = S'(B_{n,n}) \) then \( |V(G)| = 4(n + 1) \) and \( |E(G)| = 3(2n + 1) \). To define the vertex labeling \( f : V(G) \to \{0, 1, 2, 3, \ldots, 6n + 3\} \), we consider the following two cases.

**Case 1:** \( n \) is even.

\[ \begin{align*}
f(u') &= 2n - 1, \\
f(v) &= 0, \\
f(u_i') &= 2n, \\
f(v_i') &= 1, \\
f(u_1') &= 4n, \\
f(v_1') &= 6n + 3 - i; \\
f(v_1) &= f(v_n') - 1, \\
f(v_{1+i}) &= f(v_{1+i}) - 2i; \quad 1 \leq i \leq n - 1
\end{align*} \]

**Case 2:** \( n \) is odd.

**Figure 1:** Graceful labeling of \( B_{7,7}^2 \).
The vertex function $f$ defined above induces a bijective edge function $f^* : E(G) \to \{1, 2, 3, \ldots, 6n + 3\}$. Thus, $f$ is a graceful labeling of $G = S'(B_{n,n})$. Hence, $S'(B_{n,n})$ is a graceful graph.

Illustration 2.4. Graceful labeling of the graph $S'(B_{5,5})$ is shown in Figure 2.

![Figure 2: Graceful labeling of $S'(B_{5,5})$.](image)

Illustration 2.5. Graceful labeling of the graph $S'(B_{6,6})$ is shown in Figure 3.

![Figure 3: Graceful labeling of $S'(B_{6,6})$.](image)

Theorem 2.6. $S'(K_{1,n})$ is a graceful graph.

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $v$ be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3,
... \ldots, u_n \) be added vertices corresponding to \( v, v_1, v_2, v_3, \ldots, v_n \) to obtain \( S'(K_{1,n}) \). Let \( G \) be the graph \( S'(K_{1,n}) \) then \(|V(G)| = 2n + 2\) and \(|E(G)| = 3n\). We define vertex labeling \( f : V(G) \to \{0, 1, 2, \ldots, 3n\} \) as follows.

\[
\begin{align*}
  f(u) &= 1, \\
  f(v) &= 0, \\
  f(v_i) &= 2i; \quad 1 \leq i \leq n \\
  f(u_i) &= f(v_n) + i; \quad 1 \leq i \leq n
\end{align*}
\]

The vertex function \( f \) defined above induces a bijective edge function \( f^* : E(G) \to \{1, 2, 3, \ldots, 3n\} \).

Thus, \( f \) is a graceful labeling of \( G = S'(K_{1,n}) \). Hence, \( S'(K_{1,n}) \) is a graceful graph.

**Illustration 2.7.** Graceful labeling of the graph \( S'(K_{1,7}) \) is shown in Figure 4.

![Graceful labeling of the graph S'(K_{1,7})](image)

**Figure 4:** Graceful labeling of the graph \( S'(K_{1,7}) \).

### 3 Results on Odd Graceful labeling

**Theorem 3.1.** \( S'(B_{n,n}) \) is an odd graceful graph.

**Proof.** Consider \( B_{n,n} \) with the vertex set \( \{u, v, u_i, v_i, 1 \leq i \leq n\} \) where \( u_i, v_i \) are the pendant vertices. In order to obtain \( S'(B_{n,n}) \), add \( u', v', u_i', v_i' \) vertices corresponding to \( u, v, u_i, v_i \) where, \( 1 \leq i \leq n \).

If \( G = S'(B_{n,n}) \) then \(|V(G)| = 4(n + 1)\) and \(|E(G)| = 3(2n + 1)\). We define vertex labeling \( f : V(G) \to \{0, 1, 2, 3, \ldots, 12n + 5\} \) as follows.

\[
\begin{align*}
  f(u) &= 0, \\
  f(v) &= 3, \\
  f(u') &= 2, \\
  f(v') &= 5, \\
  f(u_i) &= 5 + 4i; \quad 1 \leq i \leq n \\
  f(u_i') &= 12n + 7 - 2i; \quad 1 \leq i \leq n \\
  f(v_i') &= f(u_n') + 1, \\
  f(v_i') &= f(v_n') - 2i; \quad 1 \leq i \leq n - 1 \\
  f(v_1) &= f(v_1') - 2, \\
  f(v_{1+i}) &= f(v_1) - 4i; \quad 1 \leq i \leq n - 1
\end{align*}
\]
The vertex function $f$ defined above induces a bijective edge function $f^* : E(G) \to \{1, 3, \ldots, 12n + 5\}$. Thus, $f$ is an odd graceful labeling of $G = S'(B_{n,n})$. Hence, $S'(B_{n,n})$ is an odd graceful graph.

**Illustration 3.2.** Odd graceful labeling of the graph $S'(B_{6,6})$ is shown in Figure 5.

![Figure 5: Odd graceful labeling of $S'(B_{6,6})$.](image)

**Theorem 3.3.** $D_2(B_{n,n})$ is an odd graceful graph.

**Proof.** Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i : 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Let $G$ be the graph $D_2(B_{n,n})$. Then $|V(G)| = 4(n + 1)$ and $|E(G)| = 4(2n + 1)$. We define vertex labeling $f : V(G) \to \{0, 1, 2, 3, \ldots, 16n + 7\}$ as follows.

- $f(u) = 2$,
- $f(v) = 7$,
- $f(u') = 0$,
- $f(v') = 3$,
- $f(u_i) = 16n + 11 - 4i$; $1 \leq i \leq n$
- $f(u'_i) = f(u_n) - 4i$; $1 \leq i \leq n$
- $f(v_{1+2i}) = 16 + 8i$; $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$
- $f(v_{2+2i}) = 18 + 8i$; $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$
- $f(v'_{1+2i}) = f(v_{n-1}) + 8(i + 1)$; $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$
- $f(v'_{2+2i}) = f(v_n) + 8(i + 1)$; $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$

The vertex function $f$ defined above induces a bijective edge function $f^* : E(G) \to \{1, 3, \ldots, 16n + 7\}$. Thus, $f$ is an odd graceful labeling for $G = D_2(B_{n,n})$. Hence, $D_2(B_{n,n})$ is an odd graceful graph.
Illustration 3.4. Odd graceful labeling of the graph \( D_2(B_{n,n}) \) is shown in Figure 6.

Figure 6: Odd graceful labeling of the graph \( D_2(B_{n,n}) \).

4 Concluding Remarks

Vaidya and Lekha [13] have proved that splitting graph of star \( K_{1,n} \) is an odd graceful graph while we prove that it is also a graceful graph. We proved that splitting graph of bistar \( B_{n,n} \) is both graceful and odd graceful. Moreover \( B^2_{n,n} \) is a graceful graph but not an odd graceful graph as it contains odd cycles.

Rosa [9] proved that \( K_{1,n} \) and \( B_{n,n} \) are graceful graphs but we prove that the splitting graphs of star \( K_{1,n} \) and bistar \( B_{n,n} \) are also graceful graphs. Thus, gracefulness remains invariant for the splitting graphs of \( K_{1,n} \) and \( B_{n,n} \). It is also invariant for square graph of \( B_{n,n} \). Moreover odd gracefulness is invariant for the splitting graph and the shadow graph of \( B_{n,n} \).

References


Some Star and Bistar Related Divisor Cordial Graphs

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Abstract. A divisor cordial labeling of a graph \( G \) with vertex set \( V \) is a bijection \( f \) from \( V \) to \{1, 2, ..., |V|\} such that an edge \( uv \) is assigned the label 1 if either \( f(u) \mid f(v) \) or \( f(v) \mid f(u) \) and the label 0 if \( f(u) \nmid f(v) \), then number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a divisor cordial labeling is called a divisor cordial graph. In this paper we prove that splitting graphs of star \( K_{1,n} \) and bistar \( B_{n,n} \) are divisor cordial graphs. Moreover we show that degree splitting graph of \( B_{n,n} \), shadow graph of \( B_{n,n} \) and square graph of \( B_{n,n} \) admit divisor cordial labeling.

Keywords: Divisor cordial labeling, star, bistar.

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

We begin with simple, finite, connected and undirected graph \( G = (V(G), E(G)) \) with \( p \) vertices and \( q \) edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen [1] while for number theory we refer to Burton [2]. We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition 1.1. If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.

Any graph labeling will have the following three common characteristics:
1. A set of numbers from which vertex labels are chosen;
2. A rule that assigns a value to each edge;
3. A condition that this value has to satisfy.

According to Beineke and Hegde [3] graph labeling serves as a frontier between number theory and structure of graphs. Graph labelings have many applications within mathematics as well as to several areas of computer science and communication networks. According to Graham and Sloane [4] the harmonious labelings are closely related to problems in error correcting codes while odd harmonious labeling is useful to solve undetermined equations as described by Liang and Bai [5]. The optimal linear arrangement concern to wiring network problems in electrical engineering and placement problems in production engineering can be formalised as a graph labeling problem as stated by Yegnanaryanan and Vaidhyanathan [6]. For a dynamic survey of various graph labeling problems along with an extensive bibliography we refer to Gallian [7].

**Definition 1.2.** A mapping $f : V(G) \to \{0,1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$.

**Notation 1.3.** If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Then

$v_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f$

$e_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f^*$

**Definition 1.4.** A binary vertex labeling $f$ of a graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [8]. This concept is explored by many researchers like Andar et al. [9,10], Vaidya and Dani [11] and Nasreen [12]. Motivated through the concept of cordial labeling Babujee and Shobana [13] introduced the concepts of cordial languages and cordial numbers. Some labeling schemes are also introduced with minor variations in cordial theme. Product cordial labeling, total product cordial labeling and prime cordial labeling are among mention a few. The present work is focused on divisor cordial labeling.

**Definition 1.5.** A prime cordial labeling of a graph $G$ with vertex set $V(G)$ is a bijection $f : V(G) \to \{1,2,3,\ldots,|V(G)|\}$ and the induced function $f^* : E(G) \to \{0,1\}$ is defined by $f^*(e = uv) = 1$, if $\gcd(f(u), f(v)) = 1$;

$= 0$, otherwise.

which satisfies the condition $|e_f(0) - e_f(1)| \leq 1$. A graph which admits prime cordial labeling is called a prime cordial graph.

The concept of prime cordial labeling was introduced by Sundaram et al. [14] and in the same paper they have investigated several results on prime cordial labeling. Vaidya and Vihol [15,16] as well as Vaidya and Shah [17,18] have proved many results on prime cordial labeling.
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Motivated by the concept of prime cordial labeling, Varatharajan et al. [19] introduced a new concept called divisor cordial labeling by combining the divisibility concept in Number theory and Cordial labeling concept in Graph labeling. This is defined as follows.

**Definition 1.6.** Let $G=(V(G), E(G))$ be a simple graph and $f : V(G) \to \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge $uv$, assign the label 1 if either $f(u) | f(v)$ or $f(v) | f(u)$ and the label 0 if $f(u) \not| f(v)$. $f$ is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

In the same paper [19] they have proved that path, cycle, wheel, star, $K_{2,n}$ and $K_{3,n}$ are divisor cordial graphs while $K_n$ is not divisor cordial for $n \geq 7$. Same authors in [20] have discussed divisor cordial labeling of full binary tree as well as some star related graphs.

It is important to note that prime cordial labeling and divisor cordial labeling are two independent concepts. A graph may possess one or both of these properties or neither as exhibited below.

i) $P_n$ ($n \geq 6$) is both prime cordial as proved in [14] and divisor cordial as proved in [19].

ii) $C_3$ is not prime cordial as proved in [14], but it is divisor cordial as proved in [19].

iii) We found that a 7-regular graph with 12 vertices admits prime cordial labeling, but does not admit divisor cordial labeling.

iv) Complete graph $K_7$ is not a prime cordial as stated in Gallian [7] and not divisor cordial as proved in [19].

Generally there are three types of problems that can be considered in the area of graph labeling.

1. How a particular labeling is affected under various graph operations;
2. To investigate new graph families which admit particular graph labeling;
3. Given a graph theoretic property $P$, characterize the class/classes of graphs with property $P$ that admit particular graph labeling.

The problems of second type are largely discussed while the problems of first and third types are not so often but they are of great importance. The present work is aimed to discuss the problems of first kind in the context of divisor cordial labeling.

**Definition 1.7.** For a graph $G$ the splitting graph $S'(G)$ of a graph $G$ is obtained by adding a new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$.

**Definition 1.8.** [21] Let $G=(V(G), E(G))$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \ldots S_t \cup T$ where each $S_i$ is a set of vertices having at least two vertices of the same degree and $T = V \setminus \bigcup S_i$. The degree splitting graph of $G$ denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of $S_i$ for $1 \leq i \leq t$.
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**Definition 1.9.** The *shadow graph* $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $v'$ in $G''$.

**Definition 1.10.** For a simple connected graph $G$ the *square of graph* $G$ is denoted by $G^2$ and defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$.

2. Main Results

**Theorem 2.1.** $S'(K_{1,n})$ is a divisor cordial graph.

**Proof:** Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $v$ be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3, \ldots, u_n$ are added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ to obtain $S'(K_{1,n})$. Let $G$ be the graph $S'(K_{1,n})$ then $|V(G)| = 2n + 2$ and $|E(G)| = 3n$. To define $f : V(G) \to \{1, 2, \ldots, 2n + 2\}$ we consider following three cases.

**Case 1:** $n=2$ to $8$

For $n=2$, $f(v)=4, f(u)=1, f(v_1)=3, f(v_2)=2$ and, $f(u_1)=5, f(u_2)=6$. Then $e_j(0) = 3 = e_j(1)$.

For $n=3$, $f(v)=3, f(u)=2, f(v_1)=5, f(v_2)=6, f(v_3)=7$ and $f(u_1)=1, f(u_2)=9, f(u_3)=4$. Then $e_j(0) = 5, e_j(1) = 4$.

For $n=4$, $f(v)=3, f(u)=2, f(v_1)=5, f(v_2)=6, f(v_3)=8, f(v_4)=10$ and $f(u_1)=1, f(u_2)=4, f(u_3)=7, f(u_4)=9$. Then $e_j(0) = 6 = e_j(1)$.

For $n=5$, $f(v)=3, f(u)=2, f(v_1)=5, f(v_2)=6, f(v_3)=8, f(v_4)=10, f(v_5)=12$ and $f(u_1)=1, f(u_2)=4, f(u_3)=7, f(u_4)=9, f(u_5)=11$. Then $e_j(0) = 7, e_j(1) = 8$.

For $n=6$, $f(v)=3, f(u)=2, f(v_1)=5, f(v_2)=6, f(v_3)=8, f(v_4)=10, f(v_5)=12, f(v_6)=14$ and $f(u_1)=1, f(u_2)=4, f(u_3)=7, f(u_4)=9, f(u_5)=11, f(u_6)=13$. Then $e_j(0) = 9 = e_j(1)$.

For $n=7$, $f(v)=3, f(u)=2, f(v_1)=5, f(v_2)=6, f(v_3)=8, f(v_4)=10, f(v_5)=12, f(v_6)=14, f(v_7)=16$ and $f(u_1)=1, f(u_2)=4, f(u_3)=7, f(u_4)=9, f(u_5)=11, f(u_6)=13, f(u_7)=15$. Then $e_j(0) = 10, e_j(1) = 11$.

For $n=8$, $f(v)=3, f(u)=2, f(v_1)=1, f(v_2)=6, f(v_3)=5, f(v_4)=7, f(v_5)=11, f(v_6)=13, f(v_7)=17$ and $f(u_1)=4, f(u_2)=8, f(u_3)=10, f(u_4)=14, f(u_5)=16, f(u_6)=18, f(u_7)=9, f(u_8)=15$. Then $e_j(0) = 12 = e_j(1)$.

Now for the remaining two cases let,

\[
\begin{align*}
    s &= \left\lfloor \frac{n+1}{3} \right\rfloor, \quad m = \left\lfloor \frac{2n+2}{3} \right\rfloor - 1 - s, \quad t = \left\lfloor \frac{3n}{2} \right\rfloor - (n + s + 2), \\
    x_1 &= s + t + 1, \quad x_2 = n - x_1, \quad x_3 = n - s + m - t, \quad x_4 = n - x_3.
\end{align*}
\]

**Case 2:** $n=9, 10, 11, 12$ ($t=0$)

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\[ f(v) = 2, \quad f(u) = 3, \]
\[ f(v_1) = 1, \]
\[ f(v_{1+i}) = 6i; \quad 1 \leq i \leq s \]
\[ f\left( v_{\frac{n}{2}+i+1} \right) = 5 + 6i; \quad 0 \leq i < \left\lfloor \frac{x_2}{2} \right\rfloor \]
\[ f\left( v_{\frac{n}{2}+i+2} \right) = 7 + 6i; \quad 0 \leq i < \left\lfloor \frac{x_2}{2} \right\rfloor \]
\[ f(u_{1+2i}) = 4 + 6i; \quad 0 \leq i < \left\lfloor \frac{n-s}{2} \right\rfloor \]
\[ f(u_{2+2i}) = 8 + 6i; \quad 0 \leq i < \left\lfloor \frac{n-s}{2} \right\rfloor \]
\[ f(u_{n-s-i}) = 9 + 6(i-1); \quad 1 \leq i \leq m \]

For the vertices \( f(u_{s+1}), f(u_{s+2}), \ldots, f(u_n) \) assign distinct remaining odd numbers. This assigns all the vertex labels for case 2.

**Case 3:** \( n \geq 13 \quad (t \geq 1) \)

\[ f(v) = 2, \quad f(u) = 3, \]
\[ f(v_1) = 1, \]
\[ f(v_{1+i}) = 6i; \quad 1 \leq i \leq s \]
\[ f\left( v_{\frac{n}{2}+i+1} \right) = 9 + 6(i-1); \quad 1 \leq i \leq t \]
\[ f\left( v_{\frac{n}{2}+i+2} \right) = 5 + 6i; \quad 0 \leq i < \left\lfloor \frac{x_2}{2} \right\rfloor \]
\[ f\left( v_{\frac{n}{2}+i+2} \right) = 7 + 6i; \quad 0 \leq i < \left\lfloor \frac{x_2}{2} \right\rfloor \]
\[ f(u_{1+2i}) = 4 + 6i; \quad 0 \leq i < \left\lfloor \frac{n-s}{2} \right\rfloor \]
\[ f(u_{2+2i}) = 8 + 6i; \quad 0 \leq i < \left\lfloor \frac{n-s}{2} \right\rfloor \]
\[ f(u_{n-s+i}) = f(v_{s+t+1}) + 6i; \quad 1 \leq i \leq m-t \]

For the vertices \( f(u_{s+1}), f(u_{s+2}), \ldots, f(u_n) \) assign distinct remaining odd numbers. This assigns all the vertex labels for case 3.

In view of the above labeling pattern, \( f(v) \mid f(u_1), \ldots, f(v) \mid f(u_{n-s}) \) and \( f(v_1) \mid f(v), f(v_1) \mid f(u) \). Moreover

\[ f(v) \mid f(v_2), f(v) \mid f(v_3), \ldots, f(v) \mid f(v_{s+1}), \quad f(u) \mid f(v_2), f(u) \mid f(v_3), \ldots, \quad f(u) \mid f(v_{s+1}) \]
and \( f(u) \mid f(v_{s+2}), f(u) \mid f(v_{s+3}), \ldots, f(u) \mid f(v_{s+t}) \).
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Hence, \( e_j(1) = n - s + 2 + s + s + t = n + s + 2 + \left\lceil \frac{3n}{2} \right\rceil - (n + S + 2) \).

Therefore, in last two case \( e_j(1) = \left\lceil \frac{3n}{2} \right\rceil \) and \( e_j(0) = \left\lfloor \frac{3n}{2} \right\rfloor \).

Thus, in all the cases we have \(|e_j(0) - e_j(1)| \leq 1\).

Hence, \( S'(K_{1,n}) \) is a divisor cordial graph.

**Example 2.2.** Let \( G = S'(K_{1,13}) \), \(|V(G)| = 28\) and \(|E(G)| = 39\). In accordance with Theorem 2.1 we have \( s = 4, m = 4, t = 1, x_1 = 6, x_2 = 7, x_3 = 12, x_4 = 1 \) and using the labeling pattern described in case 2. The corresponding divisor cordial labeling is shown in Figure 1. It is easy to visualize that \( e_j(0) = 19 \) and \( e_j(1) = 20 \).

![Figure 1: Divisor cordial labeling of \( G = S'(K_{1,13}) \)](image)

**Theorem 2.3.** \( S'(B_{n,n}) \) is a divisor cordial graph.

**Proof:** Consider \( B_{n,n} \) with vertex set \( \{u, v, u_i, v_i, 1 \leq i \leq n\} \) where \( u_i, v_i \) are pendant vertices. In order to obtain \( S'(B_{n,n}) \), add \( u_i', v_i', u_i', v_i' \) vertices corresponding to \( u, v, u_i, v_i \), where \( 1 \leq i \leq n \). If \( G = S'(B_{n,n}) \) then \(|V(G)| = 4(n + 1)\) and \(|E(G)| = 6n + 3\). We define vertex labeling \( f : V(G) \to \{1, 2, \ldots, 4(n + 1)\} \) as follows.

Let \( p_i \) be the highest prime number \(< 4(n + 1)\).

\[
\begin{align*}
f(u) &= 2, & f(u') &= 1, \\
f(v) &= 4, & f(v') &= p_i, \\
f(u_i) &= 6 + 2(i - 1); & 1 \leq i \leq n, \\
f(u_i') &= f(u_i) + 2i; & 1 \leq i \leq n \\
\end{align*}
\]

For the vertices \( v_1, v_2, \ldots, v_n \) and \( v_1', v_2', \ldots, v_n' \) we assign distinct odd numbers (except \( p_i \)).
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In view of the above labeling pattern we have, \( e_f(0) = 3n + 1, e_f(1) = 3n + 2 \).

Thus, \(|e_f(0) - e_f(1)| \leq 1\).

Hence, \( S'(B_{n,n}) \) is a divisor cordial graph.

Illustration 2.4. Divisor cordial labeling of the graph \( S'(B_{6,6}) \) is shown in Figure 2.

![Figure 2: Divisor cordial labeling of the graph \( S'(B_{6,6}) \)](image)

Theorem 2.5. \( DS(B_{n,n}) \) is a divisor cordial graph.

Proof: Consider \( B_{n,n} \) with \( V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\} \), where \( u_i, v_i \) are pendant vertices. Here \( V(B_{n,n}) = V_1 \cup V_2 \), where \( V_1 = \{u_i, v_i : 1 \leq i \leq n\} \) and \( V_2 = \{u, v\} \). Now in order to obtain \( DS(B_{n,n}) \) from \( G \), we add \( w_1, w_2 \) corresponding to \( V_1, V_2 \). Then \(|V(DS(B_{n,n}))| = 2n + 4 \) and

\[ E(DS(B_{n,n})) = \{uw, uw_i, vw_i \} \cup \{uu_i, vv_i, uu, vv : 1 \leq i \leq n\} \text{ so } |E(DS(B_{n,n}))| = 4n + 3. \]

We define vertex labeling \( f : V(DS(B_{n,n})) \to \{1, 2, \ldots, 2n + 4\} \) as follows.

\[
\begin{align*}
    f(u) &= 4, & f(v) &= 2n + 3, \\
    f(w_1) &= 1, & f(w_2) &= 2, \\
    f(u_i) &= 3 + 2(i - 1); & 1 \leq i \leq n \\
    f(v_i) &= 6 + 2(i - 1); & 1 \leq i \leq n
\end{align*}
\]

In view of the above defined labeling pattern we have, \( e_f(0) = 2n + 2, e_f(1) = 2n + 1 \).

Thus, \(|e_f(0) - e_f(1)| \leq 1\).

Hence, \( DS(B_{n,n}) \) is a divisor cordial graph.

Illustration 2.6. Divisor cordial labeling of the graph \( DS(B_{5,5}) \) is shown in Figure 3.
Theorem 2.7. \( D_2(B_{n,n}) \) is a divisor cordial graph.

**Proof:** Consider two copies of \( B_{n,n} \). Let \( \{u, v, u', v', 1 \leq i \leq n\} \) and \( \{u', v', u, v, 1 \leq i \leq n\} \) be the corresponding vertex sets of each copy of \( B_{n,n} \). Let \( G \) be the graph \( D_2(B_{n,n}) \) then \( |V(G)| = 4(n+1) \) and \( |E(G)| = 4(2n+1) \). We define vertex labeling \( f: V(G) \rightarrow \{1, 2, \ldots, 4(n+1)\} \) as follows.

Let \( p_1 \) be the highest prime number and \( p_2 \) be the second highest prime number such that \( p_2 < p_1 < 4(n+1) \).

