Chapter 4

*-n- Derivation in rings with involution

4.1 Introduction

An additive mapping \( x \mapsto x^* \) of a ring \( R \) into itself is called an involution on \( R \) if it satisfies the conditions; (i) \( (x^*)^* = x \), (ii) \( (xy)^* = y^*x^* \) for all \( x, y \in R \). A ring \( R \) equipped with an involution ‘*’ is called a *-ring. A ring \( R \) with involution ‘*’ is said to be *-prime if \( aRb = aRb^* = \{0\} \), where \( a, b \in R \) (equivalently \( aRb = a^*Rb = \{0\} \), where \( a, b \in R \) ) implies that either \( a = 0 \) or \( b = 0 \). It is to be noted that every prime ring having an involution ‘*’ is *-prime but the converse is not true in general. Of course, if \( R^o \) denotes the opposite ring of a prime ring \( R \), then \( R \times R^o \) equipped with the exchange involution \( *_{ex} \), defined by \( *_{ex}(x, y) = (y, x) \), is *-ex-prime but not prime. An ideal \( I \) of \( R \) is called a *-ideal of \( R \) if \( I^* = I \). Let \( R \) be a *-prime ring, \( a \in R \) such that \( aRa = \{0\} \). This implies that \( aRaR^*a = \{0\} \) also. Now *-primeness of \( R \) insures that \( a = 0 \) or \( aRa = \{0\} \). Now \( aRa = \{0\} \) together with \( aRa = \{0\} \) gives us \( a = 0 \). Thus we conclude that every *-prime ring is semiprime.

We introduce the notion of *-n-derivation in the *-ring \( R \), where \( n \) is a positive integer, and also investigate its various properties in Section 4.2. In fact, it is shown that if a prime *-ring \( R \) admits a nonzero *-n-derivation (resp. reverse *-n-derivation), then \( R \) is commutative. Further, some related properties of *-n-derivation in semiprime *-ring have also been investigated. Finally a structure theorem for *-n-derivation has also been obtained.

Section 4.3 is devoted to the extension of Posner’s first theorem in the setting of *-prime rings of characteristic different from 2. It is shown that if \( R \) is a *-prime ring of characteristic not 2 and \( d_1, d_2 \) derivations of \( R \) such that the iterate \( d_1d_2 \) is also a
derivation of $R$ and at least one of $d_1$ and $d_2$ commutes with ‘$\ast$’, then $d_1 = 0$ or $d_2 = 0$.

4.2 $\ast$-$n$-derivation in ring with involution

An additive mapping $d : R \rightarrow R$ is said to be a derivation (resp. reverse derivation) on $R$ if $d(xy) = d(x)y + xd(y)$ (resp. $d(xy) = d(y)x + yd(x)$) holds for all $x, y \in R$. Let $R$ be a $\ast$-ring. An additive mapping $d : R \rightarrow R$ is said to be a $\ast$-derivation (resp. $\ast$-reverse derivation) on $R$ if $d(xy) = d(x)y^\ast + xd(y)$ (resp. $d(xy) = d(y)x^\ast + yd(x)$) holds for all $x, y \in R$. If $R$ is a commutative $\ast$-ring then $d : R \rightarrow R$ defined by $d(x) = a(x - x^\ast)$, where $a \in R$, is a $\ast$-derivation on $R$ (for reference see [32]). An additive map $T : R \rightarrow R$ is called a left (resp. right) $\ast$-multiplier if $T(xy) = T(x)y^\ast$ (resp. $T(xy) = x^\ast T(y)$) holds for all $x, y \in R$. There has been a great deal of work concerning commutativity of prime and semiprime rings admitting certain types of derivations (for reference see [10] - [23], [33], [75], [83] etc., where further references can be found). Very recently Ali [4] defined symmetric $\ast$-biderivation, symmetric left (resp. right) $\ast$-bimultiplier and studied some properties of prime $\ast$-rings and semiprime $\ast$-rings, admitting symmetric $\ast$-biderivation and symmetric left (resp. right) $\ast$-bimultiplier. Motivated by these concepts and the notion of $n$-derivation given by Park (see [72]) we introduce the concept of $\ast$-$n$-derivation (reverse $\ast$-$n$-derivation) and $\ast$-$n$-multiplier in the setting of $\ast$-rings.

Let $n$ be any fixed positive integer. An $n$-additive (i.e.; additive in each argument) mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called a $\ast$-$n$-derivation of $R$ if the relations

$$D(x_1x'_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x'_1)^\ast + x_1 D(x'_1, x_2, \cdots, x_n)$$

$$D(x_1, x_2x'_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x'_2)^\ast + x_2 D(x_1, x'_2, \cdots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \cdots, x_nx'_n) = D(x_1, x_2, \cdots, x_n)(x'_n)^\ast + x_n D(x_1, x_2, \cdots, x'_n)$$

hold for all $x_1, x'_1, x_2, x'_2, \cdots, x_n, x'_n \in R$.