- \( f(u) = 2 \), \( f(u') = 1 \),
- \( f(v) = p_1 \), \( f(v') = p_2 \),
- \( f(u_i) = 6 + 2(i-1) \); \( 1 \leq i \leq n \)
- \( f(u'_i) = f(u_n) + 2i \); \( 1 \leq i \leq n \)
- \( f(v_1) = 4 \),

For the vertices \( v_2, v_3, \ldots, v_n \) and \( v'_1, v'_2, \ldots, v'_n \) we assign distinct odd numbers (except \( p_1 \) and \( p_2 \)).

In view of the above defined labeling pattern we have, \( e_f(0) = 4n + 2 = e_f(1) \).

Thus, \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( D_2(B_{n,n}) \) is a divisor cordial graph.

**Illustration 2.8.** Divisor cordial labeling of graph \( D_2(B_{5,5}) \) is shown in Figure 4.
Some Star and Bistar Related Divisor Cordial Graphs

**Figure 4:** Divisor cordial labeling of graph $D_5(B_{5,s})$

**Theorem 2.9.** $B_{n,n}^2$ is a divisor cordial graph.

**Proof:** Consider $B_{n,n}$ with vertex set $\{u,v,u_i,v_i,1 \leq i \leq n\}$ where $u_i,v_i$ are pendant vertices. Let $G$ be the graph $B_{n,n}^2$ then $|V(G)|=2n+2$ and $|E(G)|=4n+1$.

We define vertex labeling $f: V(G) \rightarrow \{1,2,\ldots,2n+2\}$ as follows.

Let $p_1$ be the highest prime number $\leq 2n+2$.

$f(u) = 1$, \quad \quad f(v) = p_1$

$f(u_i) = 2$

$f(v_i) = 4 + 2(i-1); \quad 1 \leq i \leq n$

For the vertices $u_2, u_3, \ldots, u_n$ we assign distinct odd numbers (except $p_1$).

In view of the above defined labeling pattern we have, $e_f(0) = 2n, e_f(1) = 2n+1$

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, $B_{n,n}^2$ is a divisor cordial graph

**Illustration 2.10.** Divisor cordial labeling of the graph $B_{7,7}^2$ is shown in Figure 5.

**Figure 5:** Divisor cordial labeling of the graph $B_{7,7}^2$
3. Concluding Remarks
As all the graphs are not divisor cordial graphs it is very interesting and challenging as well to investigate divisor cordial labeling for the graph or graph families which admit divisor cordial labeling. Here we have contributed some new results by investigating divisor cordial labeling for some star and bistar related graphs.

Varatharajan et al. [19] have proved that $K_{1,n}$ and $B_{n,n}$ are divisor cordial graphs while we prove that the splitting graphs of star $K_{1,n}$ and bistar $B_{n,n}$ are also divisor cordial graphs. Thus divisor cordiality remains invariant for the splitting graphs of $K_{1,n}$ and $B_{n,n}$. It is also invariant for degree splitting graph of $B_{n,n}$, shadow graph of $B_{n,n}$ and square graph of $B_{n,n}$.

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Some Star and Bistar Related Divisor Cordial Graphs


Prime cordial labeling of some wheel related graphs

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Abstract

A prime cordial labeling of a graph $G$ with the vertex set $V(G)$ is a bijection $f : V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\}$ such that each edge $uv$ is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits prime cordial labeling is called prime cordial graph. In this paper we prove that the gear graph $G_n$ admits prime cordial labeling for $n \geq 4$. We also show that the helm $H_n$ for every $n$, the closed helm $CH_n$ (for $n \geq 5$) and the flower graph $F_l$ (for $n \geq 4$) are prime cordial graphs.

Keywords: Prime cordial labeling, gear graph, helm, closed helm, flower graph.

\textsuperscript{2010 MSC:} 05C78.

1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notations we follow Gross and Yellen \cite{5}. We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition 1.1. If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.

Any graph labeling will have following three common characteristics:
1. a set of numbers from which vertex labels are chosen;
2. a rule that assigns a value to each edge;
3. a condition that this value has to satisfy.

According to Beineke and Hegde \cite{11} graph labeling serves as a frontier between number theory and structure of graphs. Graph labelings have many applications within mathematics as well as to several areas of computer science and communication networks. According to Graham and Sloane \cite{4} the harmonious labellings are closely related to problems in error correcting codes while odd harmonious labeling is useful to solve undetermined equations as described by Liang and Bai \cite{6}. The optimal linear arrangement concern to wiring network problems in electrical engineering and placement problems in production engineering can be formalised as a graph labeling problem as stated by Yegnanaryanan and Vaidhyananathan \cite{13}. The watershed transform is an important morphological tool used for image segmentation. An improved algorithm using Graceful labeling for watershed image segmentation is also proposed by Sridevi \textit{et al.} \cite{7}. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian \cite{3}.

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Definition 1.2. A mapping \( f : V(G) \rightarrow \{0, 1\} \) is called binary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

Definition 1.3. If for an edge \( e = uv \), the induced edge labeling \( f^* : E(G) \rightarrow \{0, 1\} \) is given by \( f^*(e) = |f(u) - f(v)| \). Then

\[
\begin{align*}
  v_f(i) &= \text{number of vertices of } G \text{ having label } i \text{ under } f, \\
  e_f(i) &= \text{number of edges of } G \text{ having label } i \text{ under } f^*
\end{align*}
\]

where \( i = 0 \) or \( 1 \).

Definition 1.4. A binary vertex labeling \( f \) of a graph \( G \) is called a cordial labeling if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). A graph \( G \) is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [2]. Some labeling schemes are also introduced with minor variations in cordial theme. Product cordial labeling, total product cordial labeling and prime cordial labeling are among mention a few. The present work is focused on prime cordial labeling.

Definition 1.5. A prime cordial labeling of a graph \( G \) with vertex set \( V(G) \) is a bijection \( f : V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\} \) and the induced function \( f^* : E(G) \rightarrow \{0, 1\} \) is defined by

\[
\begin{align*}
  f^*(e = uv) &= 1, \text{ if } \gcd(f(u), f(v)) = 1; \\
  &= 0, \text{ otherwise.}
\end{align*}
\]

satisfies the condition \( |e_f(0) - e_f(1)| \leq 1 \). A graph which admits prime cordial labeling is called a prime cordial graph.

The concept of prime cordial labeling was introduced by Sundaram et al [8] and in the same paper they have investigated several results on prime cordial labeling. Vaidya and Vihol [9] as well as Vaidya and Shah [12] have discussed prime cordial labeling in the context of some graph operations. Prime cordial labeling for some wheel related graphs have been discussed by Vaidya and Vihol in [10]. Vaidya and Shah [11] have investigated many results on prime cordial labeling. Same authors in [12] have proved that the wheel graph \( W_n \) admits prime cordial labeling for \( n \geq 8 \). The present work is aimed to investigate some new results on prime cordial labeling for some wheel related graphs.

Definition 1.6. The wheel \( W_n \) is defined to be the join \( K_1 + C_n \). The vertex corresponding to \( K_1 \) is known as apex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges. We continue to recognize apex of wheel as the apex of respective graphs corresponding to definitions 1.6 to 1.9.

Definition 1.7. The gear graph \( G_n \) is obtained from the wheel by subdividing each of its rim edge.

Definition 1.8. The helm \( H_n \) is the graph obtained from a wheel \( W_n \) by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree \( n \), \( n \) vertices of degree 4 and \( n \) pendant vertices.

Definition 1.9. The closed helm \( CH_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to form a cycle. It contains three types of vertices: an apex of degree \( n \), \( n \) vertices of degree 4 and \( n \) vertices of degree 3.

Definition 1.10. The flower \( Fl_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree 2\( n \), \( n \) vertices of degree 4 and \( n \) vertices of degree 2.

2 Main Results

Theorem 2.1. Gear graph \( G_n \) is a prime cordial graph for \( n \geq 4 \).

Proof. Let \( W_n \) be the wheel with apex vertex \( v \) and rim vertices \( v_1, v_2, \ldots, v_n \). To obtain the gear graph \( G_n \), subdivide each rim edge of wheel by the vertices \( u_1, u_2, \ldots, u_n \). Where each \( u_i \) is added between \( v_i \) and \( v_{i+1} \) for \( i = 1, 2, \ldots, n - 1 \) and \( u_n \) is added between \( v_1 \) and \( v_n \). Then \( |V(G_n)| = 2n + 1 \) and \( |E(G_n)| = 3n \). To define \( f : V(G) \rightarrow \{1, 2, 3, \ldots, 2n + 1\} \), we consider following four cases.
Case 1: \( n = 3 \)
In \( G_3 \) to satisfy the edge condition for prime cordial labeling it is essential to label four edges with label 0 and five edges with label 1 out of nine edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most three edges and 1 labels for at least six edges. That is, \(|e_f(0) - e_f(1)| = 3 > 1\). Hence, \( G_3 \) is not prime cordial graph.

Case 2: \( n = 4 \) to 9, 11, 14, 19
For \( n = 4 \), \( f(v) = 0 \), \( f(v_1) = 3, f(v_2) = 9, f(v_3) = 4, f(v_4) = 8 \) and \( f(u_1) = 1, f(u_2) = 7, f(u_3) = 2, f(u_4) = 5 \).
Then \( e_f(0) = e_f(1) \).

For \( n = 5 \), \( f(v) = 6, f(v_1) = 9, f(v_2) = 5, f(v_3) = 4, f(v_4) = 8, f(v_5) = 3 \) and \( f(u_1) = 7, f(u_2) = 10, f(u_3) = 2, f(u_4) = 1, f(u_5) = 11 \). Then \( e_f(0) = 8, e_f(1) = 7 \).

For \( n = 6 \), \( f(v) = 6, f(v_1) = 9, f(v_2) = 8, f(v_3) = 4, f(v_4) = 11, f(v_5) = 1, f(v_6) = 10 \) and \( f(u_1) = 12, f(u_2) = 2, f(u_3) = 13, f(u_4) = 5, f(u_5) = 7, f(u_6) = 3 \). Then \( e_f(0) = 9 = e_f(1) \).

For \( n = 7 \), \( f(v) = 2, f(v_1) = 7, f(v_2) = 4, f(v_3) = 6, f(v_4) = 12, f(v_5) = 8, f(v_6) = 10, f(v_7) = 14 \) and \( f(u_1) = 5, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 1, f(u_6) = 11, f(u_7) = 13, f(u_8) = 17 \). Then \( e_f(0) = 12 = e_f(1) \).

For \( n = 9 \), \( f(v) = 2, f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 8, f(v_6) = 10, f(v_7) = 14, f(v_8) = 16 \) and \( f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 1, f(u_6) = 11, f(u_7) = 13, f(u_8) = 17, f(u_9) = 19 \). Then \( e_f(0) = 13, e_f(1) = 14 \).

For \( n = 11 \), \( f(v) = 2, f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 8, f(v_6) = 10, f(v_7) = 14, f(v_8) = 16, f(v_9) = 20, f(v_{11}) = 22 \) and \( f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 21, f(u_6) = 11, f(u_7) = 13, f(u_8) = 17, f(u_9) = 19, f(u_{10}) = 23, f(u_{11}) = 1 \). Then \( e_f(0) = 16, e_f(1) = 17 \).

For \( n = 14 \), \( f(v) = 2, f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 8, f(v_6) = 10, f(v_7) = 14, f(v_8) = 16, f(v_{10}) = 20, f(v_{11}) = 22 \) and \( f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 11, f(u_6) = 13, f(u_7) = 17, f(u_8) = 19, f(u_{10}) = 23, f(u_{11}) = 25, f(u_{13}) = 29, f(u_{14}) = 1 \). Then \( e_f(0) = 21 = e_f(1) \).

For \( n = 19 \), \( f(v) = 2, f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 12, f(v_5) = 8, f(v_6) = 10, f(v_7) = 14, f(v_8) = 16, f(v_{10}) = 20, f(v_{11}) = 22 \) and \( f(u_1) = 7, f(u_2) = 3, f(u_3) = 9, f(u_4) = 15, f(u_5) = 11, f(u_6) = 13, f(u_7) = 17, f(u_8) = 19, f(u_{10}) = 23, f(u_{11}) = 25, f(u_{14}) = 31, f(u_{14}) = 35, f(u_{14}) = 37, f(u_{14}) = 1 \). Then \( e_f(0) = 29, e_f(1) = 28 \).

Now for the remaining two cases let,
\[
s = \left\lfloor \frac{n}{3} \right\rfloor, k = \left\lfloor \frac{2n+1}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor, t = \left( n + \left\lfloor \frac{2n+1}{3} \right\rfloor - 2 \right) - \left\lfloor \frac{3n}{2} \right\rfloor,
\]
\[
m = \left\lfloor \frac{2n+1}{2} \right\rfloor - (2 + s + t), h_e = \text{largest even number not divisible by 3 } \leq 2n,
\]
\( h_o = \text{largest odd number not divisible by 3 } \leq 2n + 1 \).

\( f(v) = 2, \)
\( f(v_1) = 4, \quad f(v_2) = 5, \)
\( f(v_{2+i}) = 6i; \quad 1 \leq i \leq s \)
\( f(u_1) = 7, \)
\( f(u_{1+i}) = 3 + 6(i-1); \quad 1 \leq i \leq k \)

If \( k = s \), then \( f(u_{k+2}) = 1 \) or \( f(u_n) = 1 \).

Case 3: \( t = 0 \) (\( n = 10, 12, 13, 15, 17 \))
For \( m \) odd, consider \( x_1 = \left\lfloor \frac{m}{2} \right\rfloor \), \( x_2 = x_3 = x_4 = \left\lfloor \frac{m}{2} \right\rfloor \) and for \( m \) even consider, \( x_1 = x_2 = x_3 = x_4 = \frac{m}{2} \)

\( f(v_{k+2}) = 8 + 6(i-1); \quad 1 \leq i \leq x_1 \)
\( f(v_{k+2}) = 10 + 6(i-1); \quad 1 \leq i \leq x_3 \)
\( f(u_{k+2}) = 11 + 6(i-1); \quad 1 \leq i \leq x_2 \)
\( f(u_{k+2}) = 13 + 6(i-1); \quad 1 \leq i \leq x_4 \)
which assigns all the vertex labels for case 3.

Case 4: \( t \geq 1 \) (\( n = 16, 18, n \geq 20 \))
For \( m \) odd, consider \( x_1 = x_2 = x_3 = \frac{m}{2} \), \( x_4 = \frac{m-3}{2} \) and for \( m \) even consider, \( x_1 = x_2 = x_3 = x_4 = \)
\[ m - 2 \]
\[ \frac{f(v_{i+1+2i})}{2} = 8 + 6(i - 1); \quad 1 \leq i \leq x_1 \]
\[ f(v_{i+2+2i}) = 10 + 6(i - 1), \quad 1 \leq i \leq x_3 \]
\[ f(u_{i+1+2i}) = 11 + 6(i - 1); \quad 1 \leq i \leq x_2 \]
\[ f(u_{i+2+2i}) = 13 + 6(i - 1); \quad 1 \leq i \leq x_4 \]

For the vertices \( u_{n-1}, u_{n-2}, \ldots, u_{n-(t-1)} \) we assign even numbers (not congruent 0 mod 3) in descending order starting from \( h_e \) respectively while for \( u_n, v_n, u_{n-1}, v_{n-2}, \ldots, v_{n-t} \) we assign odd numbers (not congruent 0 mod 3) in descending order starting from \( h_0 \) respectively such that \( f^*(v_j u_{j-i}) \) or \( f^*(v_j u_{j+i}) \) do not generate edge label 0. Which assigns all the vertex labels for case 4.

In view of the above defined labeling pattern for cases 3 and 4, we have
\[ e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor \quad \text{and} \quad e_f(1) = \left\lceil \frac{3n}{2} \right\rceil. \]

Thus, we have \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( G_n \) is a prime cordial graph for \( n \geq 4 \).

\[ \square \]

**Example 2.1.** For the graph \( G_{20} \), \( |V(G_{20})| = 41 \) and \( |E(G_{20})| = 60 \). In accordance with Theorem 2.1 we have \( s = 6, t = 7, m = 11, x_1 = x_2 = x_3 = 5, x_4 = 4 \) and using the labeling pattern described in case 4. The corresponding prime cordial labeling is shown in Fig. 1. It is easy to visualise that \( e_f(0) = 30 = e_f(1) \).

![Fig. 1](image)

**Theorem 2.2.** Helm graph \( H_n \) is a prime cordial graph for every \( n \).

**Proof.** Let \( v \) be the apex, \( v_1, v_2, \ldots, v_n \) be the vertices of degree 4 and \( u_1, u_2, \ldots, u_n \) be the pendant vertices of \( H_n \). Then \( |V(H_n)| = 2n + 1 \) and \( |E(H_n)| = 3n \). To define \( f : V(G) \to \{1, 2, 3, \ldots, 2n + 1\} \), we consider following three cases.

**Case 1:** \( n = 3 \) to 9

For \( n = 3 \), \( f(v) = 6, f(v_1) = 2, f(v_2) = 4, f(v_3) = 3 \) and \( f(u_1) = 1, f(u_2) = 7, f(u_3) = 5 \). Then \( e_f(0) = 4, e_f(1) = 5 \).

For \( n = 4 \), \( f(v) = 6, f(v_1) = 3, f(v_2) = 2, f(v_3) = 4, f(v_4) = 8 \) and \( f(u_1) = 1, f(u_2) = f(u_3) = 9, f(u_4) = 5, f(u_5) = 7 \). Then \( e_f(0) = 6 = e_f(1) \).

For \( n = 5 \), \( f(v) = 6, f(v_1) = 3, f(v_2) = 2, f(v_3) = 4, f(v_4) = 8, f(v_5) = 10 \) and \( f(u_1) = 11, f(u_2) = 1, f(u_3) = 5, f(u_4) = 7, f(u_5) = 9 \). Then \( e_f(0) = 8, e_f(1) = 7 \).

For \( n = 6 \), \( f(v) = 2, f(v_1) = 3, f(v_2) = 6, f(v_3) = 8, f(v_4) = 10, f(v_5) = 7, f(v_6) = 11 \) and \( f(u_1) = 1, f(u_2) = 4, f(u_3) = 12, f(u_4) = 5, f(u_5) = 9, f(u_6) = 13 \). Then \( e_f(0) = 9 = e_f(1) \).

For \( n = 7 \), \( f(v) = 2, f(v_1) = 3, f(v_2) = 6, f(v_3) = 4, f(v_4) = 10, f(v_5) = 5, f(v_6) = 11, f(v_7) = 1 \) and
\( f(u_1) = 9, f(u_2) = 8, f(u_3) = 12, f(u_4) = 14, f(u_5) = 7, f(u_6) = 13, f(u_7) = 15. \) Then \( e_f(0) = 11, e_f(1) = 10. \)

For \( n = 8, f(v) = 2, f(v_1) = 6, f(v_2) = 8, f(v_3) = 4, f(v_4) = 10, f(v_5) = 5, f(v_6) = 9, f(v_7) = 13, f(v_8) = 17\) and \( f(u_1) = 3, f(u_2) = 12, f(u_3) = 14, f(u_4) = 16, f(u_5) = 7, f(u_6) = 11, f(u_7) = 15, f(u_8) = 1. \) Then \( e_f(0) = 12 = e_f(1). \)

For \( n = 9, f(v) = 2, f(v_1) = 6, f(v_2) = 8, f(v_3) = 4, f(v_4) = 10, f(v_5) = 5, f(v_6) = 9, f(v_7) = 13, f(v_8) = 17\) and \( f(u_1) = 1, f(u_2) = 12, f(u_3) = 14, f(u_4) = 16, f(u_5) = 7, f(u_6) = 11, f(u_7) = 12, f(u_8) = 15, f(u_9) = 19. \) Then \( e_f(0) = 17, e_f(1) = 16. \)

**Case 2:** \( n \) is even, \( n \geq 10\)

\[
\begin{align*}
&f(v) = 2, \quad f(v_1) = 10, \\
&f(v_2) = 4, \quad f(v_3) = 8, \\
&f(v_{3+i}) = 12 + 2(i-1); \quad 1 \leq i \leq \frac{n}{2} - 4 \\
&f(v_2) = 6, \quad f(v_{2+i}) = 1, \\
&f(u_2) = 3, \quad f(u_{2+i}) = 2n + 1, \\
&f(u_i) = 2n - 2(i-1); \quad 1 \leq i \leq \frac{n}{2} - 1 \\
&f(v_{n-1}) = 5 + 4i; \quad 0 \leq i \leq \frac{n}{2} - 1 \\
&f(u_{n-1}) = 7 + 4i; \quad 0 \leq i \leq \frac{n}{2} - 1
\end{align*}
\]

**Case 3:** \( n \) is odd, \( n \geq 11\)

\[
\begin{align*}
&f(v) = 2, \quad f(v_1) = 10, \\
&f(v_2) = 4, \quad f(v_3) = 8, \\
&f(v_{3+i}) = 12 + 2(i-1); \quad 1 \leq i \leq \frac{n-1}{2} - 4 \\
&f(v_{n-1}) = 6, \quad f(v_{n-1}) = 3, \\
&f(u_{n-1}) = 1, \\
&f(u_i) = 2n - 2(i-1); \quad 1 \leq i \leq \frac{n-1}{2} \\
&f(v_{n-1}) = 5 + 4i; \quad 0 \leq i < \frac{n-1}{2} \\
&f(u_{n-1}) = 7 + 4i; \quad 0 \leq i < \frac{n-1}{2}
\end{align*}
\]

In view of the above defined labeling pattern for cases 2 and 3,

If \( 2n - 1 \equiv 0 \pmod{3} \) then \( e_f(0) = \left\lceil \frac{3n}{2} \right\rceil \) and \( e_f(1) = \left\lfloor \frac{3n}{2} \right\rfloor, \)

otherwise \( e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor \) and \( e_f(1) = \left\lceil \frac{3n}{2} \right\rceil. \)

Thus, we have \( |e_f(0) - e_f(1)| \leq 1. \)

Hence, \( H_n \) is a prime cordial graph for every \( n. \)

\[\square\]

**Example 2.2.** The graph \( H_{13} \) and its prime cordial labeling is shown in Fig. 2.

![Fig. 2](image-url)

**Theorem 2.3.** Closed helm \( CH_n \) is a prime cordial graph for \( n \geq 5. \)
Proof. Let $v$ be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 3 of $CH_n$. Then $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$. To define $f : V(G) \to \{1, 2, 3, \ldots, 2n + 1\}$, we consider following three cases.

**Case 1: $n = 3, 4$**

In $CH_3$ to satisfy the edge condition for prime cordial labeling it is essential to label six edges with label 0 and six edges with label 1 out of twelve edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 labels for at least eight edges. That is, $|e_f(0) - e_f(1)| = 4 > 1$. Hence, $CH_3$ is not prime cordial graph.

In $CH_4$ to satisfy the edge condition for prime cordial labeling it is essential to label eight edges with label 0 and eight edges with label 1 out of sixteen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most seven edges and 1 labels for at least nine edges. That is, $|e_f(0) - e_f(1)| = 2 > 1$. Hence, $CH_4$ is not prime cordial graph.

**Case 2: $n = 5, 6$**

For $n = 5$, $f(v) = 6$, $f(v_1) = 2$, $f(v_2) = 4$, $f(v_3) = 3$, $f(v_4) = 9$, $f(v_5) = 11$ and $f(u_1) = 10$, $f(u_2) = 8$, $f(u_3) = 1$, $f(u_4) = 7$, $f(u_5) = 5$. Then $e_f(0) = 10 = e_f(1)$. For $n = 6$, $f(v) = 6$, $f(v_1) = 1$, $f(v_2) = 5$, $f(v_3) = 10$, $f(v_4) = 4$, $f(v_5) = 12$, $f(v_6) = 3$ and $f(u_1) = 11$, $f(u_2) = 13$, $f(u_3) = 8$, $f(u_4) = 2$, $f(u_5) = 9$, $f(u_6) = 7$. Then $e_f(0) = 12 = e_f(1)$.

**Case 3: $n \geq 7$**

$f(v) = 2$, \quad $f(v_1) = 4$,
$f(v_2) = 6$, \quad $f(v_3) = 3$,
$f(v_4) = 12$, \quad $f(v_5) = 8$,
f$v_6 = 10$,
f$v_{6+i} = 14 + 2(i-1)$; \quad $1 \leq i \leq n - 6$
$f(u_1) = 1$, \quad $f(u_2) = 5$,
$f(u_3) = 7$, \quad $f(u_4) = 9$,
f$u_5 = 13$, \quad $f(u_6) = 11$,
f$u_{6+i} = 15 + 2(i-1)$; \quad $1 \leq i \leq n - 6$

In view of the above defined labeling pattern we have $e_f(0) = 2n = e_f(1)$.

Thus, we have $|e_f(0) - e_f(1)| \leq 1$.

Hence, $CH_n$ is a prime cordial graph for $n \geq 5$.

Example 2.3. The graph $CH_{10}$ and its prime cordial labeling is shown in Fig. 3.

![Fig. 3](image-url)

Theorem 2.4. Flower graph $Fl_n$ is a prime cordial graph for $n \geq 4$. 
Proof. Let $v$ be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 of $F_l n$. Then $|V(F_l n)| = 2n + 1$ and $|E(F_l n)| = 4n$. To define $f : V(G) \to \{1, 2, 3, \ldots, 2n + 1\}$, we consider following four cases.

Case 1: $n = 3$
In $F_3$, to satisfy the edge condition for prime cordial labeling it is essential to label six edges with label 0 and six edges with label 1 out of twelve edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 labels for at least eight edges. That is, $|e_f(0) - e_f(1)| = 4 > 1$. Hence, $F_3$ is not prime cordial graph.

Case 2: $n = 4$ to 9.

For $F_{4a}$, $f(v) = 6$, $f(v_1) = 4$, $f(v_2) = 2$, $f(v_3) = 9$, $f(v_4) = 3$ and $f(u_1) = 7$, $f(u_2) = 8$, $f(u_3) = 5$, $f(u_4) = 1$. Then $e_f(0) = 8 = e_f(1)$.

For $F_{4b}$, $f(v) = 6$, $f(v_1) = 2$, $f(v_2) = 4$, $f(v_3) = 8$, $f(v_4) = 10$, $f(v_5) = 3$ and $f(u_1) = 11$, $f(u_2) = 7$, $f(u_3) = 5$, $f(u_4) = 1 = f(u_5) = 9$. Then $e_f(0) = 10 = e_f(1)$.

For $F_{4c}$, $f(v) = 6$, $f(v_1) = 2$, $f(v_2) = 4$, $f(v_3) = 8$, $f(v_4) = 10$, $f(v_5) = 12$, $f(v_6) = 3$ and $f(u_1) = 5$, $f(u_2) = 7$, $f(u_3) = 9$, $f(u_4) = 11$, $f(u_5) = 13$, $f(u_6) = 1$. Then $e_f(0) = 12 = e_f(1)$.

For $F_{4d}$, $f(v) = 2$, $f(v_1) = 3$, $f(v_2) = 12$, $f(v_3) = 10$, $f(v_4) = 8$, $f(v_5) = 14$, $f(v_6) = 4$, $f(v_7) = 6$ and $f(u_1) = 1$, $f(u_2) = 5$, $f(v_3) = 11$, $f(u_4) = 7$, $f(u_5) = 13$, $f(u_6) = 15$, $f(u_7) = 9$. Then $e_f(0) = 14 = e_f(1)$.

For $F_{4e}$, $f(v) = 2$, $f(v_1) = 3$, $f(v_2) = 6$, $f(v_3) = 4$, $f(v_4) = 8$, $f(v_5) = 10$, $f(v_6) = 14$, $f(v_7) = 16$, $f(v_8) = 12$ and $f(u_1) = 1$, $f(u_2) = 9$, $f(u_3) = 7$, $f(u_4) = 5$, $f(u_5) = 11$, $f(u_6) = 13$, $f(u_7) = 15$, $f(u_8) = 17$. Then $e_f(0) = 16 = e_f(1)$.

For $F_{4f}$, $f(v) = 2$, $f(v_1) = 3$, $f(v_2) = 12$, $f(v_3) = 4$, $f(v_4) = 8$, $f(v_5) = 10$, $f(v_6) = 16$, $f(v_7) = 18$, $f(v_8) = 6$ and $f(u_1) = 17$, $f(u_2) = 1$, $f(u_3) = 5$, $f(u_4) = 7$, $f(u_5) = 11$, $f(u_6) = 13$, $f(u_7) = 15$, $f(u_8) = 19$, $f(u_9) = 9$. Then $e_f(0) = 18 = e_f(1)$.

Case 3: $n$ is even, $n \geq 10$

For $2n + 1 \equiv 0(\text{mod } 3)$

For $2n + 1 \equiv 1(\text{mod } 3)$

For $2n + 1 \equiv 2(\text{mod } 3)$

Case 4: $n$ is odd, $n \geq 11$

For $2n + 1 \equiv 0(\text{mod } 3)$

For $2n + 1 \equiv 1(\text{mod } 3)$
For $2n + 1 \equiv 2 \pmod{3}$

$$f \left( \frac{u_{2n+1}}{2} + 1 \right) = 1,$$

$$f \left( \frac{u_{2n+1}}{2} + i \right) = 2n + 1 - 4(i - 1); \quad 1 \leq i \leq 2$$

In view of the above defined labeling pattern we have $e_f(0) = 2n = e_f(1)$.

Thus, we have $|e_f(0) - e_f(1)| \leq 1$.

Hence, $F_l^n$ is a prime cordial graph for $n \geq 4$.

Example 2.4. The graph $F_l^{11}$ and its prime cordial labeling is shown in Fig. 4.

3 Open problems

- To investigate necessary and sufficient conditions for a graph to admit a prime cordial labeling.
- To investigate some new graph or graph families which admit prime cordial labeling.
- To obtain forbidden subgraph(s) characterisation for prime cordial labeling.

4 Conclusion

As all the graphs are not prime cordial graphs it is very interesting and challenging as well to investigate prime cordial labeling for the graph or graph families which admit prime cordial labeling. Here we have contributed some new results by investigating prime cordial labeling for some wheel related graphs.

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Further Results on Divisor Cordial Labeling

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Abstract. A divisor cordial labeling of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to \{1, 2, ..., $|V|$\} such that an edge $uv$ is assigned the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 otherwise, then number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a divisor cordial labeling is called a divisor cordial graph. In this paper we prove that helm $H_n$, flower graph $F_l$ and Gear graph $G_n$ are divisor cordial graphs. Moreover we show that switching of a vertex in cycle $C_n$, switching of a rim vertex in wheel $W_n$ and switching of the apex vertex in helm $H_n$ admit divisor cordial labeling.

Keywords: Labeling, Divisor cordial labeling, Switching of a vertex

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen [1] while for number theory we refer to Burton [2]. We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition 1.1. If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.

According to Beineke and Hegde [3] graph labeling serves as a frontier between number theory and structure of graphs. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. Yegnanaryan and Vaidhyanathan [4] have discussed applications of edge balanced graph labeling, edge magic labeling and (1,1) edge magic graphs. For a dynamic
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survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [5].

**Definition 1.2.** A mapping \( f : V(G) \rightarrow \{0,1\} \) is called *binary vertex labeling* of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

**Notation 1.3.** If for an edge \( e=uv \), the induced edge labeling \( f^* : E(G) \rightarrow \{0,1\} \) is given by \( f^*(e) = |f(u) - f(v)| \). Then

\[
v_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f
\]
\[
e_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f^*
\]

**Definition 1.4.** A binary vertex labeling \( f \) of a graph \( G \) is called a *cordial labeling* if

\[
|v_f(0) - v_f(1)| \leq 1 \quad \text{and} \quad |e_f(0) - e_f(1)| \leq 1
\]

A graph \( G \) is *cordial* if it admits cordial labeling.

The above concept was introduced by Cahit [6]. After this many labeling schemes are also introduced with minor variations in cordial theme. The product cordial labeling, total product cordial labeling and prime cordial labeling are among mention a few. The present work is focused on divisor cordial labeling.

**Definition 1.5.** A *prime cordial labeling* of a graph \( G \) with vertex set \( V(G) \) is a bijection \( f : V(G) \rightarrow \{1,2,3,...,|V(G)|\} \) and the induced function \( f^* : E(G) \rightarrow \{0,1\} \) is defined by

\[
f^*(e = uv) = \begin{cases} 1 & \text{if } \gcd(f(u), f(v)) = 1; \\ 0 & \text{otherwise.} \end{cases}
\]

which satisfies the condition \( |e_f(0) - e_f(1)| \leq 1 \). A graph which admits prime cordial labeling is called a *prime cordial graph*.

The concept of prime cordial labeling was introduced by Sundaram et al. [7] and in the same paper they have investigated several results on prime cordial labeling. Vaidya and Vihol [8, 9] as well as Vaidya and Shah [10, 11, 12] have proved many results on prime cordial labeling.

Motivated through the concept of prime cordial labeling Varatharajan et al. [13] introduced a new concept called divisor cordial labeling which is a combination of divisibility of numbers and cordial labelings of graphs.

**Definition 1.6.** Let \( G = (V(G), E(G)) \) be a simple graph and \( f : V(G) \rightarrow \{1,2,...,|V(G)|\} \) be a bijection. For each edge \( uv \), assign the label 1 if \( f(u) | f(v) \) or \( f(v) | f(u) \) and the label 0 otherwise. The function \( f \) is called a *divisor cordial labeling* if \( |e_f(0) - e_f(1)| \leq 1 \). A graph with a divisor cordial labeling is called a *divisor cordial graph*. 
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In [13] authors have proved that path, cycle, wheel, star, $K_{2,n}$ and $K_{3,n}$ are divisor cordial graphs while $K_n$ is not divisor cordial for $n \geq 7$. The divisor cordial labeling of full binary tree as well as some star related graphs are reported by Varatharajan et al. [14] while some star and bistar related graphs are proved to be divisor cordial graphs by Vaidya and Shah [15].

It is important to note that prime cordial labeling and divisor cordial labeling are two independent concepts. A graph may possess one or both of these properties or neither as exhibited below.

i) $P_n$ ($n \geq 6$) is both prime cordial as proved in [7] and divisor cordial as proved in [13].

ii) $C_3$ is not prime cordial as proved in [7] but it is divisor cordial as proved in [13].

iii) We found that a 7-regular graph with 12 vertices admits prime cordial labeling but does not admit divisor cordial labeling.

iv) Complete graph $K_7$ is not a prime cordial as stated in Gallian [5] and not divisor cordial as proved in [13].

**Definition 1.7.** The helm $H_n$ is the graph obtained from a wheel $W_n$ by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree $n$, $n$ vertices of degree 4 and $n$ pendant vertices.

**Definition 1.8.** The flower $F_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree $2n$, $n$ vertices of degree 4 and $n$ vertices of degree 2.

**Definition 1.9.** Let $e=uv$ be an edge of the graph $G$ and $w$ is not a vertex of $G$. The edge $e$ is called subdivided when it is replaced by edges $e' = uw$ and $e'' = wv$.

**Definition 1.10.** The gear graph $G_n$ is obtained from the wheel by subdividing each of its rim edge.

**Definition 1.11.** A vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$, removing all the edges incident to $v$ and adding edges joining $v$ to every other vertex which are not adjacent to $v$ in $G$.

2. Divisor Cordial Labeling of Some Wheel Related Graphs

**Theorem 2.1.** $H_n$ is a divisor cordial graph for every $n$.

**Proof:** Let $v$ be the apex, $v_1,v_2,\ldots,v_n$ be the vertices of degree 4 and $u_1,u_2,\ldots,u_n$ be the pendant vertices of $H_n$. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. We define vertex labeling as $f : V(G) \to \{1,2,3,\ldots,2n+1\}$ as follows.

$f(v) = 1$,

For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor = k$,
Assign the labels $v_i$ and $u_i$ such that $2f(v_i) = f(u_i)$ and $f(v_{i+1}) \neq f(v_i)$.

Now for remaining vertices, $v_{k+1}, v_{k+2}, \ldots, v_n$ and $u_{k+1}, u_{k+2}, \ldots, u_n$ assign the labels such that $f(v_j) \neq f(v_{j+1})$ where $k \leq j \leq n-1$, $f(v_n) \neq f(v_1)$ and $f(v_j) \neq f(u_j)$ where $k < j \leq n$.

In view of above labeling pattern we have, $e_f(1) = \left\lfloor \frac{3n}{2} \right\rfloor, e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, $H_n$ is a divisor cordial graph for each $n$.

**Example 2.2.** The graph $H_{13}$ and its divisor cordial labeling is shown in Figure 1.

![Figure 1: Divisor cordial labeling of $H_{13}$](image)

**Theorem 2.3.** $F_l$ is a divisor cordial graph for each $n$.

**Proof**: Let $v$ be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 of $F_l$. Then $|V(F_l)| = 2n + 1$ and $|E(F_l)| = 4n$. We define vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \ldots, 2n+1\}$ as follows.

- $f(v) = 1$,
- $f(v_1) = 2$,
- $f(u_1) = 3$,
- $f(v_{i+1}) = 5 + 2(i-1); \quad 1 \leq i \leq n-1$
- $f(u_{i+1}) = 4 + 2(i-1); \quad 1 \leq i \leq n-1$

In view of the above labeling pattern we have, $e_f(0) = 2n = e_f(1)$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, $F_l$ is a divisor cordial graph for each $n$.  

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Example 2.4. Divisor cordial labeling of the graph $F_{l_1}$ is shown in Figure 2.

![Figure 2: Divisor cordial labeling of $F_{l_1}$](image)

Theorem 2.5. $G_n$ is a divisor cordial graph for every $n$.

Proof: Let $W_n$ be the wheel with apex vertex $v$ and rim vertices $v_1, v_2, \ldots, v_n$. To obtain the gear graph $G_n$, subdivide each rim edge of wheel by the vertices $u_1, u_2, \ldots, u_n$. Where each $u_i$ is added between $v_i$ and $v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $u_n$ is added between $v_1$ and $v_n$. Then $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$. We define vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \ldots, 2n + 1\}$ as follows.

Our aim is to generate $\left\lfloor \frac{3n}{2} \right\rfloor$ edges having label 1 and $\left\lceil \frac{3n}{2} \right\rceil$ edges having label 0. $f(v) = 1$, which generates $n$ edges having label 1.

Now it remains to generate $k = \left\lfloor \frac{3n}{2} \right\rfloor - n$ edges with label 1.

For the vertices $v_i, u_i, v_j, u_j, \ldots$ assign the vertex label as per following ordered pattern.

Upto it generate $k$ edges with label 1.

$2, 2^2, 2^3, \ldots, 2^h,$

$3, 3 \times 2, 3 \times 2^2, \ldots, 3 \times 2^i,$

$5, 5 \times 2, 5 \times 2^2, \ldots, 5 \times 2^i,$

$\ldots, \ldots, \ldots, \ldots, \ldots,$

$\ldots, \ldots, \ldots, \ldots, \ldots,$

where $(2m-1)2^{k_m} \leq 2n + 1$ and $m \geq 1, k_m \geq 0$. 


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Observe that \((2m - 1)2^a | (2m - 1)2^{a+1}\) and \((2m - 1)2^a\) does not divide \(2m + 1\). Then for remaining vertices of \(G_n\), assign the vertex label such that the consecutive vertices do not generate edge label 1.

In view of above labeling pattern we have, \(e_f(1) = \left\lceil \frac{3n}{2} \right\rceil\), \(e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor\).

Thus, \(|e_f(0) - e_f(1)| \leq 1\).

Hence, \(G_n\) is a divisor cordial graph for each \(n\).

Example 2.6. Divisor cordial labeling of the graph \(G_{20}\) is shown in Figure 3.

![Figure 3: Divisor cordial labeling of \(G_{20}\)](image)

3. Switching of a Vertex and Divisor Cordial Labeling

**Theorem 3.1.** Switching of a vertex in cycle \(C_n\) admits divisor cordial labeling.

**Proof:** Let \(v_1, v_2, \ldots, v_n\) be the successive vertices of \(C_n\) and \(G_v\) denotes graph obtained by switching of vertex \(v\) of \(G = C_n\). Without loss of generality let the switched vertex be \(v_1\).

We note that \(|V(G_v)| = n\) and \(|E(G_v)| = 2n - 5\). We define vertex labeling

\[ f : V(G_v) \to \{1, 2, \ldots, n\}\]

as follows:

\[ f(v_1) = 1, \quad f(v_{n+i}) = 1 + i. \]

In view of the above labeling pattern we have, \(e_f(1) = n - 3, e_f(0) = n - 2\).

Thus, \(|e_f(0) - e_f(1)| \leq 1\).
Hence, the graph obtained by switching of a vertex in cycle $C_n$ is a divisor cordial labeling.

**Example 3.2.** The graph obtained by switching of a vertex in cycle $C_8$ and its divisor cordial labeling is shown in Figure 4.

![Figure 4: Switching of a vertex in $C_8$ and its divisor cordial labeling](image)

**Theorem 3.3.** Switching of a rim vertex in a wheel $W_n$ admits divisor cordial labeling.

**Proof:** Let $v$ be the apex vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel $W_n$. Let $G_{v_i}$ denote the graph obtained by switching of a rim vertex $v_i$ of $G = W_n$. We note that $|V(G_{v_i})| = n+1$ and $|E(G_{v_i})| = 3n - 6$. To define vertex labeling $f : V(G_{v_i}) \rightarrow \{1, 2, \ldots, n+1\}$, we consider the following two cases.

**Case 1:** $n = 4$

- $f(v) = 1$, $f(v_1) = 5$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$. Then $e_f(0) = 3 = e_f(1)$.

**Case 2:** $n \geq 5$

- $f(v) = 2$, $f(v_1) = 1$,
- $f(v_2) = 3$, $f(v_3) = 6$,
- $f(v_4) = 4$, $f(v_5) = 5$,
- $f(v_{5+i}) = 6 + i; \quad 1 \leq i \leq n-5$

In view of the above-defined labeling pattern for case 2,

If $n$ is even then $e_f(0) = \frac{3n - 6}{2} = e_f(1)$, otherwise $e_f(0) = \left\lfloor \frac{3n - 6}{2} \right\rfloor = e_f(1) - 1$.

Thus in both the cases we have, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by switching of a rim vertex in a wheel $W_n$ is a divisor cordial labeling.
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**Example 3.4.** The graph obtained by switching of a rim vertex in the wheel $W_9$ and its divisor cordial labeling is shown in Figure 5.

![Figure 5: Switching of a rim vertex in $W_9$ and its divisor cordial labeling](image)

**Theorem 3.5.** Switching of the apex vertex in helm $H_n$ admits divisor cordial labeling.

**Proof:** Let $v$ be the apex, $v_1,v_2,\ldots,v_n$ be the vertices of degree 4 and $u_1,u_2,\ldots,u_n$ be the pendant vertices of $H_n$. Let $G_v$ denotes graph obtained by switching of an apex vertex $v$ of $G = H_n$. We note that $|V(G_v)| = 2n + 1$ and $|E(G_v)| = 3n$. We define vertex labeling $f : V(G_v) \rightarrow \{1, 2, \ldots, 2n + 1\}$ as follows:

Our aim is to generate $\left\lfloor \frac{3n}{2} \right\rfloor$ edges having label 1 and $\left\lfloor \frac{3n}{2} \right\rfloor$ edges having label 0.

$f(v) = 1$, which generates $n$ edges having label 1.

Now it remains to generate $k = \left\lfloor \frac{3n}{2} \right\rfloor - n$ edges with label 1.

For the vertices $v_1,v_2,\ldots,v_l$ assign the vertex label as per following ordered pattern up to it generate $k$ edges with label 1.

\[
2, \quad 2^2, \quad 2^3, \quad \ldots, \quad 2^{k_m}, \\
3, \quad 3 \times 2, \quad 3 \times 2^2, \quad \ldots, \quad 3 \times 2^{k_m}, \\
5, \quad 5 \times 2, \quad 5 \times 2^2, \quad \ldots, \quad 5 \times 2^{k_m}, \\
\ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \\
\ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots,
\]

where $(2m - 1)2^{k_m} \leq 2n + 1$ and $m \geq 1, k_m \geq 0$. 

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Observe that \((2m - 1)2^\alpha | (2m - 1)2^{\alpha + 1}\) and \((2m - 1)2^\beta\) does not divide \(2m + 1\).

Then for remaining vertices \(v_{i+1}, v_{i+2}, \ldots, v_n\) and \(u_1, u_2, \ldots, u_n\) assign the vertex label such that no edge label generate 1.

In view of above labeling pattern we have, \(e_f(1) = \left\lfloor \frac{3n}{2} \right\rfloor, e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor \cdot\)

Thus, \(|e_f(0) - e_f(1)| \leq 1\).

Hence, the graph obtained by switching of the apex vertex in helm \(H_n\) admits divisor cordial labeling.

**Example 3.6.** The graph obtained by switching of the apex vertex in helm \(H_{11}\) and its divisor cordial labeling is shown in Figure 6.

![Figure 6: Switching of the apex vertex in \(H_{11}\) and its divisor cordial labeling](image)

4. Concluding Remarks
The divisor cordial labeling is a variant of cordial labeling. It is very interesting to investigate graph or graph families which are divisor cordial as all the graphs do not admit divisor cordial labeling. Here it has been proved that helm \(H_n\), flower graph \(F_n\) and Gear graph \(G_n\) are divisor cordial graphs. The graphs \(C_n\) and \(W_n\) are proved to be divisor cordial graphs by Varatharajan et al. [13] while we prove the graphs obtained by switching of a vertex in \(C_n\), switching of a rim vertex in \(W_n\) and switching of the apex vertex in \(H_n\) are divisor cordial graphs. Hence \(C_n\), \(W_n\) and \(H_n\) are switching invariant graphs for divisor cordial labeling.

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DOMINATION INTEGRITY OF SHADOW GRAPHS

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ABSTRACT

The domination integrity $DI(G)$ of a simple connected graph $G$ is a measure of vulnerability of a graph and it is defined as $DI(G) = \min \{|X| + m(G - X) : X$ is a dominating set $\}$, where $m(G - X)$ is the order of a maximum component of $G - X$. Here we determine the domination integrity of shadow graphs of path $P_n$, cycle $C_n$, complete bipartite graph $K_{m,n}$ and bistar $B_{n,n}$.

Keywords: integrity, domination integrity, shadow graph

AMS Subject classification (2010): 05C69, 05C76.

1. INTRODUCTION

We begin with simple, finite, connected and undirected graph $G$ with vertex set $V(G)$ and edge set $E(G)$. For any undefined terminology and notation related to the concept of domination we refer to Haynes et al. [12] while for the fundamental concepts of graph theory we rely upon Harary [11]. In the remaining portion of this section we will give brief summary of definitions and information related to the present work.
The vulnerability of network have been studied in various contexts including road transportation system, information security, structural engineering and communication network. A graph structure is vulnerable if ‘any small damage produces large consequences’. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations (junctions) or communication links (connections). In the theory of graphs, the vulnerability implies a lack of resistance (weakness) of graph network arising from deletion of vertices or edges or both. Many graph theoretic parameters have been used to describe the vulnerability of communication networks including binding number, rate of disruption, toughness, neighbor-connectivity, integrity, mean integrity, edge-connectivity and tenacity. In the analysis of the vulnerable communication network two quantities are playing vital role, namely (i) the number of elements that are not functioning (ii) the size of the largest remaining (survived) sub network within which mutual communication can still occur. In adverse relationship it is desirable that an opponent's network would be such that above referred two quantities can be made simultaneously small. Here the first parameter provides an information about nodes which can be targeted for more disruption while the later gives the impact of damage after disruption. To estimate these quantities Barefoot et al. [1] have introduced the concept of integrity, which is defined as follows.

**Definition 1.1** The integrity of a graph $G$ is denoted by $I(G)$ and defined by 
$$I(G) = \min \{ |S| + m(G - S) : S \subseteq V(G) \},$$
where $m(G - S)$ is the order of a maximum component of $G - S$.

Many results are reported in a survey article on integrity by Bagga et al. [2]. Some general results on the interrelations between integrity and other graph parameters are investigated by Goddard and Swart [3] while Mamut and Vumar [4] have determined the integrity of middle graph of some graphs. It is also observed that bigger the integrity of network, more reliable functionality of the network after any disruption.
caused by nonfunctional devices (elements). In some respect connectivity is useful for local weaknesses while integrity gives brief account of overall vulnerability of the graph.

**Definition 1.2** A subset \( S \) of \( V(G) \) is called a *dominating set* if for every \( v \in V - S \), there exists a \( u \in S \) such that \( v \) is adjacent to \( u \).

Domination and integrity both are important concepts in their own ways. Theory of domination plays vital role in determining decision making bodies of minimum strength or weakness of a network when certain part of it is paralysed. In the case of disruption of a network, the damage will be more when vital node are under siege. This motivated the study of domination integrity when the sets of nodes disturbed are dominating sets. Sundareswaran and Swaminathan [5] introduced the concept of domination integrity of a graph as a new measure of vulnerability which is defined as follows.

**Definition 1.3** The *domination integrity* of a connected graph \( G \) denoted by \( DI(G) \) and defined as
\[
DI(G) = \min \{|X| + m(G - X) : X \text{ is a dominating set}\},
\]
where \( m(G - X) \) is the order of a maximum component of \( G - X \).

Sundareswaran and Swaminathan [5] have investigated domination integrity of some standard graphs. In the same paper they have investigated domination integrity of Binomial trees and Complete k-ary trees while in [6] they have investigated domination integrity of middle graph of some standard graphs. Same authors in [7] have investigated the domination integrity of powers of cycles while in [8] they have discussed domination integrity of trees. Vaidya and Kothari [9] have discussed domination integrity in the context of some graph operations. Same authors [10] have investigated domination integrity of splitting graph of path \( P_n \) and cycle \( C_n \).

Generally there are three types of problems that can be considered in the field of domination.
1. To find upper or lower bound of any particular dominating parameter with respect to graph parameters like $\delta(G)$, $\Delta(G)$, $\alpha_0(G)$, $\beta_0(G)$, $\kappa(G)$, $\omega(G)$, $\text{diam}(G)$, etc.
2. To study any domination parameter in the context of various graph (s) or graph families.
3. How a particular dominating parameter is affected under various graph operations.

The problems of first two types are largely discussed while the problems of third type are not so often but they are of great importance. The present work is aimed to discuss the problems of third kind in the context of domination integrity.

**Definition 1.4** The *shadow graph* $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $v'$ in $G'$ to the neighbours of the corresponding vertex $u''$ in $G''$.

The present work is intended to obtain domination integrity of shadow graphs of path $P_n$, cycle $C_n$, complete bipartite $K_{m,n}$ and bistar $B_{n,n}$.

### 2. MAIN RESULTS

**Proposition 2.1** [5]

(i) \[
\text{DI} \left( P_n \right) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor + 1; & n = 2, 3, 4, 5, \\
\left\lfloor \frac{n}{3} \right\rfloor + 2; & n \geq 6.
\end{cases}
\]

(ii) \[
\text{DI} \left( C_n \right) = \begin{cases} 
3; & n = 3, 4, \\
\left\lfloor \frac{n}{3} \right\rfloor + 2; & n \geq 5.
\end{cases}
\]
(iii) \( DI (K_{m,n}) = \min \{m, n\} + 1. \)

\[
\begin{align*}
3; & \quad n = 2, 3 \\
5; & \quad n = 4, 5 \\
7; & \quad n = 6, 7 \\
9; & \quad n = 8, 9 \\
11; & \quad n = 10, 11.
\end{align*}
\]

**Theorem 2.2** \( DI \left( D_2 (P_n) \right) = \) \[
\begin{align*}
3; & \quad n = 2, 3 \\
5; & \quad n = 4, 5 \\
7; & \quad n = 6, 7 \\
9; & \quad n = 8, 9 \\
11; & \quad n = 10, 11.
\end{align*}
\]

**Proof:** Consider two copies of \( P_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of the first copy of \( P_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of the second copy of \( P_n \). Let \( G \) be the graph \( D_2 (P_n) \). Then \( |V(G)| = 2n \) and \( |E(G)| = 4(n - 1) \).

For \( n = 2i \) and \( n = 2i + 1 \) where \( 1 \leq i \leq 5 \) consider \( S = \{u_{2k}, v_{2k} : 1 \leq k \leq i\} \). \( S \) is a dominating set of \( G \) as \( u_{2i-1}, v_{2i-1} \in N(u_{2i}) \) and \( u_{2i+1}, v_{2i+1} \in N(v_{2i}) \) for \( i = 1, 2, 3, 4 \). Moreover \( |S| = 2i \) and \( m(G - S) = 1 \).

Now we discuss the minimality of \(|S| + m(G - S)\).

There does not exist any dominating set \( S_1 \) of \( G \) such that \( |S_1| < |S| \) and \( m(G - S_1) = 1 \). It can be checked that for any dominating set \( S_2 \) of \( G \) with \( m(G - S_2) \geq 2 \) then \(|S| + m(G - S) \leq |S_2| + m(G - S_2)\).

Therefore,
\[
\min \{|X| + m(G - X) : X \text{ is a dominating set}\} = |S| + m(G - S) = 2i + 1.
\]

Hence, \( DI \left( D_2 (P_n) \right) = \) \[
\begin{align*}
3; & \quad n = 2, 3 \\
5; & \quad n = 4, 5 \\
7; & \quad n = 6, 7 \\
9; & \quad n = 8, 9 \\
11; & \quad n = 10, 11.
\end{align*}
\]

**Theorem 2.3** For \( n \geq 12 \),
\[
DI \left( D_2 (P_n) \right) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\
\frac{2(n-1)}{3} + 5; & \text{if } n \equiv 1 \pmod{3} \\
\frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]
**Proof:** Consider two copies of $P_n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of $P_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the second copy of $P_n$. Let $G$ be the graph $D_2(P_n)$.

- If $n \equiv 0 (\text{mod } 3)$ (i.e. $n = 3k$), consider $S = \{u_{2+3i}, v_{2+3i} / 0 \leq i \leq k-1\}$ and $|S| = 2k = \frac{2n}{3}$.

- If $n \equiv 1 (\text{mod } 3)$ (i.e. $n = 3k+1$), consider $S = \{u_{2+3i}, v_{2+3i} / 0 \leq i \leq k-1\} \cup \{v_{n-1}\}$ and $|S| = 2k+1 = \frac{2(n-1)}{3} + 1$.

- If $n \equiv 2 (\text{mod } 3)$ (i.e. $n = 3k+2$), consider $S = \{u_{2+3i}, v_{2+3i} / 0 \leq i \leq k\}$ and $|S| = 2k+2 = \frac{2(n-2)}{3} + 2$.

In all the above cases $S$ is a dominating set for $G$ as $u_{1+3t}, u_{3+3t} \in N(u_{2+3t})$ and $v_{1+3t}, v_{3+3t} \in N(v_{2+3t})$ for $t \in \mathbb{N} \cup \{0\}$ moreover $m(G - S) = 4$.

Now we discuss minimality of $|S| + m(G - S)$.

If we consider any dominating set $S_I$ of $G$ with $m(G - S_I) > 4$, then due to construction of $G = D_2(P_n)$ (i.e. to convert $G - S_I$ into disconnect graph the set $S_I$ must contain $u_i$ and $v_i$ both) and as $S_I$ is dominating set, $|S_I| > |S|$. Hence, $|S| + m(G - S) < |S_I| + m(G - S_I)$. If $S_2$ is any dominating set of $G$ with $m(G - S_2) < 4$, then $|S_2| > |S|$. Hence, $|S| + m(G - S) < |S_2| + m(G - S_2)$.

Therefore,

$$|S| + m(G - S) = \min \{|X| + m(G - X) : X \text{ is a dominating set}\} = DI(D_2(P_n)).$$

Hence, for $n \geq 12$

$$DI\left(D_2(P_n)\right) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \equiv 0 (\text{mod } 3) \\
\frac{2(n-1)}{3} + 5; & \text{if } n \equiv 1 (\text{mod } 3) \\
\frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2 (\text{mod } 3).
\end{cases}$$
Remark 2.4 From Proposition 2.1, Theorem 2.2 and Theorem 2.3 we can calculate the difference between domination integrity of $P_n$ and $D_2(P_n)$ as follows:

$$DI\left(D_2(P_n)\right) - DI\left(P_n\right) = \begin{cases} 0; & n = 3 \\ 1; & n = 2, 5 \\ 2; & n = 4 \\ 3; & n = 6 \\ 4; & n = 8, 9 \\ 5; & n = 10, 11 \\ \left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 12 \text{ and } n \equiv 0 \text{ (mod 3)} \text{ or } n \equiv 2 \text{ (mod 3)} \\ \left\lceil \frac{n}{3} \right\rceil + 1; & n \geq 12 \text{ and } n \equiv 1 \text{ (mod 3)}. \end{cases}$$

Theorem 2.5

$$DI\left(D_2(P_n)\right) = \begin{cases} 5; & n = 3, 4 \\ 7; & n = 5, 6 \\ 9; & n = 7, 8 \\ 10; & n = 9 \\ 11; & n = 10. \end{cases}$$

Proof: Consider two copies of $C_n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of $C_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the second copy of $C_n$. Let $G$ be the graph $D_2(C_n)$.

To prove this result we consider following three cases.

Case 1: $n = 3$ to 8

For $n = 2i + 1$ and $n = 2i + 2$ where $i = 1$ to 3, consider $S = \{u_{1+2k}, v_{1+2k}; 0 \leq k \leq i\}$ and $|S| = 2(i+1)$. $S$ is dominating set for $G$ as $u_{2+2k}, v_{2+2k} \in N(u_{1+2k})$ and $u_n, v_n \in N(u_1)$ with $m(G - S) = 1$. If $S_1$ is any dominating set of $G$ with $m(G - S_1) > 1$, then due to construction
of $D_2(C_n)$ and as $S_1$ is a dominating set, $|S_1| + m(G - S_1) > |S| + m(G - S)$. Moreover there does not exist any dominating set $S_2$ with $m(G - S_2) = 1$ and $|S_2| < |S|$. Hence, $|S| + m(G - S) = 2i + 2 + 1$

$$= \min \{|X| + m(G - X) : X \text{ is a dominating set}\}.$$

Therefore, $DI(D_2(C_n)) = \begin{cases} 5; & n = 3, 4 \\ 7; & n = 5, 6 \\ 9; & n = 7, 8. \end{cases}$

**Case 2: $n = 9$**

Consider $S = \{u_1, u_4, u_7, v_1, v_4, v_7\}$. Then $|S| = 6$ and $m(G - S) = 4$. Clearly $S$ is a dominating set of $D_2(C_9)$. Moreover for any other dominating set $S_1$ of $(D_2(C_9))$, we have $|S_1| + m(G - S_1) > |S| + m(G - S) = 10$. Hence, $DI(D_2(C_9)) = 10$.

**Case 3: $n = 10$**

Consider $S = \{u_1, u_3, u_5, u_7, u_9, v_1, v_3, v_5, v_7, v_9\}$. Then $|S| = 10$ and $m(G - S) = 1$. Clearly $S$ is a dominating set of $D_2(C_{10})$. Moreover for any other dominating set $S_1$ of $D_2(C_{10})$, we have $|S_1| + m(G - S_1) > |S| + m(G - S) = 11$. Hence, $DI(D_2(C_{10})) = 11$.

Therefore from above cases, $DI(D_2(C_n)) = \begin{cases} 5; & n = 3, 4 \\ 7; & n = 5, 6 \\ 9; & n = 7, 8 \\ 10; & n = 9 \\ 11; & n = 10. \end{cases}$

**Theorem 2.6** For $n \geq 11$,

$$DI(D_2(P_n)) = \begin{cases} \frac{2n}{3} + 4; & \text{if } n \equiv 0(\text{mod } 3) \\ \frac{2(n-1)}{3} + 6; & \text{if } n \equiv 1(\text{mod } 3) \\ \frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2(\text{mod } 3). \end{cases}$$
Proof: Consider two copies of $C_n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of $C_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the second copy of $C_n$. Let $G$ be the graph $D_2(C_n)$.

- If $n \equiv 0 \pmod{3}$ (i.e. $n = 3k$), consider $S = \{u_{2+3t}, v_{2+3t} / 0 \leq i \leq k-1\}$ and $|S| = 2k = \frac{2n}{3}$.

- If $n \equiv 1 \pmod{3}$ (i.e. $n = 3k+1$) or $n \equiv 2 \pmod{3}$ (i.e. $n = 3k+2$) consider $S = \{u_{2+3t}, v_{2+3t} / 0 \leq i \leq k\} \cup \{v_{n-1}\}$. Then

  $$|S| = 2k + 2 = \frac{2(n-1)}{3} + 2 \quad \text{for } n \equiv 1 \pmod{3} \quad \text{and}$$

  $$|S| = 2k + 2 = \frac{2(n-2)}{3} + 2 \quad \text{for } n \equiv 2 \pmod{3}.$$ 

In all the above cases $S$ is a dominating set for $G$ as $u_{2+3t}, v_{2+3t} \in N(u_{1+3t}), u_{3+3t}, v_{3+3t} \in N(u_{4+3t})$ for $t \in \mathbb{N} \cup \{0\}$ and $u_n, v_n \in N(u_1)$ moreover $m(G - S) = 4$.

Now we discuss minimality of $|S| + m(G - S)$.

If we consider any dominating set $S_1$ of $G$ with $m(G - S_1) > 4$, then due to construction of $G = D_2(C_n)$ (i.e. to convert $G - S_1$ into disconnect graph the set $S_1$ must contain $u_i$ and $v_i$ both) and as $S_1$ is a dominating set, $|S_1| > |S|$. Hence, $|S| + m(G - S) < |S_1| + m(G - S_1)$.

If $S_2$ is any dominating set of $G$ with $m(G - S_2) < 4$, then $|S_2| > |S|$. Hence, $|S| + m(G - S) < |S_2| + m(G - S_2)$.

Therefore,

$$|S| + m(G - S) = \min \{ |X| + m(G - X) : X \text{ is a dominating set} \} = DI(D_2(C_n)).$$

Hence, for $n \geq 11$,

$$DI(D_2(P_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\
\frac{2(n-1)}{3} + 6; & \text{if } n \equiv 1 \pmod{3} \\
\frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2 \pmod{3}.
\end{cases}$$
**Remark 2.7** From Proposition 2.1, Theorem 2.5 and Theorem 2.6 we can calculate the difference between domination integrity of \( C_n \) and \( D_2(C_n) \) as follows:

\[
DI \left( D_2 \left( C_n \right) \right) - DI \left( C_n \right) = \begin{cases} 
2; & n = 3, 4 \\
3; & n = 5, 6 \\
4; & n = 7, 8 \\
5; & n = 10 \\
6; & n = 9 \\
\left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 11.
\end{cases}
\]

**Theorem 2.8.** \( DI \left( D_2(K_{m,n}) \right) = 2m + 1 \), where \( m \leq n \).

**Proof:** Consider two copies of complete bipartite graph \( K_{m,n} \) where \( m \leq n \) with \( A = \{v_1, v_2, \ldots, v_m\} \) and \( B = \{u_1, u_2, \ldots, u_n\} \) are two partitions of first copy of \( K_{m,n} \) where \( C= \{v'_1, v'_2, \ldots, v'_m\} \) and \( D = \{u'_1, u'_2, \ldots, u'_n\} \) are partitions of second copy of \( K_{m,n} \). Let \( G \) be the graph \( D_2(K_{m,n}) \).

Consider \( S=\{v'_1, v'_2, \ldots, v'_m, v'_1, v'_2, \ldots, v'_m\} \). Then \( |S| = 2m \) and \( m(G-S) = 1 \).

\( S \) is a dominating set of \( G \) as \( u_i, u'_i \in N(v_1) \) for \( 1 \leq i \leq n \). Now we discuss the minimality of \( |S| + m(G-S) \).

For above \( S \), \( m(G-S) = 1 \) which is minimum. If we consider any dominating set \( S_1 \) of \( G \) with \( |S_1|=t \leq 2m=|S| \), then \( m(G-S_1) = 2(m+n)-t \).

Therefore \( |S_1| + m(G-S_1) = t + 2(m + n) - t = 2(m + n) > 2m + 1 = |S| + m(G-S) \).

Hence, \( |S| + m(G-S) = 2m + 1 = min\{ |X| + m(G-X) : X \text{ is a dominating set} \} = DI \left( D_2(K_{m,n}) \right) \).

Therefore \( DI \left( D_2(K_{m,n}) \right) = 2m + 1 \), where \( m \leq n \).
Theorem 2.9. $DI(B_{n,n}) = 3$.

Proof: Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices.

Consider $S = \{u, v\}$. Then $|S| = 2$ and $m(G - S) = 1$. Clearly $S$ is a dominating set of $B_{n,n}$ as $u_i \in N(u)$ and $v_i \in N(v)$ for $1 \leq i \leq n$. Moreover if $S_1$ is any dominating set of $B_{n,n}$ other than $S$, then $|S_1| > |S|$. Hence,

$$|S| + m(G - S) = 2 + 3 = 5$$

Therefore $DI(B_{n,n}) = 3$.

Theorem 2.10. $DI(D_2(B_{n,n})) = 5$.

Proof: Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$ where $u_i, v_i, u'_i, v'_i$ are pendant vertices. Let $G$ be the graph $D_2(B_{n,n})$.

Consider $S = \{u, v, v', v'\}$. Then $|S| = 4$ and $m(G - S) = 1$. $S$ is a dominating set of $G$ as $u_i, v_i \in N(u)$ and $v_i, v'_i \in N(v)$ for $1 \leq i \leq n$. Now we discuss the minimality of $|S| + m(G - S)$. For above $S$, $m(G - S) = 1$ which is minimum.

We claim that there does not exist any dominating set $S_1$ of $G$ such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Due to construction of $D_2(B_{n,n})$ if $S_1$ is any dominating set, then $S_1 \cap S \neq \emptyset$.

**Case 1:** $S_1 \subset S$

If we consider any dominating set $S_1 \subset S$, then clearly $m(G - S_1) = n > 1$ and $|S_1| + m(G - S_1) > |S| + m(G - S)$.

**Case 2:** If $S_1$ is any one of following sets $\{u, v, u_i\}$, $\{u, v', v_i\}$, $\{u', v', u_i\}$, $\{u', v, v_i\}$ for any fixed $i$, then $m(G - S_1) = n > 1$.

Hence, $|S_1| + m(G - S_1) = n + 3 > |S| + m(G - S)$.

Therefore $|S| + m(G - S) = min\{|X| + m(G - X) : X \text{ is a dominating set}\} = DI(D_2(B_{n,n}))$.

Hence, $DI(D_2(B_{n,n})) = 5$.  


The rapid growth of various modes of communication have emerged as a search for sustainable and secured network. The vulnerability of network is an important issue with special reference to defence objectives. We take up this problem in the context of expansion of graph network by means of shadow of a graph and discuss domination integrity of $D_2(P_n)$, $D_2(C_n)$, $D_2(K_{m,n})$ and $D_2(B_{n,n})$.

References


DOMINATION INTEGRITY OF TOTAL GRAPHS

S. K. VAIDYA¹, N. H. SHAH² §

Abstract. The domination integrity of a simple connected graph $G$ is a measure of vulnerability of a graph. Here we determine the domination integrity of total graphs of path $P_n$, cycle $C_n$ and star $K_{1,n}$.

Keywords: Integrity, Domination Integrity, total graph.

AMS Subject Classification: 05C38,05C69,05C76.

1. Introduction

We begin with simple, finite, connected and undirected graph $G$ with vertex set $V(G)$ and edge set $E(G)$. For any undefined terminology and notation related to the concept of domination we refer to Haynes et al. [7] while for the fundamental concepts in graph theory we rely upon Harary [6]. In the remaining portion of this section we will give brief summary of definitions and information related to the present work.

The vulnerability of network have been studied in various contexts including road transportation system, information security, structural engineering and communication network. A graph structure is vulnerable if ‘any small damage produces large consequences’. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations (junctions) or communication links (connections). In the theory of graphs, the vulnerability implies a lack of resistance(weakness) of graph network arising from deletion of vertices or edges or both. Communication networks must be so designed that they do not easily get disrupted under external attack and even if they get disturbed then they should be easily reconstructible. Many graph theoretic parameters have been introduced to describe the vulnerability of communication networks including binding number, rate of disruption, toughness, neighbor-connectivity, integrity, mean integrity, edge-connectivity and tenacity. In the analysis of the vulnerable communication network two quantities are playing vital role, namely (i) the number of elements that are not functioning (ii) the size of the largest remaining (survived) sub network within which mutual communication can still occur. In adverse relationship it is desirable that an opponent’s network would be such that the above referred two quantities can be made simultaneously small. Here the first parameter provides an information about nodes which can be targeted for more disruption while the later gives the impact of damage after disruption. To estimate these quantities

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Barefoot et al. [2] have introduced the concept of integrity, which is defined as follows.

**Definition 1.1** The integrity of a graph $G$ is denoted by $I(G)$ and defined by $I(G) = \min \{ |S| + m(G - S) : S \subseteq V(G) \}$ where $m(G - S)$ is the order of a maximum component of $G - S$.

Many results are reported in a survey article on integrity by Bagga et al. [1]. Some general results on the interrelations between integrity and other graph parameters are investigated by Goddard and Swart [5] while Mamut and Vumar [8] have determined the integrity of middle graph of some graphs. It is also observed that bigger the integrity of network, more reliable functionality of the network after any disruption caused by non-functional devices (elements). The connectivity is useful to identify local weaknesses in some respect while integrity gives brief account of vulnerability of the graph network.

**Definition 1.2** A subset $S$ of $V(G)$ is called dominating set if for every $v \in V - S$, there exist a $u \in S$ such that $v$ is adjacent to $u$.

**Definition 1.3** The minimum cardinality of a minimal dominating set in $G$ is called the domination number of $G$ denoted as $\gamma(G)$ and the corresponding minimal dominating set is called a $\gamma$-set of $G$.

The theory of domination plays vital role in determining decision making bodies of minimum strength or weakness of a network when certain part of it is paralysed. In the case of disruption of a network, the damage will be more when vital node are under siege. This motivated the study of domination integrity when the sets of nodes disturbed are dominating sets. Sundareswaran and Swaminathan [9] have introduced the concept of domination integrity of a graph as a new measure of vulnerability which is defined as follows.

**Definition 1.4** The domination integrity of a connected graph $G$ denoted by $DI(G)$ and defined as $DI(G) = \min \{|X| + m(G - X) : X \text{ is a dominating set}\}$ where $m(G - X)$ is the order of a maximum component of $G - X$.

Sundareswaran and Swaminathan [9] have investigated domination integrity of some standard graphs. In the same paper they have investigated domination integrity of Bimomial trees and Complete k-ary trees while in [10] they have investigated domination integrity of middle graph of some standard graphs. Same authors in [11] have investigated the domination integrity of powers of cycles while in [12] they have discussed domination integrity of trees. Vaidya and Kothari [13, 14] have discussed domination integrity in the context of some graph operations and also investigated domination integrity of splitting graph of path $P_n$ and cycle $C_n$ while Vaidya and Shah [15] have investigated domination integrity of shadow graphs of $P_n, C_n, K_{m,n}$ and $B_{n,n}$.

Generally following types of problems are generally considered in the field of domination.
1. Introduce new type of domination parameters by combining domination with other graph theoretical property.
2. To find upper or lower bound of any particular dominating parameter with respect to graph parameters like $\delta(G), \Delta(G), \alpha_0(G), \beta_0(G), \kappa(G), \omega(G), \text{diam}(G)$ etc.
3. To obtain exact domination number for some graphs or graph families.
4. Characterize the graph or graph family which satisfies certain dominating parameter.
5. Study of algorithmic and complexity results for particular dominating parameter.
6. How a particular dominating parameter is affected under various graph operations.

The problems of first five types are largely discussed while the problems of sixth type are not so often but they are of great importance. The present work is aimed to discuss the problems of sixth kind in the context of domination integrity. We investigate domination integrity for total graphs of $P_n$, $C_n$ and $K_{1,n}$.

The concept of total graph $T(G)$ of graph $G$ was introduced by Behzad [3] which is defined as follows:

**Definition 1.5** The total graph $T(G)$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$.

It is obvious that $T(G)$ always contains both $G$ and its line graph $L(G)$ as a induced subgraphs. Also $T(G)$ is the largest graph formed by adjacent and incidence relation between graph elements.

Dündar and Aytaç [4] discussed the integrity of total graphs via certain graph parameters while we discuss the domination integrity of total graphs.

**Proposition 1.6** [4]

(i) $\gamma(T(P_n)) = \left\{ \begin{array}{ll}
\left\lfloor \frac{|V(T(P_n))|}{5} \right\rfloor + 1; & \text{if } |V(T(P_n))| \equiv 0 \pmod{5} \\
\left\lfloor \frac{|V(T(P_n))|}{5} \right\rfloor + 1; & \text{otherwise}
\end{array} \right.$

(ii) $\gamma(T(C_n)) = \left\{ \begin{array}{ll}
\left\lfloor \frac{|V(T(C_n))|}{5} \right\rfloor + 1; & \text{if } |V(T(C_n))| \equiv 0 \pmod{5} \\
\left\lfloor \frac{|V(T(C_n))|}{5} \right\rfloor + 1; & \text{otherwise}
\end{array} \right.$

2. **Main Results**

**Proposition 2.1.** [9]

(i) $DI(P_n) = \left\{ \begin{array}{ll}
\left\lfloor \frac{n}{2} \right\rfloor + 1; & n = 2, 3, 4, 5 \\
\left\lfloor \frac{n}{3} \right\rfloor + 1; & n \geq 6
\end{array} \right.$

(ii) $DI(C_n) = \left\{ \begin{array}{ll}
3; & n = 3, 4 \\
\left\lfloor \frac{n}{3} \right\rfloor + 2; & n \geq 5
\end{array} \right.$

(iii) $DI(K_{m,n}) = \min\{m,n\} + 1$

**Theorem 2.2.** $DI(T(P_n)) = n + 1$ for $n = 2$ to 7.

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n$ and $u_1, u_2, \ldots, u_{n-1}$ be the added vertices corresponding to edges $e_1, e_2, \ldots, e_{n-1}$ to obtain $T(P_n)$. Let $G$ be the graph
For different possibilities of Case 4:

Proof. Consider $S$ then $S$ is dominating set of $T(P_3)$ and $m(G - S) = 2$, so $|S| + m(G - S) = 4$. There does not exist any dominating set $S_1$ of $T(P_3)$ such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $DI(T(P_3)) = 4$.

Case 3: $n = 4$

Consider $S = \{v_2, u_2\}$ then $S$ is dominating set of $T(P_4)$ and $m(G - S) = 2$, so $|S| + m(G - S) = 5$. If $S_1$ is any dominating set of $T(P_4)$ with $|S_1| \leq 2$ then $m(G - S_1) = 5$ so $|S_1| + m(G - S_1) > 5$. If we consider any dominating set $S_2$ of $T(P_4)$ such that $m(G - S_2) = 1$ then $|S_2| \geq 4$ hence, $|S_2| + m(G - S_2) \geq 5$. Therefore, $DI(T(P_4)) = 5$.

Case 4: $n = 5$

Consider $S = \{v_2, u_2, u_4\}$ then $S$ is dominating set of $T(P_5)$ and $m(G - S) = 2$, so $|S| + m(G - S) = 6$. If $S_1$ is any dominating set of $T(P_5)$ with $|S_1| = 3$ then $m(G - S_1) \geq 4$ so $|S_1| + m(G - S_1) > 6$. If $S_2$ is any dominating set of $T(P_5)$ with $|S_2| = 2$ then $m(G - S_2) = 7$ so $|S_2| + m(G - S_2) = 9 > 6$. If we consider any dominating set $S_3$ of $T(P_5)$ such that $m(G - S_3) = 1$ then $|S_3| \geq 6$ hence, $|S_3| + m(G - S_3) \geq 7$. Therefore, $DI(T(P_5)) = 6$.

Case 5: $n = 6$

Consider $S = \{v_2, u_2, u_4, v_5\}$ then $S$ is dominating set of $T(P_6)$ and $m(G - S) = 3$, so $|S| + m(G - S) = 7$. If $S_1$ is any dominating set of $T(P_6)$ with $m(G - S_1) \geq 4$ then $|S_1| + m(G - S_1) > 7$. If $S_2$ is any dominating set of $T(P_6)$ with $|S_2| = 2$ then clearly $|S_2| + m(G - S_2) > 7$. Therefore, $DI(T(P_6)) = 7$.

Case 6: $n = 7$

Consider $S = \{v_2, u_2, u_4, v_5, v_7\}$ then $S$ is dominating set of $T(P_7)$ and $m(G - S) = 3$, so $|S| + m(G - S) = 8$. If $S_1$ is any dominating set of $T(P_7)$ with $m(G - S_1) \geq 4$ then $|S_1| + m(G - S_1) = 9 > 6$. If $S_2$ is any dominating set of $T(P_7)$ with $|S_2| = 2$ then clearly $|S_2| + m(G - S_2) > 8$. Therefore, $DI(T(P_7)) = 8$.

Hence $DI(T(P_n)) = n + 1$ for $n = 2$ to 7.

\[\text{Theorem 2.3. For } n \geq 8,\]

\[DI(T(P_n)) = \begin{cases} \frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\ \left\lfloor \frac{2n}{3} \right\rfloor + 4; & \text{if } n \equiv 1 \pmod{3} \\ \frac{2n}{3} + 4; & \text{if } n \equiv 2 \pmod{3} \end{cases}\]

\[\text{Proof. Let } v_1, v_2, \ldots, v_n \text{ be the vertices of path } P_n \text{ and } u_1, u_2, \ldots, u_{n-1} \text{ be the added vertices corresponding to edges } e_1, e_2, \ldots, e_{n-1} \text{ to obtain } T(P_n). \text{ Let } G \text{ be the graph } T(P_n). \]

Proposition 1.6 gives the value of $\gamma(T(P_n))$, here we provide $D(\gamma - \text{set})$ for $T(P_n)$ for different possibilities of $n$ as below:

- If $n \equiv 0 \pmod{5}$ (i.e. $n = 5k$), consider $D = \{u_{2+5i}, u_{4+5i} | 0 \leq i < k\}$.
- If $n \equiv 1 \pmod{5}$ (i.e. $n = 5k + 1$) or $n \equiv 2 \pmod{5}$ (i.e. $n = 5k + 2$), consider $D = \{v_{2+5i}, u_{4+5i}, v_i | 0 \leq i < k\}$.
- If $n \equiv 3 \pmod{5}$ (i.e. $n = 5k + 3$), consider $D = \{v_{2+5i}, u_{4+5j} | 0 \leq i \leq k, 0 \leq j < k\}$. 
If \( n \equiv 4(\text{mod } 5) \) (i.e. \( n = 5k + 4 \)), consider \( D = \{v_{2+5i}, u_{4+5j}, v_n | 0 \leq i \leq k, 0 \leq j < k \}. \)

Hence, \( \gamma(T(P_n)) = \begin{cases} 
\frac{2n-1}{5}; & \text{if } 2n - 1 \equiv 0(\text{mod } 5) \\
\left\lfloor \frac{2n-1}{5} \right\rfloor + 1; & \text{otherwise}
\end{cases} \)

Clearly, \( DI(T(P_n)) \leq |D| + m(G - D) \).................................(i)

Now we define another subset \( S \) of \( V(T(P_n)) \) as below:

- If \( n \equiv 0(\text{mod } 3) \) (i.e. \( n = 3k \)), consider \( S = \{v_{2+3i}, u_{2+3j} | 0 \leq i < k \} \) and \( |S| = 2k \).
- If \( n \equiv 1(\text{mod } 3) \) (i.e. \( n = 3k + 1 \)) or \( n \equiv 2(\text{mod } 3) \) (i.e. \( n = 3k + 2 \)), consider \( S = \{v_{2+3i}, u_{2+3j}, v_n | 0 \leq i < k \} \) and \( |S| = 2k + 1 \).

In all the above cases \( S \) is a dominating set for \( G \) as \( u_{1+3t}, u_{3+3t} \in N(u_{2+3t}) \) and \( v_{1+3t}, v_{3+3t} \in N(v_{2+3t}) \) for \( t \in \mathbb{N} \cup \{0\} \) moreover \( m(G - S) = 4 \).

In order to compare the values of parameters \( |D| + m(G - D) \) and \( |S| + m(G - S) \) and to check the minimality of \( |S| + m(G - S) \), we prepare the Table 1 for random values of \( n \) between 8 to 20.

**Table 1**

| \( n \) | \( 2n - 1 \) | \( |D| \) | \( m(G - D) \) | \( |D| + m(G - D) \) | \( |S| \) | \( m(G - S) \) | \( |S| + m(G - S) \) |
|---|---|---|---|---|---|---|---|
| 8 | 15 | 3 | 12 | 15 | 5 | 4 | 9 |
| 9 | 17 | 4 | 13 | 17 | 6 | 4 | 10 |
| 10 | 19 | 4 | 15 | 19 | 7 | 4 | 11 |
| 11 | 21 | 5 | 16 | 21 | 7 | 4 | 11 |
| 16 | 31 | 7 | 24 | 31 | 11 | 4 | 15 |
| 20 | 39 | 8 | 31 | 39 | 13 | 4 | 17 |

From columns 5 and 8 of Table 1, we can observe that for \( D (\gamma - \text{set}) \) and dominating set \( S \),
\[ |S| + m(G - S) < |D| + m(G - D) \].........................(ii)

We have verified that the above relation (ii) is valid even for larger values of \( n \).

From (i) and (ii), we have,
\[ DI(T(P_n)) \leq |S| + m(G - S) < |D| + m(G - D). \]

Hence, \( DI(T(P_n)) \leq |S| + m(G - S) \).................................(iii)

We claim that \( DI(T(P_n)) = |S| + m(G - S) \).

If we consider any dominating set \( S_1 \) of \( G \) such that, \( |D| \leq |S_1| < |S| \) then due to construction of \( T(P_n) \), it generates large value of \( m(G - S_1) \) such that,
\[ |S| + m(G - S) < |S_1| + m(G - S_1). \]

Let \( S_2 \) be dominating set of \( G \) with minimal cardinality such that \( m(G - S_2) = 3 \) then,
\[ |S| + m(G - S) \leq |S_2| + m(G - S_2), \text{ for } 8 \leq n \leq 13 \text{ and} \]
\[ |S| + m(G - S) < |S_2| + m(G - S_2), \text{ for } n \geq 14. \]

Moreover if \( S_3 \) is any dominating set of \( G \) with \( m(G - S_3) = 2 \) or \( m(G - S_3) = 1 \) then clearly,
\[ |S| + m(G - S) < |S_3| + m(G - S_3) \]
From above discussion we can say that among all dominating sets of \( G \), \( S \) is such that 
\[ |S| + m(G - S) \] 
is minimum. 
Therefore, 
\[ |S| + m(G - S) = \min \{|X| + m(G - X)| X \text{ is a dominating set} \} \]
\[ = DI(T(P_n)). \]

Hence, for \( n \geq 8 \)
\[ DI(T(P_n)) = \begin{cases} \frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\ \lceil \frac{2n}{3} \rceil + 4; & \text{if } n \equiv 1 \pmod{3} \\ \lfloor \frac{2n}{3} \rfloor + 4; & \text{if } n \equiv 2 \pmod{3} \end{cases} \]

**Corollary 2.4.** \( DI(T(P_n)) = DI(P_n) = \begin{cases} 1; & n = 2, 3 \\ 2; & n = 4, 5 \\ 3; & n = 6, 7 \end{cases} \]

**Theorem 2.5.** \( DI(T(C_n)) = \begin{cases} 6; & n = 3, 4 \\ 7; & n = 5 \\ 8; & n = 6 \\ 9; & n = 7 \\ 10; & n = 8, 9 \\ 11; & n = 10 \end{cases} \)

**Proof.** In view of Proposition 2.1, Theorem 2.2 and Theorem 2.3 the result is obvious. □

Let \( v_1, v_2, \ldots, v_n \) be the vertices of cycle \( C_n \) and \( u_1, u_2, \ldots, u_n \) be the added vertices corresponding to edges \( e_1, e_2, \ldots, e_n \) to obtain \( T(C_n) \). Let \( G \) be the graph \( T(C_n) \). Then \( |V(G)| = 2n \) and \( |E(T(G))| = 4n \).

To prove this result we consider following two cases.

**Case 1: \( n = 3, 4 \)**

For \( n = 3 \), consider \( S = \{v_2, v_3, u_3\} \) then \( m(G - S) = 3 \). There does not exist any dominating set \( S_1 \) of \( T(C_3) \) such that \( |S_1| + m(G - S_1) < |S| + m(G - S) \). Hence, \( DI(T(C_3)) = 6 \).

For \( n = 4 \), consider \( S = \{v_2, v_4, u_2, u_4\} \) then \( m(G - S) = 2 \). There does not exist any dominating set \( S_1 \) of \( T(C_4) \) such that \( |S_1| + m(G - S_1) < |S| + m(G - S) \). Hence, \( DI(T(C_4)) = 6 \).

**Case 2: \( n = 5 \) to 10**

To explain this case we prepare the following Table 2.

The Table 2 gives dominating set \( S \) and corresponding values of \( m(G - S) \) for \( n = 5 \) to 10.

It can be observed that among all dominating sets of \( G \), above given \( S \) gives the minimum value of \( |S| + m(G - S) \).

Hence, for \( n = 3 \) to 10
Table 2

| n   | S                      | |S| | m(G - S) | |S| + m(G - S) |
|-----|------------------------|---|----|--------|-------------|
| 5   | \{v_2, u_2, u_4, v_5\} | 4 | 3  | 7      |             |
| 6   | \{v_2, u_2, u_5, u_6\} | 4 | 4  | 8      |             |
| 7   | \{v_2, u_2, u_4, v_5, u_7, v_7\} | 6 | 3  | 9      |             |
| 8   | \{v_2, u_2, u_4, v_5, u_7, v_7, v_8\} | 7 | 3  | 10     |             |
|     | \{v_2, u_2, u_5, u_6, v_8, u_8\} | 6 | 4  | 10     |             |
| 9   | \{v_2, u_2, v_5, u_6, v_7, u_7\} | 6 | 4  | 10     |             |
| 10  | \{v_2, u_2, u_4, v_5, u_7, v_7, u_9, v_{10}\} | 8 | 3  | 11     |             |

\[ \text{DI}(T(C_n)) = \begin{cases} 
6; & n = 3, 4 \\
7; & n = 5 \\
8; & n = 6 \\
9; & n = 7 \\
10; & n = 8, 9 \\
11; & n = 10 
\end{cases} \]

**Theorem 2.6.** For \( n \geq 11 \)

\[ \text{DI}(T(C_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \geq 11 & \text{and } n \equiv 0(\text{mod } 3) \\
\frac{2(n+2)}{3} + 4; & \text{if } n \geq 11 & \text{and } n \equiv 1(\text{mod } 3) \\
\frac{2(n+1)}{3} + 4; & \text{if } n \geq 11 & \text{and } n \equiv 2(\text{mod } 3) 
\end{cases} \]

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of cycle \( C_n \) and \( u_1, u_2, \ldots, u_n \) be the added vertices corresponding to edges \( e_1, e_2, \ldots, e_n \) to obtain \( T(C_n) \). Let \( G \) be the graph \( T(C_n) \). Then \( |V(G)| = 2n \) and \( |E(T(G))| = 4n \).

Proposition 1.6 gives the value of \( \gamma(T(C_n)) \), here we provide \( D(\gamma - \text{set}) \) for \( T(C_n) \) for different possibilities of \( n \) as below:

- If \( n \equiv 0(\text{mod } 5) \) (i.e. \( n = 5k \)), consider \( D = \{v_{2+5i}, u_{4+5i} | 0 \leq i < k \} \).
- If \( n \equiv 1(\text{mod } 5) \) (i.e. \( n = 5k + 1 \)) or \( n \equiv 2(\text{mod } 5) \) (i.e. \( n = 5k + 2 \)), consider \( D = \{v_{2+5i}, u_{4+5i}, v_n | 0 \leq i < k \} \).
- If \( n \equiv 3(\text{mod } 5) \) (i.e. \( n = 5k + 3 \)), consider \( D = \{v_{2+5i}, u_{4+5j} | 0 \leq i \leq k, 0 \leq j < k \} \).
- If \( n \equiv 4(\text{mod } 5) \) (i.e. \( n = 5k + 4 \)), consider \( D = \{v_{2+5i}, u_{4+5j}, v_n | 0 \leq i \leq k, 0 \leq j < k \} \).

Hence, \( \gamma(T(C_n)) = \begin{cases} 
\frac{2n}{5}; & \text{if } 2n \equiv 0(\text{mod } 5) \\
\left\lfloor \frac{2n}{5} \right\rfloor + 1; & \text{otherwise} 
\end{cases} \)

Clearly, \( \text{DI}(T(C_n)) \leq |D| + m(G - D) \) .................. (iv)

Now we define another subset \( S \) of \( V(T(C_n)) \) as below:

- If \( n \equiv 0(\text{mod } 3) \) (i.e. \( n = 3k \)) and \( n \equiv 2(\text{mod } 3) \) (i.e. \( n = 3k - 1 \)), consider \( S = \{v_{2+3i}, u_{2+3i} | 0 \leq i < k \} \) and \( |S| = 2k \).
If \( n \equiv 1 \pmod{3} \) (i.e. \( n = 3k + 1 \)), consider \( S = \{v_{2+3i}, u_{2+3i} | 0 \leq i < k\} \cup \{v_n, u_n\} \) and \( |S| = 2(k + 1) = \frac{2(n+2)}{3} \).

In all the above cases \( S \) is a dominating set for \( G \) as \( u_{1+3t}, u_{3+3t} \in N(u_{2+3t}) \) and \( v_{1+3t}, v_{3+3t} \in N(v_{2+3t}) \) for \( t \in \mathbb{N} \cup \{0\} \) moreover \( m(G - S) = 4 \).

In order to compare the values of parameters \( |D| + m(G - D) \) and \( |S| + m(G - S) \) as well as to check the minimality of \( |S| + m(G - S) \), we prepare the Table 3 for random values of \( n \) between 11 to 25.

### Table 3

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From columns 5 and 8 of Table 3, we can observe that for \( D \ (\gamma - set) \) and dominating set \( S \),

\[ |S| + m(G - S) < |D| + m(G - D) \]

We have verified that the above relation (v) is valid even for larger values of \( n \).

From (iv) and (v), we have,

\[ DI(T(C_n)) \leq |S| + m(G - S) < |D| + m(G - D) \]

Hence, \( DI(T(C_n)) \leq |S| + m(G - S) \) \( \vdots \) \( vi \)

We claim that \( DI(T(C_n)) = |S| + m(G - S) \).

If we consider any dominating set \( S_1 \) of \( G \) such that, \( |D| \leq |S_1| < |S| \) then due to construction of \( T(C_n) \), it generates large value of \( m(G - S_1) \) such that,

\[ |S| + m(G - S) < |S_1| + m(G - S_1) \]

Let \( S_2 \) be dominating set of \( G \) with minimal cardinality such that \( m(G - S_2) = 3 \) then,

\[ |S| + m(G - S) \leq |S_2| + m(G - S_2) \]

for \( n = 13 \) and \( |S| + m(G - S) < |S_2| + m(G - S_2) \), for \( n = 11, 12 \) and \( n \geq 14 \).

Moreover if \( S_3 \) is any dominating set of \( G \) with \( m(G - S_3) = 2 \) or \( m(G - S_3) = 1 \) then clearly,

\[ |S| + m(G - S) < |S_3| + m(G - S_3) \]

From above discussion we can say that among all dominating sets of \( G \), \( S \) is such that \( |S| + m(G - S) \) is minimum.

Therefore,

\[ |S| + m(G - S) = min\{|X| + m(G - X)|X \text{ is a dominating set}\} = DI(T(C_n)) \]

Hence, for \( n \geq 11 \)
In view of Proposition 2.1, Theorem 2.5 and Theorem 2.6 the proof is obvious.

\[ DI(T(C_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 0(\text{mod } 3) \\
\frac{2(n+2)}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 1(\text{mod } 3) \\
\frac{2(n+1)}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 2(\text{mod } 3)
\end{cases} \]

Corollary 2.7. \( DI(T(C_n)) - DI(C_n) = \begin{cases} 
3; & n = 3, 4, 5 \\
4; & n = 6, 7 \\
5; & n = 8, 9, 10 \\
\frac{n}{3} + 2; & n \geq 11 \text{ and } n \equiv 0(\text{mod } 3) \\
\lceil \frac{n}{3} \rceil + 2; & n \geq 11 \text{ and } n \equiv 1(\text{mod } 3) \text{ or } n \equiv 2(\text{mod } 3)
\end{cases} \)

Proof. In view of Proposition 2.1, Theorem 2.5 and Theorem 2.6 the proof is obvious.

Theorem 2.8. \( DI(T(K_{1,n})) = n + 2. \)

Proof. Let \( v \) be the apex vertex of \( K_{1,n} \) and \( v_1, v_2, \ldots, v_n \) be the pendant vertices of \( K_{1,n} \) and \( u_1, u_2, \ldots, u_n \) be the added vertices corresponding to edges \( e_1, e_2, \ldots, e_n \) to obtain \( T(K_{1,n}). \) Let \( G \) be the graph \( T(K_{1,n}). \)

Consider \( S = \{v, u_1, u_2, \ldots, u_n\} \) then \( |S| = n + 1 \) and \( m(G - S) = 1. \) Clearly \( S \) is a dominating set of \( G \) and \( |S| + m(G - S) = n + 2. \)

For \( S_1 = \{v, u_1, u_2, \ldots, u_{n-1}\} \) then \( |S_1| = n \) and \( m(G - S_1) = 2 \) and \( |S_1| + m(G - S_1) = n + 2. \)

For \( S_2 = \{v, u_1, u_2, \ldots, u_{n-2}\} \) then \( |S_2| = n - 1 \) and \( m(G - S_2) = 4 \) and \( |S_2| + m(G - S_2) = n + 3. \)

Similarly for any other dominating set \( S_3 \) of \( G, \) \( |S| + m(G - S) \leq |S_3| + m(G - S_3). \)

\[ |S| + m(G - S) = \min \{|X| + m(G - X)|X \text{ is a dominating set}\} \]

\[ = DI(T(K_{1,n})). \]

Hence, \( DI(T(K_{1,n})) = n + 2. \)

3. Concluding Remarks

The rapid growth of various modes of communication have emerged as a search for sustainable and secured network. The vulnerability of network is an important issue with special reference to defence objectives. We take up this problem in the context of expansion of graph network by means of total graph of a graph and investigate domination integrity of \( T(P_n), T(C_n) \) and \( T(K_{1,n}) \) and from Corollary 2.4 and Corollary 2.7 we conclude that the domination integrity increases in such circumstances. To investigate the domination integrity for line graph, shadow graph and the graph obtained by switching of a vertex in the context of \( P_n \) and \( C_n \) is an open area of research.

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References

Research Article

Some New Results on Prime Cordial Labeling

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A prime cordial labeling of a graph \(G\) with the vertex set \(V(G)\) is a bijection \(f: V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\}\) such that each edge \(uv\) is assigned the label 1 if \(\gcd(f(u), f(v)) = 1\) and 0 if \(\gcd(f(u), f(v)) > 1\); then the number of edges labeled with 1 and the number of edges labeled with 1 differ by at most 1. A graph which admits a prime cordial labeling is called a prime cordial graph. In this work we give a method to construct larger prime cordial graph using a given prime cordial graph \(G\). In addition to this we have investigated the prime cordial labeling for double fan and degree splitting graphs of path as well as bistar. Moreover we prove that the graph obtained by duplication of an edge (spoke as well as rim) in wheel \(W_n\) admits prime cordial labeling.

1. Introduction

We consider a finite, connected, undirected, and simple graph \(G = (V(G), E(G))\) with \(p\) vertices and \(q\) edges which is also denoted as \(G(p, q)\). For standard terminology and notations related to graph theory we follow Balakrishnan and Ranganathan [1] while for any concept related to number theory we refer to Burton [2]. In this section we provide brief summary of definitions and other required information for our investigations.

**Definition 1.** The **Graph labeling** is an assignment of numbers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges), then the labeling is called a **vertex labeling** (edge labeling).

Many labeling schemes have been introduced so far and they are explored as well by many researchers. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. Various applications of graph labeling are reported in the work of Yegnanarayanan and Vaidhyanathan [3]. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [4].

**Definition 2.** A labeling \(f: V(G) \rightarrow \{0, 1\}\) is called **binary vertex labeling** of \(G\) and \(f(v)\) is called the label of the vertex \(v\) of \(G\) under \(f\).

**Notation 1.** If for an edge \(e = uv\), the induced edge labeling \(f^*: E(G) \rightarrow \{0, 1\}\) is given by \(f^*(e) = |f(u) - f(v)|\). Then

\[
\begin{align*}
\nu_f(i) & = \text{number of vertices of } G \text{ having label } i \text{ under } f \\
\epsilon_f(i) & = \text{number of edges of } G \text{ having label } i \text{ under } f^*,
\end{align*}
\]

where \(i = 0\) or 1.

**Definition 3.** A binary vertex labeling \(f\) of a graph \(G\) is called a **cordial labeling** if \(|\nu_f(0) - \nu_f(1)| \leq 1\) and \(|\epsilon_f(0) - \epsilon_f(1)| \leq 1\). A graph \(G\) is **cordial** if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [5].

The notion of prime labeling was originated by Entringer and was introduced by Tout et al. [6].

**Definition 4.** A **prime labeling** of a graph \(G\) is an injective function \(f: V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}\) such that for, every...
A prime cordial labeling of a graph $G$ with vertex set $V(G)$ is a bijection $f : V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\}$ and if the induced function $f^* : E(G) \rightarrow \{0, 1\}$ is defined by

\[
f^*(e = uv) = 1, \quad \text{if } \gcd (f(u), f(v)) = 1,
\]

\[
f^*(e = uv) = 0, \quad \text{otherwise,}
\]

then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits prime cordial labeling is called a prime cordial graph.

Many graphs are proved to be prime cordial in the work of Sundaram et al. [10]. Prime cordial labeling for some cycle related graphs has been discussed by Vaidya and Vihol [11]. Prime cordial labeling in the context of some graph operations has been discussed by Vaidya and Vihol [12] and Vaidya and Shah [13, 14]. Vaidya and Shah [14] have proved that the wheel graph $W_n$ admits prime cordial labeling for $n \geq 8$ while same authors in [15] have discussed prime cordial labeling for some wheel related graphs. Babitha and Baskar Babujee [16] have exhibited prime cordial labeling for some cycle related graphs and discussed the duality of prime cordial labeling. The same authors in [17] have derived some results. Theorem 12.

Let $G(p, q)$ be a prime cordial graph and let $f_1, f_2$ be the prime cordial labeling of $G$. Let $w_1, w_2 \in V$ be the vertices of $G$ such that $f_1(w_1) = 2$ and $f_2(w_2) = 4$. Consider the $K_{2n}$ with bipartition $V = V_1 \cup V_2$ with $V_1 = \{v_1, v_2\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$. Now identify the vertices $v_1$ to $w_1$ and $v_2$ to $w_2$ and denote the resultant graph as $G_1$. Then $V(G_1) = V(G) \cup \{u_1, u_2, \ldots, u_n\}$ and $E(G_1) = E(G) \cup \{w_1u_i, w_2u_i | 1 \leq i \leq n\}$ so $|V(G_1)| = p + n$ and $|E(G_1)| = q + 2n$. To define $f : V(G_1) \rightarrow \{1, 2, 3, \ldots, p + n\}$, we consider the following three cases.

Case (i) $n$ is even and $G$ is of any size $q$. Since $G$ is a prime cordial graph, we assign vertex labels such that $f(u_i) = f_1(w_i)$, where $u_i \in V(G) = V(G_1) \cap V(G)$ and $i \leq i \leq p$:

\[
f(u_i) = p + i, \quad i \leq i \leq n.
\]

Since $n$ is even and $f_1(w_1) = 2$ and $f_1(w_2) = 4$, $w_1$ and $w_2$ are adjacent to each $u_i$, $1 \leq i \leq n$. And this vertex assignment generates $n$ edges with label 1 and $n$ edges with label 0. Following Table 1 gives edge condition for prime cordial labeling for $G_1$ under $f$.

From Table 1, we have $|e_f(0) - e_f(1)| \leq 1$.

Case (ii) $(n, p, q)$ are odd with $e_f(0) = |q/2|$. Here $p$ and $q$ both are odd and $G$ is a prime cordial graph with $e_f(0) = |q/2|$. Since $G$ is a prime cordial graph, we keep the vertex label of all the vertices of $G$ in $G_1$ as it is. Therefore $f(u_i) = f_1(w_i)$, where $w_i \in V(G)$ and $i \leq i \leq p$:

\[
f(u_i) = p + i, \quad i \leq i \leq n.
\]

Since $n$ is odd and $f_1(w_1) = 2$ and $f_1(w_2) = 4$, $w_1$ and $w_2$ are adjacent to each $u_i$, $1 \leq i \leq n$. And this vertex assignment generates $n + 1$ edges with label 0 and $n - 1$ edges with label 1.
Therefore edge conditions for $G_1$ under $f$ are $e_f(0) = \lfloor q/2 \rfloor + n + 1$ and $e_f(1) = \lfloor q/2 \rfloor + n - 1$. Therefore, $e_f(0) - 1 = e_f(1)$. Hence, $|e_f(0) - e_f(1)| = 1$ for graph $G_1$.

Case (iii) ($n$ is odd, $p$ is even, and $q$ is odd with $e_f(0) = \lfloor q/2 \rfloor$). Here $p$ is even, $q$ is odd, and $G$ is a prime cordial graph with $e_f(0) = \lfloor q/2 \rfloor$.

Since $G$ is a prime cordial graph, we keep the vertex label of all the vertices of $G$ in $G_1$ as it is. Therefore $f(w_i) = f_1(w_i)$, where $w_i \in V(G)$ and $i \leq i \leq p$:

$$f(u_i) = p + i, \quad i \leq i \leq n.$$  

Since $n$ is odd and $f_1(w_1) = 2$ and $f_1(w_2) = 4, w_1$ and $w_2$ are adjacent to each other, $i \leq i \leq n$. And this vertex assignment generates $n - 1$ edges with label 0 and $n + 1$ edges with label 1.

Therefore edge conditions for $G_1$ under $f$ are $e_f(0) = \lfloor q/2 \rfloor + n - 1$ and $e_f(1) = \lfloor q/2 \rfloor + n + 1$. Therefore, $e_f(0) - e_f(1) = 1$. Hence, $|e_f(0) - e_f(1)| \leq 1$ for graph $G_1$.

Hence, in all the cases discussed above, $G_1$ admits prime cordial labeling.

Illustration 1. Consider the graph $G$ as shown in Figure 1, with $p = 7$ and $q = 9$. $G$ is a prime cordial graph with $e_f(0) = 4, e_f(1) = 5$. Take $n = 3$ and construct graph $G_1$. In accordance with Case (ii) of Theorem 12, a prime cordial labeling of $G_1$ is as shown in Figure 1. Here $e_f(0) = 8, e_f(1) = 7$.

Theorem 13. Double fan $DF_n$ is a prime cordial graph for $n = 8$ and $n \geq 10$.

Proof. Let $DF_n$ be the double fan with apex vertices $u_1, u_2$ and $v_1, v_2, \ldots, v_n$ are vertices common path. Then $|V(DF_n)| = n + 2$ and $|E(DF_n)| = 3n - 1$. To define $f : V(G) \rightarrow \{1, 2, 3, \ldots, n + 2\}$, we consider the following cases.

Case 1 ($n = 3$ to 7 and $n = 9$). In order to satisfy the edge condition for prime cordial labeling in $DF_3$, it is essential to label four edges with label 0 and four edges with label 1 out of eight edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most one edge and 1 label for at least seven edges. That is, $|e_f(0) - e_f(1)| = 6 > 1$. Hence, $DF_3$ is not a prime cordial graph.

In order to satisfy the edge condition for prime cordial labeling in $DF_4$, it is essential to label five edges with label 0 and six edges with label 1 out of eleven edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most three edges and 1 label for at least eight edges. That is, $|e_f(0) - e_f(1)| = 5 > 1$. Hence, $DF_4$ is not a prime cordial graph.

In order to satisfy the edge condition for prime cordial labeling in $DF_5$, it is essential to label seven edges with label 0 and seven edges with label 1 out of fourteen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 label for at least ten edges. That is, $|e_f(0) - e_f(1)| = 6 > 1$. Hence, $DF_5$ is not a prime cordial graph.

In order to satisfy the edge condition for prime cordial labeling in $DF_6$, it is essential to label eight edges with label 0 and nine edges with label 1 out of seventeen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most six edges and 1 label for at least eleven edges. That is, $|e_f(0) - e_f(1)| = 5 > 1$. Hence, $DF_6$ is not a prime cordial graph.

In order to satisfy the edge condition for prime cordial labeling in $DF_7$, it is essential to label ten edges with label 0 and ten edges with label 1 out of twenty edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most eight edges and 1 label for at least twelve edges. That is, $|e_f(0) - e_f(1)| = 4 > 1$. Hence, $DF_7$ is not a prime cordial graph.

In order to satisfy the edge condition for prime cordial labeling in $DF_8$, it is essential to label thirteen edges with label 0 and thirteen edges with label 1 out of twenty-six edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most twelve edges and 1 label for at least fourteen edges. That is, $|e_f(0) - e_f(1)| = 2 > 1$. Hence, $DF_8$ is not a prime cordial graph.

Case 2 ($n = 8, 10, 11, 12$). For $n = 8$, $f(u_1) = 2, f(u_2) = 6$ and $f(v_1) = 4, f(v_2) = 8, f(v_3) = 3, f(v_4) = 9, f(v_5) = 10, f(v_6) = 5, f(v_7) = 7$, and $f(v_8) = 1$. Then $e_f(0) = 11, e_f(1) = 12$.

For $n = 10$, $f(u_1) = 2, f(u_2) = 6$ and $f(v_1) = 3, f(v_2) = 9, f(v_3) = 12, f(v_4) = 8, f(v_5) = 4, f(v_6) = 10, f(v_7) = 1, f(v_8) = 5, f(v_9) = 7$, and $f(v_{10}) = 11$. Then $e_f(0) = 15, e_f(1) = 14$.

For $n = 11$, $f(u_1) = 2, f(u_2) = 6$ and $f(v_1) = 3, f(v_2) = 9, f(v_3) = 12, f(v_4) = 8, f(v_5) = 4, f(v_6) = 10, f(v_7) = 5, f(v_8) = 7, f(v_9) = 11, f(v_{10}) = 13$, and $f(v_{11}) = 1$. Then $e_f(0) = 16, e_f(1) = 16$.

For $n = 12$, $f(u_1) = 2, f(u_2) = 6$ and $f(v_1) = 3, f(v_2) = 9, f(v_3) = 12, f(v_4) = 8, f(v_5) = 4, f(v_6) = 14, f(v_7) = 10, f(v_8) = 5, f(v_9) = 7, f(v_{10}) = 11, f(v_{11}) = 13$, and $f(v_{12}) = 1$. Then $e_f(0) = 18, e_f(1) = 17$.

Now for the remaining three cases let

$$k = \left\lfloor \frac{n + 2}{2} \right\rfloor, \quad m = \left\lfloor \frac{n + 2}{3} \right\rfloor, \quad t_1 = \left\lfloor \frac{3n - 1}{2} \right\rfloor, \quad t_2 = 3k - 7 + \left\lfloor \frac{m}{2} \right\rfloor, \quad t_3 = \text{largest even number} \leq n + 2, \quad t_4 = \text{largest odd number} \leq n + 2.$$  

Case $3 (t_1 = t_2)$. Consider

$$f(u_1) = 2, \quad f(u_2) = 6.$$  

(7)
For the vertices $v_1, v_2, v_3, \ldots, v_n$ we assign the vertex labels in the following order: $1, t_3, t_3 - 2, t_3 - 4, \ldots, 14, 12, 10, 8, 4, 3, 5, 7, 9, \ldots, t_4 - 2, t_4$.

**Case 4** ($t_1 > t_2$). Consider

$$f(u_1) = 2, \quad f(u_2) = 6. \quad (8)$$

Let $t_5 = t_1 - t_2$. Consider

$$f(v_1) = 3, \quad f(v_2) = 9, \quad f(v_3) = 15, \quad f(v_4) = 21, \quad \vdots \quad f(v_{t_5}) = 3(2t_5 - 1), \quad f(v_{t_5+1}) = f(v_{t_5}) + 6.$$  

Now for remaining vertices $v_{t_5+2}, v_{t_5+3}, \ldots, v_n$ we assign the labels $1, t_5, t_5 - 2, t_5 - 4, \ldots, 14, 12, 10, 8, 4, 3, 5, 7, \ldots$ all the odd numbers in ascending order.

**Case 5** ($t_2 > t_1$). Let $t_6 = t_2 - t_1$.

**Sub-Case 1.** $n$ is even. Consider

$$f(u_1) = 2, \quad f(u_2) = 6. \quad (10)$$

For the vertices $v_1, v_2, v_3, \ldots, v_n$ we assign the vertex labels in the following order: $n + 2, n + 1, n - 1, n - 2, n - 3, n + 2 - 2t_6, n + 2 - 2(t_6 + 1), n + 2 - 2(t_6 + 2), \ldots, 10, 8, 4, 3, 5, 7, \ldots$ remaining odd numbers in ascending order.

**Sub-Case 2.** $n$ is odd. Consider

$$f(u_1) = 2, \quad f(u_2) = 6. \quad (11)$$

For the vertices $v_1, v_2, v_3, \ldots, v_n$ we assign the vertex labels in the following order: $n + 2, n + 1, n - 1, n - 2, n - 3, n + 2 - 2t_6, n + 2 - 2(t_6 + 1), n + 2 - 2(t_6 + 2), \ldots, 10, 8, 4, 3, 5, 7, \ldots$ remaining odd numbers in ascending order.

In view of the above defined labeling pattern for Cases 3, 4, and 5, we have

$$e_f(0) = \left\lceil \frac{3n - 1}{2} \right\rceil, \quad e_f(1) = \left\lfloor \frac{3n - 1}{2} \right\rfloor. \quad (12)$$

Thus, we have $|e_f(0) - e_f(1)| \leq 1$. Hence, $DF_n$ is a prime cordial graph for $n = 8$ and $n \geq 10$.

**Illustration 2.** For the graph $DF_{15}$, $|V(DF_{15})| = 17$ and $|E(DF_{15})| = 44$. In accordance with Theorem 13 we have $k = 5, m = 5, t_1 = 22, \text{ and } t_2 = 20$. Here $t_1 > t_2$ so labeling pattern described in Case 4 will be applicable and $t_5 = 2$. The corresponding prime cordial labeling is shown in Figure 2. Here $e_f(0) = 22 = e_f(1)$.

**Illustration 3.** For the graph $DF_{37}$, $|V(DF_{37})| = 39$ and $|E(DF_{37})| = 110$. In accordance with Theorem 13, we have $k = 19, m = 13, t_1 = 55, \text{ and } t_2 = 57$. Here $t_2 > t_1$ and $n = 37$ so labeling pattern described in Sub-Case 2 of Case 5 will be applicable and $t_6 = 2$. And corresponding labeling pattern is as below:

$$f(u_1) = 2, \quad f(u_2) = 6. \quad (13)$$
For the vertices \( v_1, v_2, \ldots, v_{33} \) we assign the vertex labels 39, 38, 37, 36, 35, 34, 32, 30, 28, 26, 24, 22, 20, 18, 16, 14, 12, 10, 8, 4, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, and 33, respectively, where \( e_f(0) = 55 = e_f(1) \). Therefore \( DF_{G'} \) is a prime cordial graph.

**Theorem 14.** The graph obtained by duplication of an arbitrary rim edge by an edge in \( W_n \) is a prime cordial graph, where \( n \geq 6 \).

**Proof.** Let \( v_0 \) be the apex vertex of \( W_n \) and let \( v_1, v_2, \ldots, v_n \) be the rim vertices. Without loss of generality we duplicate the rim edge \( e = v_1v_2 \) by an edge \( e' = u_1u_2 \) and call the resultant graph as \( G \). Then \(|V(G)| = n+3 \) and \(|E(G)| = 2n+5 \). To define \( f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, n+3\} \), we consider the following four cases.

**Case 1** \((n = 3, 4, 5)\). For \( n = 3 \), to satisfy the edge condition for prime cordial labeling, it is essential to label five edges with label 0 and six edges with label 1 out of eleven edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 label for at least seven edges. That is, \(|e_f(0) - e_f(1)| = 3 > 1 \). Hence, for \( n = 3 \), \( G \) is not a prime cordial graph.

For \( n = 4 \), to satisfy the edge condition for prime cordial labeling, it is essential to label six edges with label 0 and seven edges with label 1 out of thirteen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 label for at least nine edges. That is, \(|e_f(0) - e_f(1)| = 5 > 1 \). Hence, for \( n = 4 \), \( G \) is not a prime cordial graph.

For \( n = 5 \), to satisfy the edge condition for prime cordial labeling it is essential to label seven edges with label 0 and eight edges with label 1 out of fifteen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most six edges and 1 label for at least nine edges. That is, \(|e_f(0) - e_f(1)| = 3 > 1 \). Hence, for \( n = 5 \), \( G \) is not a prime cordial graph.

**Case 2** \((n = 6 \text{ to } 10)\). For \( n = 6 \), \( f(u_4) = 4 \), \( f(u_5) = 8 \) and \( f(v_0) = 6 \), \( f(v_4) = 10 \), \( f(v_5) = 2 \), \( f(v_2) = 3 \), \( f(v_3) = 7 \), \( f(v_4) = 5 \), \( f(v_5) = 1 \), and \( f(v_6) = 2 \). Then \( e_f(0) = 8 \), \( e_f(1) = 9 \).

For \( n = 7 \), \( f(u_4) = 4 \), \( f(u_5) = 8 \) and \( f(v_0) = 6 \), \( f(v_1) = 3 \), \( f(v_2) = 9 \), \( f(v_3) = 7 \), \( f(v_4) = 5 \), \( f(v_5) = 10 \), \( f(v_6) = 1 \), and \( f(v_7) = 2 \). Then \( e_f(0) = 4 \), \( e_f(1) = 5 \).

For \( n = 8 \), \( f(u_4) = 11 \), \( f(u_5) = 10 \) and \( f(v_0) = 6 \), \( f(v_1) = 3 \), \( f(v_2) = 2 \), \( f(v_3) = 8 \), \( f(v_4) = 4 \), \( f(v_5) = 7 \), \( f(v_6) = 1 \), and \( f(v_7) = 5 \). Then \( e_f(0) = 10 \), \( e_f(1) = 11 \).

For \( n = 9 \), \( f(u_4) = 11 \), \( f(u_5) = 12 \) and \( f(v_0) = 2 \), \( f(v_1) = 1 \), \( f(v_2) = 3 \), \( f(v_3) = 6 \), \( f(v_4) = 8 \), \( f(v_5) = 4 \), \( f(v_6) = 10 \), \( f(v_7) = 5 \), \( f(v_8) = 7 \), and \( f(v_9) = 9 \). Then \( e_f(0) = 11 \), \( e_f(1) = 12 \).

For \( n = 10 \), \( f(u_4) = 13 \), \( f(u_5) = 12 \) and \( f(v_0) = 2 \), \( f(v_1) = 9 \), \( f(v_2) = 3 \), \( f(v_3) = 6 \), \( f(v_4) = 4 \), \( f(v_5) = 8 \), \( f(v_6) = 10 \), \( f(v_7) = 5 \), \( f(v_8) = 1 \), \( f(v_9) = 7 \), and \( f(v_{10}) = 11 \). Then \( e_f(0) = 12 \), \( e_f(1) = 13 \).

**Case 3** \((n \text{ is even}, n \geq 12)\). Consider

\[
\begin{align*}
&f(v_0) = 2, \\
&f(v_1) = 5, \quad f(v_2) = 10, \\
&f(v_3) = 4, \quad f(v_4) = 8, \\
&f(v_{n+1}) = 12 + 2(i-1), \quad 1 \leq i \leq (n/2) - 5 \\
&f(v_{n/2}) = 6, \quad f(v_{(n/2)+1}) = 3, \\
&f(v_{(n/2)+2}) = 9, \\
&f(v_{n-1}) = 1, \quad f(v_n) = 7, \\
&f(v_{(n/2)+3}) = 11 + 2(i-1), \quad 1 \leq i \leq (n/2) - 4 \\
&f(u_1) = n + 3, \quad f(u_2) = n + 2.
\end{align*}
\]
In view of the above defined labeling pattern for Case 3, we have
\(e_f(0) = n + 3\) and \(e_f(1) = n + 2\) for \(n \equiv 4 (\text{mod } 7)\) and
\(e_f(0) = n + 2\) and \(e_f(1) = n + 3\) for \(n \not\equiv 4 (\text{mod } 7)\).

Case 4 \((n \text{ is odd, } n \geq 11)\). Consider
\[
\begin{align*}
  f(v_0) &= 2, \\
  f(v_1) &= 10, \\
  f(v_2) &= 4, \\
  f(v_3) &= 8, \\
  f(v_{3i}) &= 12 + 2(i - 1), \quad 1 \leq i \leq ((n - 1)/2) - 4 \\
  f(v_{n-1}/2) &= 6, \\
  f(v_{(n+1)/2}) &= 3, \\
  f(v_{n+3}/2) &= 1, \\
  f(v_{n+1-i}) &= 5 + 2(i - 1), \quad 1 \leq i \leq (n-3)/2 \\
  f(u_1) &= n + 2, \\
  f(u_2) &= n + 3.
\end{align*}
\]

In view of the above defined labeling pattern for Case 4, we have
\(e_f(0) = n + 3\) and \(e_f(1) = n + 2\) for \(n \equiv 3 (\text{mod } 5)\) and
\(e_f(0) = n + 2\) and \(e_f(1) = n + 3\) for \(n \not\equiv 4 (\text{mod } 7)\).

Illustration 4. Let \(G\) be the graph obtained by duplication of an arbitrary rim edge by an edge in wheel \(W_{13}\). Then
\(|V(G)| = 16\) and \(|E(G)| = 31\). In accordance with Theorem 14, Case 4 will be applicable and the corresponding prime cordial labeling is shown in Figure 3. Here \(e_f(0) = 16\), \(e_f(1) = 15\).

Theorem 15. The graph obtained by duplication of an arbitrary spoke edge by an edge in wheel \(W_n\) is prime cordial graph, where \(n = 7\) and \(n \geq 9\).

Proof. Let \(v_0\) be the apex vertex of \(W_n\) and let \(v_1, v_2, \ldots, v_n\) be the rim vertices. Without loss of generality we duplicate the spoke edge \(e' = v_0 v_1\) by an edge \(e = u_1 u_2\) and call the resultant graph \(G\). Then \(|V(G)| = n + 3\) and \(|E(G)| = 3n + 2\). To define \(f : V(G) \to \{1, 2, 3, \ldots, n + 3\}\), we consider following three cases.

Case 1 \((n = 3\) to 6 and \(n = 8\)). For \(n = 3\), to satisfy the edge condition for prime cordial labeling it is essential to label five edges with label 0 and six edges with label 1 out of eleven edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 label for at least seven edges. That is, \(|e_f(0) - e_f(1)| = 3 > 1\). Hence, for \(n = 3\), \(G\) is not a prime cordial graph.

For \(n = 4\), to satisfy the edge condition for prime cordial labeling, it is essential to label seven edges with label 0 and seven edges with label 1 out of fourteen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most four edges and 1 label for at least ten edges. That is, \(|e_f(0) - e_f(1)| = 6 > 1\). Hence, for \(n = 4\), \(G\) is not a prime cordial graph.

For \(n = 5\), to satisfy the edge condition for prime cordial labeling, it is essential to label eight edges with label 0 and nine edges with label 1 out of seventeen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most six edges and 1 label for at least eleven edges. That is, \(|e_f(0) - e_f(1)| = 5 > 1\). Hence, for \(n = 5\), \(G\) is not a prime cordial graph.

For \(n = 6\), to satisfy the edge condition for prime cordial labeling, it is essential to label ten edges with label 0 and ten edges with label 1 out of nineteen edges. But all the possible assignments of vertex labels will give rise to 0 labels for at most eight edges and 1 label for at least twelve edges. That is, \(|e_f(0) - e_f(1)| = 4 > 1\). Hence, for \(n = 6\), \(G\) is not a prime cordial graph.

For \(n = 7\), \(n = 9\) to 12, and \(n = 14, 16, 18, 20, 22\). For \(n = 7\), \(f(u_1) = 2, f(u_2) = 1\) and \(f(v_0) = 6, f(v_1) = 9, f(v_2) = 3, f(v_3) = 4, f(v_4) = 8, f(v_5) = 10, f(v_6) = 5\), and \(f(v_7) = 7\). Then \(e_f(0) = 12, e_f(1) = 11\).

For \(n = 9\), \(f(u_1) = 2, f(u_2) = 1\) and \(f(v_0) = 6, f(v_1) = 3, f(v_2) = 12, f(v_3) = 4, f(v_4) = 8, f(v_5) = 10, f(v_6) = 5, f(v_7) = 7, f(v_8) = 11, f(v_9) = 6\). Then \(e_f(0) = 15, e_f(1) = 14\).

For \(n = 10\), \(f(u_1) = 2, f(u_2) = 1\) and \(f(v_0) = 6, f(v_1) = 3, f(v_2) = 12, f(v_3) = 4, f(v_4) = 8, f(v_5) = 10, f(v_6) = 5, f(v_7) = 7, f(v_8) = 11, f(v_9) = 13, f(v_{10}) = 9\). Then \(e_f(0) = 16, e_f(1) = 14\).

For \(n = 11\), \(f(u_1) = 2, f(u_2) = 1\) and \(f(v_0) = 6, f(v_1) = 5, f(v_2) = 4, f(v_3) = 8, f(v_4) = 10, f(v_5) = 14, f(v_6) = 12, f(v_7) = 9, f(v_8) = 3, f(v_9) = 13, f(v_{10}) = 11, f(v_{11}) = 7\). Then \(e_f(0) = 18, e_f(1) = 17\).
For $n = 12$, $f(u_1) = 2$, $f(u_2) = 1$ and $f(v_6) = 6$, $f(v_i) = 5$, $f(v_8) = 4$, $f(v_4) = 8$, $f(v_9) = 10$, $f(v_3) = 14$, $f(v_6) = 12$, $f(v_2) = 9$, $f(v_6) = 3$, $f(v_2) = 13$, $f(v_9) = 15$, $f(v_4) = 11$, and $f(v_6) = 7$. Then $e_f(0) = 19$, $e_f(1) = 19$.

For $n = 14$, $f(u_1) = 2$, $f(u_2) = 1$ and $f(v_9) = 6$, $f(v_i) = 5$, $f(v_9) = 4$, $f(v_4) = 8$, $f(v_9) = 10$, $f(v_3) = 12$, $f(v_6) = 14$, $f(v_i) = 16$, $f(v_9) = 3$, $f(v_1) = 9$, $f(v_1) = 17$, $f(v_2) = 13$, $f(v_3) = 11$, and $f(v_4) = 7$. Then $e_f(0) = 22$, $e_f(1) = 22$.

For $n = 16$, $f(u_1) = 2$, $f(u_2) = 1$ and $f(v_9) = 6$, $f(v_i) = 19$, $f(v_9) = 4$, $f(v_4) = 8$, $f(v_9) = 10$, $f(v_3) = 12$, $f(v_6) = 14$, $f(v_i) = 16$, $f(v_9) = 18$, $f(v_1) = 3$, $f(v_1) = 9$, $f(v_1) = 17$, $f(v_2) = 5$, $f(v_3) = 11$, $f(v_1) = 13$, $f(v_6) = 15$, $f(v_9) = 17$, and $f(v_9) = 19$. Then $e_f(0) = 28$, $e_f(1) = 28$.

For $n = 20$, $f(u_1) = 2$, $f(u_2) = 1$ and $f(v_9) = 6$, $f(v_i) = 19$, $f(v_9) = 4$, $f(v_4) = 8$, $f(v_9) = 10$, $f(v_3) = 12$, $f(v_6) = 14$, $f(v_i) = 16$, $f(v_9) = 18$, $f(v_1) = 20$, $f(v_1) = 10$, $f(v_1) = 14$, $f(v_2) = 5$, $f(v_3) = 11$, $f(v_1) = 13$, $f(v_6) = 15$, $f(v_9) = 17$, and $f(v_9) = 19$. Then $e_f(0) = 28$, $e_f(1) = 28$.

For $n = 22$, $f(u_1) = 2$, $f(u_2) = 1$ and $f(v_9) = 6$, $f(v_i) = 25$, $f(v_9) = 4$, $f(v_4) = 8$, $f(v_9) = 10$, $f(v_3) = 12$, $f(v_6) = 14$, $f(v_i) = 16$, $f(v_9) = 18$, $f(v_1) = 20$, $f(v_1) = 10$, $f(v_1) = 14$, $f(v_2) = 5$, $f(v_3) = 11$, $f(v_1) = 13$, $f(v_6) = 15$, $f(v_9) = 17$, and $f(v_9) = 19$. Then $e_f(0) = 28$, $e_f(1) = 28$.

For the next case let $t_1 = \lfloor (n + 3)/2 \rfloor$, $m = \lfloor (n + 3)/3 \rfloor$, $t_2 = \lfloor m/2 \rfloor$, $k_1 = \lfloor 3n + 3/2 \rfloor$, $k_2 = 2t_1 + t_2 - 4$.

$$t = k_1 - k_2, \quad t_3 = \begin{cases} t - 2, & n = 13, 15, 17, \\ t - 1, & n = 19, 21, \quad n \geq 23. \end{cases}$$ (16)

Case 3 ($t_1 - t \geq 3(n = 13, 15, 17, 19, 21$ and $n \geq 23$)). Consider:

$$f(u_1) = 2, \quad f(u_2) = 5,$$ $f(v_9) = 6,$ $f(v_9) = 3,$ $f(v_2) = 4,$ $f(v_9) = 1,$ $f(v_{2i}) = 8 + 2(i - 1), \quad 1 \leq i \leq t$ (17)

$$f(v_{2i+1}) = \begin{cases} 9, & \text{if } t \equiv 4 \pmod{7}, \\ 7, & \text{otherwise} \end{cases}$$

$$f(v_{2i+2}) = \begin{cases} 9, & \text{if } t \equiv 4 \pmod{7}, \\ 7, & \text{otherwise} \end{cases}$$

$$f(v_{2i+3}) = 9 + 2i, \quad 1 \leq i \leq t_3.$$ (18)

In view of the above defined labeling pattern for Case 3, we have $e_f(0) = \lfloor (3n + 2)/2 \rfloor$ and $e_f(1) = \lfloor (3n + 2)/2 \rfloor$.

Thus for Cases 2 and 3 we have $|e_f(0) - e_f(1)| \leq 1$.

Hence, $G$ is a prime cordial graph for $n = 7$ and $n \geq 9$. 

Illustration 5. Let $G_1$ be the graph obtained by duplication of arbitrary spoke edge by an edge of wheel $W_5$. Then $|V(G)| = 26$ and $|E(G)| = 71$. In accordance with Theorem 15 we have $t_1 = 13$, $m = 8$, $t_2 = 4$, $k_1 = 35$, $k_2 = 26$, and $t = 9$. Here $t_1 - t = 4 > 3$ so labeling pattern described in Case 3 will be applicable. The corresponding prime cordial labeling is shown in Figure 4. It is easy to visualise that $e_f(0) = 35$, $e_f(1) = 36$.

Theorem 16. $DS(P_n)$ is a prime cordial graph.

Proof. Consider $P_n$ with $V(P_n) = \{v_i : 1 \leq i \leq n\}$. Here $V(P_n) = V_1 \cup V_2$, where $V_1 = \{v_i : 2 \leq i \leq n - 1\}$ and $V_2 = \{v_1, v_n\}$. Now in order to obtain $DS(P_n)$ from $G$, we add $w_1, w_2$ corresponding to $V_1, V_2$. Then $|V(DS(P_n))| = n + 2$ and $E(DS(P_n)) = \{v_1, v_2, w_1, w_2\} \cup \{v_1, v_2, v_1, v_1, v_n, v_n, v_n, v_n\}$ so $|E(DS(P_n))| = 2n - 1$. We define vertex labeling $f: V(DS(P_n)) \rightarrow \{1, 2, \ldots, n + 2\}$ as follows.

Let $p_i$ be the highest prime number $< n + 2$ and $k = \lfloor (n + 2)/2 \rfloor$.

Consider

$$f(w_1) = 2, \quad f(w_2) = 3,$$ $f(v_1) = 1,$ $f(v_n) = 9,$ $f(v_{n-1}) = p_1$. (19)

For $0 \leq i < k - 1$,

$$f(v_{2i+1}) = \begin{cases} (n + 2) - 2i, & \text{if } n \text{ is even} \\ (n + 1) - 2i, & \text{if } n \text{ is odd} \end{cases}$$ (20)

And for vertices $v_{k+2}, v_{k+3}, \ldots, v_{n-2}$ we assign distinct odd numbers ($< n + 2$) in ascending order starting from 5.

In view of the above defined labeling pattern, if $n$ is even number, then $e_f(0) = n, e_f(1) = n - 1$; otherwise $e_f(0) = n - 1, e_f(1) = n$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DS(P_n)$ is a prime cordial graph.

Illustration 6. Prime cordial labeling of the graph $DS(P_7)$ is shown in Figure 5.

Theorem 17. $DS(B_n, v)$ is a prime cordial graph.

Proof. Consider $B_n$, with $V(B_n) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$, where $u_i, v_i$ are pendant vertices. Here $V(B_n) = V_1 \cup V_2$, where $V_1 = \{u, v, v_i : 1 \leq i \leq n\}$ and $V_2 = \{u_i, v_i\}$. Now in order to obtain $DS(B_n, v)$ from $G$, we add $w_1, w_2$ corresponding to $V_1, V_2$. Then $|V(DS(B_n, v))| = 2n + 4$ and $E(DS(B_n, v)) = \{u_1, u_2, v_2, w_2\} \cup \{u, v, v_i, w_1, w_1, v_i : 1 \leq i \leq n\}$.
so \(|E(DS(B_{n,n}))| = 4n + 3\). We define vertex labeling \(f : V(DS(B_{n,n})) \rightarrow \{1, 2, \ldots, 2n + 4\}\) as follows:

\[
\begin{align*}
  f(u) &= 6, \quad f(v) = 1, \\
  f(w_1) &= 2, \quad f(w_2) = 3, \\
  f(u_1) &= 4, \\
  f(u_{i+1}) &= 8 + 2(i - 1), \quad 1 \leq i \leq n - 1, \\
  f(v_i) &= 5 + 2(i - 1), \quad 1 \leq i \leq n.
\end{align*}
\]

(21)

In view of the above defined labeling pattern we have \(e_f(0) = 2n + 1\), \(e_f(1) = 2n + 2\).

Thus, \(|e_f(0) - e_f(1)| \leq 1\).

Hence, \(DS(B_{n,n})\) is a prime cordial graph.

Illustration 7. Prime cordial labeling of the graph \(DS(B_{5,5})\) is shown in Figure 6.

3. Conclusion

A new approach for constructing larger prime cordial graph from the existing prime cordial graph is investigated.
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Cordial Labeling for Some Bistar Related Graphs

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Abstract

In this paper we prove that square graph and shadow graph of bistar admit cordial labeling. Moreover we prove that splitting graph of bistar as well as degree splitting graph of bistar are cordial graphs.

Keywords: Cordial labeling, square graph, shadow graph, splitting graph, degree splitting graph.

AMS Subject Classification(2010): 05C78.

1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen [7]. Various labeling schemes have been introduced so far and explored as well by many researchers. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. Some diversified applications of graph labeling have been studied by Yegnanaryanan and Vaidhyanathan [17]. A dynamic survey on different graph labeling problems with an extensive bibliography can be found in Gallian [5].

The concept of cordial labeling was introduced by Cahit [4] as a weaker version of graceful [10] and harmonious [6] labeling. In the same paper Cahit investigated some classes of cordial graphs and a necessary condition for an eulerian graph to be cordial graph. Ho et al. [9] have also proved many results on cordial labeling. This concept is explored by many researchers like Andar et al. [1, 2], Vaidya and Dani [12, 13]. Lawrence and Koilraj [8] studied cordial labeling for the splitting graph of some standard graphs. Babujee and Shobana [3] introduced the concepts of cordial languages and cordial numbers. Some labeling schemes are also introduced with minor variations in cordial theme like product cordial labeling, total product cordial labeling and prime cordial labeling.

Many results on star and bistar related graphs in the context of various graph labeling problems have been proved by Vaidya and Shah [14–16]. In the present work, we investigate some results on cordial labelings of some bistar related graphs.
We provide a brief summary of the definitions and other information which are useful for the present investigations.

**Definition 1.1.** The graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (edge labeling).

**Definition 1.2.** A mapping \( f : V(G) \rightarrow \{0, 1\} \) is called a binary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

If for an edge \( e = uv \), the induced edge labeling \( f^* : E(G) \rightarrow \{0, 1\} \) is given by \( f^*(e) = |f(u) - f(v)| \) then we introduce following notations,

\[
\begin{aligned}
&v_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f, \\
&e_f(i) = \text{number of edges of } G \text{ having label } i \text{ under } f^*
\end{aligned}
\]

where \( i = 0 \) or \( 1 \)

**Definition 1.3.** A binary vertex labeling \( f \) of a graph \( G \) is called a cordial labeling if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). A graph \( G \) is cordial if it admits cordial labeling.

**Definition 1.4.** The bistar \( B_{n,n} \) is graph obtained by joining the center (apex) vertices of two copies of \( K_{1,n} \) by an edge.

**Definition 1.5.** For a simple connected graph \( G \) the square of graph \( G \) is denoted by \( G^2 \) and defined as the graph with the same vertex set as of \( G \) and two vertices are adjacent in \( G^2 \) if they are at a distance 1 or 2 apart in \( G \).

**Definition 1.6.** The shadow graph \( D_2(G) \) of a connected graph \( G \) is constructed by taking two copies of \( G \) say \( G' \) and \( G'' \). Join each vertex \( u \) in \( G' \) to the neighbours of the corresponding vertex \( u' \) in \( G'' \).

**Definition 1.7.** For a graph \( G \) the splitting graph \( S'(G) \) of a graph \( G \) is obtained by adding a new vertex \( v' \) corresponding to each vertex \( v \) of \( G \) such that \( N(v) = N(v') \).

**Definition 1.8.** [11] Let \( G = (V(G), E(G)) \) be a graph with \( V = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_t \cup T \) where each \( S_i \) is a set of vertices having at least two vertices of the same degree and \( T = V \setminus \bigcup S_i \). The degree splitting graph of \( G \) denoted by \( DS(G) \) is obtained from \( G \) by adding vertices \( w_1, w_2, w_3, \ldots, w_t \) and joining to each vertex of \( S_i \) for \( 1 \leq i \leq t \).

2 Main Results

**Theorem 2.1.** \( B_{n,n}^2 \) is a cordial graph.

**Proof:** Consider \( B_{n,n} \) with vertex set \( \{u, v, u_i, v_i : 1 \leq i \leq n\} \) where \( u_i, v_i \) are pendant vertices. Let \( G \) be the graph \( B_{n,n}^2 \) then \( |V(G)| = 2n + 2 \) and \( |E(G)| = 4n + 1 \).

We define a vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) as follows.
$f(u) = 1,$
$f(v) = 0,$
$f(u_i) = 1; \ 1 \leq i \leq n$
$f(v_i) = 0; \ 1 \leq i \leq n$

In view of the above defined labeling pattern we have,
$v_f(0) = n + 1 = v_f(1), e_f(0) = 2n$ and $e_f(1) = 2n + 1.$
Thus we proved that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1.$ Hence, $B^2_{n,n}$ is a cordial graph.

**Illustration 2.2.** Cordial labeling of the graph $B^2_{7,7}$ is shown in Figure 1.

![Figure 1: $B^2_{7,7}$ and its cordial labeling.](image)

**Theorem 2.3.** $D_2(B_{n,n})$ is a cordial graph.

**Proof:** Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i : 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Let $G$ be the graph $D_2(B_{n,n})$ then $|V(G)| = 4(n + 1)$ and $|E(G)| = 4(2n + 1)$.

We define a vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

$f(u) = 0,$
$f(u') = 1,$
$f(v) = 0,$
$f(v') = 1,$
$f(u_i) = 0; \ 1 \leq i \leq n$
$f(v_i) = 0; \ 1 \leq i \leq n$
$f(u'_i) = 1; \ 1 \leq i \leq n$
$f(v'_i) = 1; \ 1 \leq i \leq n$

In view of the above defined labeling pattern we have,
Thus we proved that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence, $D_2(B_{n,n})$ is a cordial graph.

**Illustration 2.4.** Cordial labeling of the graph $D_2(B_{5,5})$ is shown in Figure 2.

**Figure 2:** $D_2(B_{5,5})$ and its cordial labeling.

**Theorem 2.5.** $S'(B_{n,n})$ is a cordial graph.

**Proof:** Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i : 1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices. In order to obtain $S'(B_{n,n})$, add $u', v', u'_i, v'_i$ vertices corresponding to $u, v, u_i, v_i$ where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n + 1)$ and $|E(G)| = 6n + 3$.

We define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

- $f(u) = 0$,
- $f(u') = 1$,
- $f(v) = 0$,
- $f(v') = 1$,
- $f(u_i) = 0; \ 1 \leq i \leq n$
- $f(u'_i) = 0; \ 1 \leq i \leq n$
- $f(v_i) = 1; \ 1 \leq i \leq n$
- $f(v'_i) = 1; \ 1 \leq i \leq n$

In view of the above labeling pattern we have,

$v_f(0) = 2n + 2 = v_f(1)$ and $e_f(0) = 3n + 1, e_f(1) = 3n + 2$.

Thus we proved that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence, $S'(B_{n,n})$ is a cordial graph.
Illustration 2.6. Cordial labeling of the graph $S'(B_6,6)$ is shown in Figure 3.

Figure 3: $S'(B_6,6)$ and its cordial labeling.

Theorem 2.7. $DS(B_{n,n})$ is a cordial graph.

Proof: Consider $B_{n,n}$ with $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$, where $u_i, v_i$ are pendant vertices. Here $V(B_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ and $V_2 = \{u, v\}$.

Now in order to obtain $DS(B_{n,n})$ from $G$, we add $w_1, w_2$ corresponding to $V_1, V_2$. Then $|V(DS(B_{n,n}))| = 2n + 4$ and $E(DS(B_{n,n})) = \{uv, uw_2, vw_2\} \cup \{uu_i, vv_i, w_1u_i, w_1v_i : 1 \leq i \leq n\}$ so $|E(DS(B_{n,n}))| = 4n + 3$.

We define a vertex labeling $f : V(DS(B_{n,n})) \rightarrow \{0, 1\}$ as follows.

- $f(u) = 0$,
- $f(v) = 0$,
- $f(w_1) = 1$,
- $f(w_2) = 1$,
- $f(u_i) = 0; \ 1 \leq i \leq n$,
- $f(v_i) = 1; \ 1 \leq i \leq n$.

In view of the above defined labeling pattern we have,

- $v_f(0) = n + 2 = v_f(1)$ and $e_f(0) = 2n + 1, e_f(1) = 2n + 2$.

Thus we proved that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence, $DS(B_{n,n})$ is a cordial graph.

Illustration 2.8. Cordial labeling of the graph $DS(B_{5,5})$ is shown in Figure 4.
3. Concluding remarks

Here we have contributed some new results on cordial labeling for the larger graphs obtained from bistar by means of various graph operations. Similar results can be obtained for different graph families and this is an open area of research.

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Cordial Labeling of Snakes

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Abstract: We prove that the graphs alternate triangular snake, alternate quadrilateral snake, double alternate triangular snake and double alternate quadrilateral snake admit cordial labeling.

Keywords: Cordial graph, Cordial labeling, Snake.

1 Introduction

In this paper, we consider only finite, connected, undirected and simple graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notation we refer to Balakrishnan and Ranganathan [4]. A brief summary of definitions and information related to the present work is given in order to maintain the compactness.

Definition 1.1. A graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (edge labeling).

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Various labeling schemes have been introduced so far and explored as well by many researchers. Several diversified applications of graph labeling have been reported by Yegnanaryanan and Vaidhyanathan [13]. A dynamic survey on different graph labeling problems along with an extensive bibliography can be found in Gallian [6].

**Definition 1.2.** A mapping \( f : V(G) \rightarrow \{0,1\} \) is called binary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

If for an edge \( e = uv \), the induced edge labeling \( f^* : E(G) \rightarrow \{0,1\} \) is given by \( f^*(e) = |f(u) - f(v)| \) then we introduce following notations,

\[
\begin{align*}
    v_f(i) &= \text{number of vertices of } G \text{ having label } i \text{ under } f \\
    e_f(i) &= \text{number of edges of } G \text{ having label } i \text{ under } f^* \\
\end{align*}
\]

where \( i = 0 \) or \( 1 \)

**Definition 1.3.** A binary vertex labeling \( f \) of a graph \( G \) is called a cordial labeling if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). A graph \( G \) is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [5] as a weaker version of graceful and harmonious labeling. In the same paper Cahit investigated many classes of cordial graphs and a necessary condition for an Eulerian graph to be cordial graph. Andar et al. [1, 2] and Ho et al. [8] have contributed many results on cordial labeling. Vaidya and Dani [9, 10] as well as Vaidya and Vihol [11] have investigated many results on cordial labeling for the graphs arising from different graph operations. Vaidya and Shah [12] have discussed cordial labeling for some bistar related graphs. Lawrence and Koilraj [7] have discussed cordial labeling for the splitting graph of some standard graphs. Motivated through the concept of cordial labeling Babujee and Shobana [3] have introduced the concepts of cordial languages and cordial numbers. Some labeling schemes are also introduced with minor variations in cordial theme.

**Definition 1.4.** An alternate triangular snake \( A(T_n) \) is obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternately) to a new vertex \( v_i \). That is every alternate edge of path is replaced by \( C_3 \).

**Definition 1.5.** An alternate quadrilateral snake \( A(QS_n) \) is obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i, u_{i+1} \) to new vertices \( v_i, w_i \) respectively and then joining \( v_i \) and \( w_i \). That is every alternate edge of path is replaced by \( C_4 \).

**Definition 1.6.** A double alternate triangular snake \( DA(T_n) \) consists of two alternate triangular snakes that have a common path. That is, double alternate triangular snake is obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternately) to new vertices \( v_i \) and \( w_i \).

**Definition 1.7.** A double alternate quadrilateral snake \( DA(QS_n) \) consists of two alternate quadrilateral snakes that have a common path. That is, it is obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternately) to new vertices \( v_i, v'_i \) and \( w_i, w'_i \) respectively and adding the edges \( v_iw_i \) and \( v'_iw'_i \).
2 Main Results

Theorem 2.1. \( A(T_n) \) admits cordial labeling.

Proof. Let \( A(T_n) \) be alternate triangular snake obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternately) to new vertex \( v_i \) where \( 1 \leq i \leq n-1 \) for even \( n \) and \( 1 \leq i \leq n-2 \) for odd \( n \). Therefore \( V(A(T_n)) = \{u_i, v_j/1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \} \). We note that

\[
|V(A(T_n))| = \begin{cases} 
\frac{3n}{2}, & n \equiv 0 \pmod{2} \\
\frac{3n-1}{2}, & n \equiv 1 \pmod{2}
\end{cases}
\]

and

\[
|E(A(T_n))| = \begin{cases} 
2n-1, & n \equiv 0 \pmod{2} \\
2n-2, & n \equiv 1 \pmod{2}
\end{cases}
\]

To define vertex labeling \( f : V(A(T_n)) \to \{0, 1\} \) we consider following five cases.

Case 1: \( n = 2, 3 \).

For \( n = 2 \), \( A(T_2) = C_3 \), which is a cordial graph as proved by Ho et al. \[8\].

For \( n = 3 \), \( f(u_1) = 0, f(u_2) = 1, f(u_3) = 1 \) and \( f(v_1) = 0 \). Then \( v_f(0) = 2, v_f(1) = 2 \) and \( e_f(0) = 2 = e_f(1) \). Hence, \( A(T_3) \) admits cordial labeling.

Case 2: \( n \equiv 0 \pmod{4} \).

Let \( n = 4k \),

\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k-1 \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k-1 \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k-1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k-1 \\
  f(v_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n}{4} - 1 \\
  f(v_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = \frac{3n}{4} = v_f(1) \) and \( e_f(0) = n - 1, e_f(1) = n \).

Case 3: \( n \equiv 1 \pmod{4} \).

Let \( n = 4k + 1 \),

\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k-1 \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k-1 \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k-1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k-1 \\
  f(u_n) &= 0; \\
  f(v_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-1}{4} - 1 \\
  f(v_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-1}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = \left\lceil \frac{3n-1}{4} \right\rceil, v_f(1) = \left\lfloor \frac{3n-1}{4} \right\rfloor \) and \( e_f(0) = n - 1 = e_f(1) \).

Case 4: \( n \equiv 2 \pmod{4} \).
Let $n = 4k + 2$, 
\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_{n-1}) &= 0; \\
  f(u_n) &= 1; \\
  f(v_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-2}{4} - 1 \\
  f(v_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-2}{4} - 1 \\
  f(v_n) &= 1;
\end{align*}
\]
In view of the above defined labeling pattern, $v_f(0) = \left\lfloor \frac{3n}{4} \right\rfloor$, $v_f(1) = \left\lceil \frac{3n}{4} \right\rceil$ and $e_f(0) = n - 1, e_f(1) = n$.

**Case 5: $n \equiv 3 (\text{mod} \ 4)$.

Let $n = 4k + 3$, 
\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_{n-2}) &= 0; \\
  f(u_{n-1}) &= 1; \\
  f(u_n) &= 1; \\
  f(v_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-3}{4} - 1 \\
  f(v_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-3}{4} - 1 \\
  f(v_n) &= 0;
\end{align*}
\]
In view of the above defined labeling pattern, $v_f(0) = \frac{3n-1}{4} = v_f(1)$ and $e_f(0) = n - 1 = e_f(1)$.
Thus, in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.
Hence, $A(T_n)$ admits cordial labeling.

**Example 2.2.** A cordial labeling of $A(T_{11})$ is shown in Figure 1.

---

**Fig. 1** $A(T_{11})$ and its cordial labeling.
Theorem 2.3. $A(QS_n)$ is admits cordial labeling.

Proof. Let $A(QS_n)$ be an alternate quadrilateral snake obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i, u_{i+1}$ (alternately) to new vertices $v_i, w_i$ respectively and then joining $v_i$ and $w_i$ where $1 \leq i \leq n - 1$ for even $n$ and $1 \leq i \leq n - 2$ for odd $n$. Therefore $V(A(T_n)) = \{u_i, v_j, w_j/1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$. We note that

$$\|V(A(QS_n))\| = \begin{cases} 2n, & n \equiv 0(\text{mod } 2) \\ 2n - 1, & n \equiv 1(\text{mod } 2) \end{cases}$$

and

$$|E(A(QS_n))| = \begin{cases} \frac{5n-2}{2}, & n \equiv 0(\text{mod } 2) \\ \frac{5n-5}{2}, & n \equiv 1(\text{mod } 2) \end{cases}$$

To define vertex labeling $f : V(A(QS_n)) \to \{0, 1\}$ we consider following five cases.

Case 1: $n = 2, 3$.

For $n = 2$, $A(QS_2) = C_4$, which is a cordial graph as proved by Ho et al. [8].

For $n = 3$, $f(u_1) = 1, f(u_2) = 0, f(u_3) = 1$ and $f(v_1) = 1, f(w_1) = 0$. Then $v_f(0) = 2, v_f(1) = 3$ and $e_f(0) = 2, e_f(1) = 3$. Hence, $A(QS_3)$ admits cordial labeling.

Case 2: $n \equiv 0(\text{mod } 4)$.

Let $n = 4k$,

$$f(u_{1+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{2+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{3+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{4+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(v_i) = 1; \quad 1 \leq i \leq \frac{n}{2}$$
$$f(w_{1+2i}) = 1; \quad 0 \leq i \leq \frac{n}{4} - 1$$
$$f(w_{2+2i}) = 0; \quad 0 \leq i \leq \frac{n}{4} - 1$$

In view of the above defined labeling pattern, $v_f(0) = n = v_f(1)$ and $e_f(0) = n + \frac{n}{4} - 1$ and $e_f(1) = n + \frac{n}{4}$.

Case 3: $n \equiv 1(\text{mod } 4)$.

Let $n = 4k + 1$,

$$f(u_{1+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{2+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{3+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{4+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_n) = 0;$$
$$f(v_i) = 1; \quad 1 \leq i \leq \frac{n-1}{2}$$
$$f(w_{1+2i}) = 1; \quad 0 \leq i \leq \frac{n-1}{4} - 1$$
$$f(w_{2+2i}) = 0; \quad 0 \leq i \leq \frac{n-1}{4} - 1$$

In view of the above defined labeling pattern, $v_f(0) = n, v_f(1) = n - 1$ and $e_f(0) = n - 1 + \frac{n-1}{4} = \frac{5n - 5}{4} = e_f(1)$. 

Case 4: \( n \equiv 2 \pmod{4} \).
Let \( n = 4k + 2 \),

\[
\begin{align*}
    f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{2+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
    f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(v_i) &= 1; \quad 1 \leq i \leq \frac{n}{2} \\
    f(w_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-2}{4} \\
    f(w_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-2}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n = v_f(1) \) and \( e_f(0) = \frac{5n - 2}{4} = e_f(1) \).

Case 5: \( n \equiv 3 \pmod{4} \).

Let \( n = 4k + 3 \),

\[
\begin{align*}
    f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{2+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
    f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_n) &= 1; \\
    f(v_i) &= 1; \quad 1 \leq i \leq \frac{n}{2} - \frac{1}{2} \\
    f(w_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n}{4} - \frac{3}{4} \\
    f(w_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n}{4} - \frac{3}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n - 1, v_f(1) = n \) and \( e_f(0) = \frac{5n - 7}{4}, e_f(1) = \frac{5n - 3}{4} \).

Thus in each case we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( A(QS_n) \) admits cordial labeling.

\[\square\]

Example 2.4. A cordial labeling of \( A(QS_8) \) is shown in Figure 2.

![Fig. 2 A(QS_8) and its cordial labeling.](image)

Theorem 2.5. \( DA(T_n) \) admits cordial labeling.
Proof. Let $G$ be a double alternate triangular snake $DA(T_n)$ then $V(G) = \{u_i, v_j, w_j/1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \}$. We note that

$$|V(G)| = \begin{cases} 
2n, & n \equiv 0(\text{mod} \ 2) \\
2n - 1, & n \equiv 1(\text{mod} \ 2)
\end{cases} \quad \text{and} \quad |E(G)| = \begin{cases} 
3n - 1, & n \equiv 0(\text{mod} \ 2) \\
3n - 3, & n \equiv 1(\text{mod} \ 2)
\end{cases}$$

To define vertex labeling $f : V(DA(T_n)) \to \{0, 1\}$ we consider following five cases.

**Case 1:** $n = 2, 3$.

For $n = 2$, $f(u_1) = 0, f(u_2) = 1$ and $f(v_1) = 0, f(w_1) = 1$. Then $v_f(0) = 2 = v_f(1)$ and $e_f(0) = 2, e_f(1) = 3$. Hence, $DA(T_2)$ admits cordial labeling.

For $n = 3$, $f(u_1) = 0, f(u_2) = 1, f(u_3) = 1$ and $f(v_1) = 0, f(w_1) = 1$. Then $v_f(0) = 2, v_f(1) = 3$ and $e_f(0) = 3 = e_f(1)$. Hence, $DA(T_3)$ admits cordial labeling.

**Case 2:** $n \equiv 0(\text{mod} \ 4)$.

Let $n = 4k$,

$$f(u_{1+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{2+4i}) = 1; \quad 0 \leq i \leq k - 1$$
$$f(u_{3+4i}) = 1; \quad 0 \leq i \leq k - 1$$
$$f(u_{4+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(v_i) = 0; \quad 1 \leq i \leq \frac{n}{2}$$
$$f(w_i) = 1; \quad 1 \leq i \leq \frac{n}{2}$$

In view of the above defined labeling pattern, $v_f(0) = n = v_f(1)$ and $e_f(0) = \frac{3n - 2}{2}$ and $e_f(1) = \frac{3n}{2}$.

**Case 3:** $n \equiv 1(\text{mod} \ 4)$.

Let $n = 4k + 1$,

$$f(u_{1+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_{2+4i}) = 1; \quad 0 \leq i \leq k - 1$$
$$f(u_{3+4i}) = 1; \quad 0 \leq i \leq k - 1$$
$$f(u_{4+4i}) = 0; \quad 0 \leq i \leq k - 1$$
$$f(u_n) = 0;$$
$$f(v_i) = 0; \quad 1 \leq i \leq \frac{n-1}{2}$$
$$f(w_i) = 1; \quad 1 \leq i \leq \frac{n-1}{2}$$

In view of the above defined labeling pattern, $v_f(0) = n, v_f(1) = n - 1$ and $e_f(0) = \frac{3n - 3}{2}, e_f(1) = \frac{3n - 3}{2}$.

**Case 4:** $n \equiv 2(\text{mod} \ 4)$. 
Let $n = 4k + 2$,

$$
\begin{align*}
  f(u_{1+4i}) &= 0; & 0 \leq i &\leq k \\
  f(u_{2+4i}) &= 1; & 0 \leq i &\leq k \\
  f(u_{3+4i}) &= 1; & 0 \leq i &\leq k - 1 \\
  f(u_{4+4i}) &= 0; & 0 \leq i &\leq k - 1 \\
  f(v_i) &= 0; & 1 \leq i &\leq \frac{n}{2} \\
  f(w_i) &= 1; & 1 \leq i &\leq \frac{n}{2}
\end{align*}
$$

In view of the above defined labeling pattern, $v_f(0) = n = v_f(1)$ and $e_f(0) = 3n/2 - 1, e_f(1) = 3n/2$.

**Case 5**: $n \equiv 3 (\text{mod} \ 4)$.

Let $n = 4k + 3$,

$$
\begin{align*}
  f(u_{1+4i}) &= 0; & 0 \leq i &\leq k \\
  f(u_{2+4i}) &= 1; & 0 \leq i &\leq k \\
  f(u_{3+4i}) &= 1; & 0 \leq i &\leq k - 1 \\
  f(u_{4+4i}) &= 0; & 0 \leq i &\leq k - 1 \\
  f(u_n) &= 1; \\
  f(v_i) &= 0; & 1 \leq i &\leq \frac{n-1}{2} \\
  f(w_i) &= 1; & 1 \leq i &\leq \frac{n-1}{2}
\end{align*}
$$

In view of the above defined labeling pattern, $v_f(0) = n - 1, v_f(1) = n$ and $e_f(0) = \frac{3n-3}{2} = e_f(1)$.

Thus, in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DA(T_n)$ admits cordial labeling.

\[ \Box \]

**Example 2.6.** A cordial labeling of $DA(T_{10})$ is shown in Figure 3.

![DA(T10) and its cordial labeling.](image)

**Theorem 2.7.** $DA(QS_n)$ admits cordial labeling.

**Proof.** Let $G$ be a double alternate quadrilateral snake $DA(T_n)$ then $V(G) = \{u_i, v_j, w_j, v'_j, w'_j/1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{4} \rfloor \}$. We note that
To define vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) we consider following two cases.

**Case 1:** \( n \equiv 0(\text{mod} 2) \).

Let \( n = 2k \),

\[
\begin{align*}
  f(u_{i+2}) &= 0; \quad 0 \leq i \leq \frac{n}{2} - 1 \\
  f(u_{2i}) &= 1; \quad 1 \leq i \leq \frac{n}{2} \\
  f(v_i) &= 1; \quad 1 \leq i \leq \frac{n}{2} \\
  f(w_i) &= 1; \quad 1 \leq i \leq \frac{n}{2} \\
  f(v'_i) &= 0; \quad 1 \leq i \leq \frac{n}{2} \\
  f(w'_i) &= 0; \quad 1 \leq i \leq \frac{n}{2}
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = \frac{3n}{2} = v_f(1) \) and \( e_f(0) = 2n, e_f(1) = 2n - 1 \).

**Case 2:** \( n \equiv 1(\text{mod} 2) \).

Let \( n = 2k + 1 \),

\[
\begin{align*}
  f(u_{i+2}) &= 0; \quad 0 \leq i \leq \frac{n-1}{2} \\
  f(u_{2i}) &= 1; \quad 1 \leq i \leq \frac{n-1}{2} \\
  f(v_i) &= 1; \quad 1 \leq i \leq \frac{n-1}{2} \\
  f(w_i) &= 1; \quad 1 \leq i \leq \frac{n-1}{2} \\
  f(v'_i) &= 0; \quad 1 \leq i \leq \frac{n-1}{2} \\
  f(w'_i) &= 0; \quad 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = \frac{3n - 1}{2} = v_f(1) \) and \( e_f(0) = 2n - 2 = e_f(1) \).

Thus, in both the cases we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( DA(QS_n) \) admits cordial labeling.

**Example 2.8.** A cordial labeling of \( DA(QS_n) \) is shown in Figure 4.

![Fig. 4 DA(QS_9) and its cordial labeling.](image-url)
3 Concluding Remarks

The snakes are the graphs obtained from paths by attaching some graphs in various fashion. We have investigated cordial labelings for the same.

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References


On Square Divisor Cordial Graphs

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Abstract

The square divisor cordial labeling is a variant of cordial labeling and divisor cordial labeling. Here we prove that the graphs like flower $F_{1,n}$, bistar $B_{n,n}$, restricted square graph of $B_{n,n}$, shadow graph of $B_{n,n}$ as well as splitting graphs of star $K_{1,n}$ and bistar $B_{n,n}$ are square divisor cordial graphs. Moreover we show that the degree splitting graphs of $B_{n,n}$ and $P_n$ admit square divisor cordial labeling.

Keywords: Square divisor cordial labeling, Divisor cordial labeling.

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1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen [1] while for any concept of number theory we refer to Burton [2]. We will provide brief summary of definitions and other information which are prerequisites for the present investigations.

Definition 1.1 If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.

The graph labeling is described as a frontier between number theory and structure of graphs by Beineke and Hegde [3]. There are enormous applications of graph labeling in various fields including computer science and communication networks. Yegnanaryanan and Vaidhyanathan [4] described applications of edge balanced graph labeling, edge

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magic labeling and (1,1) edge magic graph. For a dynamic survey of various graph labeling problems along with an extensive bibliography we refer to Gallian [5].

**Definition 1.2** A mapping $f : V(G) \rightarrow \{0,1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$.

**Notation 1.3** If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Then

$$v_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f$$

$$e_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f^*$$

**Definition 1.4** A binary vertex labeling $f$ of a graph $G$ is called a cordial labeling if

$$|v_f(0) - v_f(1)| \leq 1 \text{ and } |e_f(0) - e_f(1)| \leq 1.$$ A graph $G$ is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [6]. This concept has been explored by many researchers and various labeling schemes are also introduced with minor variations in cordial theme. Product cordial labeling, total product cordial labeling and prime cordial labeling and divisor cordial labeling are among mention a few. The present work is focused on square divisor cordial labeling.

**Definition 1.5** A prime cordial labeling of a graph $G$ with vertex set $V(G)$ is a bijection $f : V(G) \rightarrow \{1,2,3,...,|V(G)|\}$ and the induced function $f^* : E(G) \rightarrow \{0,1\}$ is defined by

$$f^*(e = uv) = 1, \text{ if } gcd(f(u), f(v)) = 1;$$

$$= 0, \text{ otherwise}$$

which satisfies the condition $|e_f(0) - e_f(1)| \leq 1$. A graph which admits prime cordial labeling is called a prime cordial graph.

The concept of prime cordial labeling was introduced by Sundaram et al. [7] and in the same paper they have investigated several results on prime cordial labeling. Vaidya and Vihol [8,9] as well as Vaidya and Shah [10,11] have proved many results on prime cordial labeling.

Varatharajan et al. [12] have introduced a new concept called divisor cordial labeling by combining the divisibility of numbers and the concept of Cordial labeling.

**Definition 1.6** Let $G = (V(G), E(G))$ be a simple graph and $f : \{1,2,...,|V(G)|\}$ be a bijection. For each edge $uv$, assign the label 1 if $f(u) | f(v)$ or $f(v) | f(u)$ and the label 0 otherwise. $f$ is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

In the same paper [12] they have proved that path, cycle, wheel, star, $K_{2,n}$ and $K_{3,n}$ are divisor cordial graphs while $K_n$ is not divisor cordial for $n \geq 7$. Same authors in [13] have discussed divisor cordial labeling of full binary tree as well as some star related graphs. Vaidya and Shah [14] have proved that some star and bistar related graphs are
divisor cordial graph. Same authors [15] have shown that helm, flower and gear graphs admit divisor cordial labeling. Moreover the graphs obtained from switching of a vertex in various graphs are proved to be divisor cordial.

Motivated by the concept of divisor cordial labeling, Murugesan et al. [16] have introduced the concept of square divisor cordial labeling and many graphs are proved to be square divisor cordial graphs.

**Definition 1.7** Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \rightarrow \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge $uv$, assign the label 1 if $f(u)^2 | f(v)$ or $f(v)^2 | f(u)$ and the label 0 otherwise. $f$ is called a square divisor cordial labeling if $|f(0) - f(1)| \leq 1$. A graph with a square divisor cordial labeling is called a square divisor cordial graph.

**Definition 1.8** The flower $F_{2n}$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree $2n$, $n$ vertices of degree 4 and $n$ vertices of degree 2.

**Definition 1.9** Bistar $B_{n,n}$ is the graph obtained by joining the center (apex) vertices of two copies of $K_{1,n}$ by an edge.

**Definition 1.10** For a simple connected graph $G$ the square of graph $G$ is denoted by $G^2$ and defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$.

**Definition 1.11** The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $v'$ in $G''$.

**Definition 1.12** For a graph $G$ the splitting graph $S'(G)$ of a graph $G$ is obtained by adding a new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$.

**Definition 1.13** [17] Let $G = (V(G), E(G))$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup ... \cup S_t \cup T$ where each $S_i$ is a set of vertices having at least two vertices of the same degree and $T = V - \left( \bigcup_{i=1}^{t} S_i \right)$. The degree splitting graph of $G$ denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, w_3, ..., w_t$ and joining to each vertex of $S_i$ for $1 \leq i \leq t$. 
2. Main Results

**Theorem 2.1**: Flower graph $F_l_n$ is a square divisor cordial graph for each $n$.

**Proof**: Let $v$ be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 of $F_l_n$. Then $|V(F_l_n)| = 2n + 1$ and $|E(F_l_n)| = 4n$. We define vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, 2n + 1\}$ as follows.

$$
egin{align*}
  f(v) &= 1, \\
  f(v_i) &= 2, \\
  f(u_i) &= 3, \\
  f(u_{i+1}) &= 5 + 2(i - 1); \quad 1 \leq i \leq n - 1 \\
  f(u_{i+1}) &= 4 + 2(i - 1); \quad 1 \leq i \leq n - 1
\end{align*}
$$

In view of the above labeling pattern we have,

$$
|f(0) - f(1)| \leq 1.
$$

Hence $F_l_n$ is a square divisor cordial graph for each $n$.

**Illustration 2.2** Square divisor cordial labeling of the graph $F_{l_{11}}$ is shown in Fig. 1.

![Fig. 1.](image)

**Theorem 2.3**: $B_{n,n}$ is a square divisor cordial graph.

**Proof**: Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices. If $G = B_{n,n}$ then $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 1$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, 2n + 2\}$ as follows.
\[ f(u) = 1, \]
\[ f(v) = 2n+1, \]
\[ f(u_i) = 1 + i; \quad 1 \leq i \leq n \]
\[ f(v_i) = n + 1 + i; \quad 1 \leq i \leq n - 1 \]
\[ f(v_n) = 2n + 2; \]

In view of the above labeling pattern we have,
\[ e_f(0) = n + 1, e_f(1) = n. \]  Thus, \[ |e_f(0) - e_f(1)| \leq 1. \]

Hence \( B_{n,n} \) is a square divisor cordial graph.

Illustration 2.4: Square divisor cordial labeling of the graph \( B_{8,8} \) is shown in Fig. 2.

![Fig. 2](image_url)

**Theorem 2.5**: Restricted \( B_{n,n}^2 \) is a square divisor cordial graph.

**Proof**: Consider \( B_{n,n} \) with vertex set \( \{u,v,u_i,v_i, 1 \leq i \leq n\} \) where \( u_i,v_i \) are pendant vertices. Let \( G \) be the restricted \( B_{n,n}^2 \) graph with \( V(G) = V(B_{n,n}) \) and \( E(G) = E(B_{n,n}) \cup \{vu_i, uv_i / 1 \leq i \leq n\} \) then \( |V(G)| = 2n + 2 \) and \( |E(G)| = 4n + 1 \).

We define vertex labeling \( f: V(G) \rightarrow \{1,2,\ldots,2n+2\} \) as follows.

\[ f(u) = 1, \]
\[ f(v) = p_1, \]
\[ f(u_i) = 2, \]
\[ f(v_i) = 4 + 2(i-1); \quad 1 \leq i \leq n \]

For the vertices \( u_2,u_3,\ldots,u_n \) we assign distinct odd numbers (except \( p_1 \)).

In view of the above defined labeling pattern we have,
\[ e_f(0) = 2n, e_f(1) = 2n + 1. \]  Thus, \[ |e_f(0) - e_f(1)| \leq 1. \]

Hence \( B_{n,n}^2 \) is a square divisor cordial graph.

**Illustration 2.6**: Square divisor cordial labeling of restricted \( B_{7,7}^2 \) is shown in Fig. 3.
Theorem 2.7: $D_2(B_{n,n})$ is a square divisor cordial graph.

Proof: Consider two copies of $B_{n,n}$. Let \{u, v, u_1, v_1, 1 \leq i \leq n\} and \{u', v', u'_1, v'_1, 1 \leq i \leq n\} be the corresponding vertex sets of each copy of $B_{n,n}$. Let $G$ be the graph $D_2(B_{n,n})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 4(2n+1)$. We define vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, 4(n+1)\}$ as follows.

Let $p_1$ be the highest prime number, $p_2$ be the second highest prime number and $p_3$ be the third highest prime number such that $p_3 < p_2 < p_1 < 4(n+1)$.

- $f(u) = p_1$, $f(u') = 1$,
- $f(v) = p_2$, $f(v') = p_3$,
- $f(u_i) = 2 + 2(i - 1); \quad 1 \leq i \leq n$
- $f(u'_i) = f(u_i) + 2i; \quad 1 \leq i \leq n$
- $f(v_1) = 4(n+1)$, $f(v_2) = 4n$,

For the vertices $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ we assign distinct odd numbers (except $p_1$, $p_2$ and $p_3$).

In view of the above defined labeling pattern we have,

$$e_f(0) = 4n + 2 = e_f(1).$$

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence $D_2(B_{n,n})$ is a square divisor cordial graph.

Illustration 2.8: Square divisor cordial labeling of the graph $D_2(B_{5,5})$ is shown in Fig. 4.
Theorem 2.9: \( S'(K_{1,n}) \) is a square divisor cordial graph.

Proof: Let \( v_1, v_2, v_3, \ldots, v_n \) be the pendant vertices and \( v \) be the apex vertex of \( K_{1,n} \) and \( u, u_1, u_2, u_3, \ldots, u_n \) are added vertices corresponding to \( v, v_1, v_2, v_3, \ldots, v_n \) to obtain \( S'(K_{1,n}) \).

Let \( G \) be the graph \( S'(K_{1,n}) \) then \( |V(G)| = 2n + 2 \) and \( |E(G)| = 3n \). We define vertex labeling \( f : V(G) \rightarrow \{1, 2, \ldots, 2n+2\} \) as follows.

\[
\begin{align*}
  f(v) &= 2, \\
  f(u) &= 1, \\
  f(v_i) &= 4 + 2i; \quad 0 \leq i < n \\
  f(u_i) &= 3 + 2i; \quad 0 \leq i < n
\end{align*}
\]

In view of the above defined labeling pattern we have,

\[
e_f(1) = n + \left\lfloor \frac{2n + 2}{4} \right\rfloor, \quad e_f(0) = 3n - e_f(1).
\]

Thus, \( |e_f(0) - e_f(1)| \leq 1 \).

Hence \( S'(K_{1,n}) \) is a square divisor cordial graph.

Illustration 2.10: Square divisor cordial labeling of the graph \( S'(K_{1,11}) \) is shown in Fig. 5.
**Theorem 2.11:** $S'(B_{n,n})$ is a square divisor cordial graph.

**Proof:** Consider $B_{n,n}$ with vertex set $\{u,v,u_i,v_i,1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices. In order to obtain $S'(B_{n,n})$, add $u'_i, v', u'_i, v'_i$ vertices corresponding to $u, v, u_i, v_i$ where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 6n + 3$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, 4(n+1)\}$ as follows.

Let $p_1$ be the highest prime number and $p_2$ be the second highest prime number such that $p_2 < p_1 < 4n + 1$.

$f(u) = p_2$,

$f(u') = 2$,

$f(v) = 1$,

$f(v') = p_1$,

$f(u_i) = 4 + 4(i-1); \quad 1 \leq i \leq n$

$f(u'_i) = 6 + 4(i-1); \quad 1 \leq i \leq n$

$f(v_i) = 4n + 4$,

For the vertices $v_2, v_3, \ldots, v_n$ and $v'_2, v'_3, \ldots, v'_n$ we assign distinct odd numbers (except $p_1$ and $p_2$).

In view of the above labeling pattern we have,

$e_f(0) = 3n + 1, e_f(1) = 3n + 2$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence $S'(B_{n,n})$ is a square divisor cordial graph.

**Illustration 2.12:** Square divisor cordial labeling of the graph $S'(B_{6,6})$ is shown in Fig. 6.

---

![Fig. 6](image-url)
Theorem 2.13: $DS(B_{n,n})$ is a square divisor cordial graph.

Proof: Consider $B_{n,n}$ with $V(B_{n,n}) = \{u,v,u_i,v_i : 1 \leq i \leq n\}$, where $u_i,v_i$ are pendant vertices. Here $V(B_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u_i,v_i : 1 \leq i \leq n\}$ and $V_2 = \{u,v\}$. Now in order to obtain $DS(B_{n,n})$ from $G$, we add $w_i,w_2$ corresponding to $V_1,V_2$. Then $|V(DS(B_{n,n}))| = 2n + 4$ and $E(DS(B_{n,n})) = \{uv, uw_i, vw_i, w_iu_i, w_i v_i : 1 \leq i \leq n\}$ so $|E(DS(B_{n,n}))| = 4n + 3$. We define vertex labeling $f : V(DS(B_{n,n})) \to \{1,2,\ldots,2n+4\}$ as follows.

$f(u) = 4,
\quad f(v) = 2n + 3,
\quad f(w_i) = 1,
\quad f(w_2) = 2,
\quad f(u_i) = 3 + 2(i-1); \quad 1 \leq i \leq n
\quad f(v_i) = 6 + 2(i-1); \quad 1 \leq i \leq n$

In view of the above defined labeling pattern we have, $e_f(0) = 2n + 2, e_f(1) = 2n + 1$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence $DS(B_{n,n})$ is a square divisor cordial graph.

Illustration 2.14: Square divisor cordial labeling of the graph $DS(B_{5,5})$ is shown in Fig. 7.

Theorem 2.15: $DS(P_n)$ is a square divisor cordial graph.
Proof: Consider \( P_n \) with \( V(P_n) = \{v_i : 1 \leq i \leq n\} \). Here \( V(P_n) = V_1 \cup V_2 \), where \( V_1 = \{v_i : 2 \leq i \leq n - 1\} \) and \( V_2 = \{v_1, v_n\} \). Now in order to obtain \( DS(P_n) \) from \( G \), we add \( w_1, w_2 \) corresponding to \( V_1, V_2 \). Then
\[
|V(DS(P_n))| = n + 2 \quad \text{and}
\]
\[
E(DS(P_n)) = E(P_n) \cup \{v_1w_2, v_2w_2\}
\]
\[
\cup \{v_1v_i : 2 \leq i \leq n - 1\}
\]
so \(|E(DS(P_n))| = 2n - 1\).

We define vertex labeling \( f : V(DS(P_n)) \to \{1, 2, \ldots, n + 2\} \) as follows.
\[
f(v_1) = 1,
f(w_1) = 4,
f(v_2) = 3,
f(v_i) = 4 + i; \quad 1 \leq i \leq n - 2
\]

In view of the above defined labeling pattern, if \( n + 2 \equiv 0 (mod \ 16) \) then \( e_f(0) = n - 1, e_f(1) = n \), otherwise \( e_f(0) = n, e_f(1) = n - 1 \).

Thus, \(|e_f(0) - e_f(1)| \leq 1\).

Hence \( DS(P_n) \) is a square divisor cordial graph.

Illustration 2.16: Square divisor cordial labeling of the graph \( DS(P_7) \) is shown in Fig. 8.

![Fig. 8](image_url)

3. Concluding Remarks

The square divisor cordial labeling is a labeling with the blend of cordial and divisor cordial labelings. As all the graphs do not admit square divisor cordial labeling, it is very interesting to find out graph or graph families which are square divisor cordial graphs. Here we contribute some new results and many graphs are proved to be square divisor cordial graphs.
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