Similarly an $n$-additive mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called a reverse $\ast$-$n$-derivation of $R$ if the relations

$$D(x_1x'_1, x_2, \cdots, x_n) = D(x_1', x_2, \cdots, x_n)x_1^\ast + x'_1 D(x_1, x_2, \cdots, x_n)$$

$$D(x_1, x_2x'_2, \cdots, x_n) = D(x_1, x'_2, \cdots, x_n)x_2^\ast + x'_2 D(x_1, x_2, \cdots, x_n)$$

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D(x_1, x_2, \cdots, x_n x'_n) = D(x_1, x_2, \cdots, x'_n) x_n^* + x'_n D(x_1, x_2, \cdots, x_n)

hold for all \(x_1, x_2, x'_2, \cdots, x_n, x'_n \in R\).

For an example of \(\ast\)-n-derivation, consider \(C\) the ring of complex numbers with involution \(^*\) defined by \(z^* = \bar{z}\), where \(\bar{z}\) denotes the conjugate of the complex number \(z\). Now define \(D : C \times C \times \cdots \times C \rightarrow C\) such that

\[
D(z_1, z_2, \cdots, z_n) = \lambda(z_1 - \bar{z}_1)(z_2 - \bar{z}_2) \cdots (z_n - \bar{z}_n)
\]

where \(\lambda\) is any fixed complex number. One can easily verify that \(D\) is a \(\ast\)-n-derivation of \(C\).

An \(n\)-additive mapping \(T_1 : R \times R \times \cdots \times R \rightarrow R\) is called a left \(\ast\)-n-multiplier of \(R\) if

\[
T(x_1 x_1', x_2, \cdots, x_n) = T(x_1, x_2, \cdots, x_n)(x_1')^* \\
T(x_1, x_2 x_2', \cdots, x_n) = T(x_1, x_2, \cdots, x_n)(x_2')^* \\
\vdots \\
T(x_1, x_2, \cdots, x_n x'_n) = T(x_1, x_2, \cdots, x_n)(x_n')^*
\]

hold for all \(x_1, x_1', x_2, x_2', \cdots, x_n, x'_n \in R\).

An \(n\)-additive mapping \(T_2 : R \times R \times \cdots \times R \rightarrow R\) is called a right \(\ast\)-n-multiplier of \(R\) if

\[
T(x_1 x_1', x_2, \cdots, x_n) = x_1^* T(x_1', x_2, \cdots, x_n) \\
T(x_1, x_2 x_2', \cdots, x_n) = x_2^* T(x_1, x_2', \cdots, x_n) \\
\vdots \\
T(x_1, x_2, \cdots, x_n x'_n) = x_n^* T(x_1, x_2, \cdots, x'_n)
\]

hold for all \(x_1, x_1', x_2, x_2', \cdots, x_n, x'_n \in R\).

For examples of left \(\ast\)-n-multiplier and right \(\ast\)-n-multiplier, consider \(S\) to be a commutative ring which is not a zero ring and \(R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\} \). Define
Let $D$ be a $*$-derivation of $*$-ring $R$. If $D$ is also a permuting map, then all the above $n$-conditions used in the definition of $*$-$n$-derivation are equivalent and in this case $D$ is called permuting $*$-$n$-derivation of $*$-ring $R$ i.e., an $n$-additive permuting map $D : R \times R \times \cdots \times R \rightarrow R$ is called a permuting $*$-$n$-derivation of $*$-ring $R$ if

$$D(x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x_1')^* + x_1 D(x_1', x_2, \cdots, x_n)$$

hold for all $x_1, x_2, \cdots, x_n \in R$. It is obvious that every permuting $*$-$n$-derivation of $*$-ring $R$ is also a $*$-$n$-derivation but its converse is not true. For justification, let us consider the following example: Let $S$ be a noncommutative ring. Set

$$T_1, T_2 : R \times R \times \cdots \times R \rightarrow R$$

and $r \mapsto r^*$ of $R$ into itself, where $r \in R$ such that

$$T_1 \left( \begin{bmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$T_2 \left( \begin{bmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & z_1 z_2 \cdots z_n & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\left( \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \right)^* = \left( \begin{bmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix} \right).$$

One can easily verify that ‘$*$’ is an involution on $R$. Also it is straightforward to check that $T_1$ is a nonzero left $*$-$n$-multiplier but not a right $*$-$n$-multiplier of the $*$-ring $R$ and $T_2$ is a nonzero right $*$-$n$-multiplier but not a left $*$-$n$-multiplier of the $*$-ring $R$. Finally an $n$-additive mapping $T : R \times R \times \cdots \times R \rightarrow R$ is called an $*$-$n$-multiplier of $R$ if it is both a left $*$-$n$-multiplier and a right $*$-$n$-multiplier of $R$. For an example of $*$-$n$-multiplier, consider $\mathbb{C}$ the ring of complex numbers with involution ‘$*$’ defined by $z^* = \bar{z}$, where $\bar{z}$ denotes the conjugate of the complex number $z$. Now define $T : \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} \rightarrow \mathbb{C}$ such that $T(z_1, z_2, \cdots, z_n) = \mu \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n$, where $\mu$ is any fixed complex number. One can easily verify that $T$ is a $*$-$n$-multiplier of $\mathbb{C}$.
Let $R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, 0 \in S \right\}$. Define $D : R \times R \times \cdots \times R \to R$ and $r \mapsto r^*$ of $R$ into itself, where $r \in R$ such that

\[
D \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

One can easily show that $R$ is a $*$-ring and $D$ is a $*$-$n$-derivation of $R$. However $D$ is not a permuting $*$-$n$-derivation of $R$. Similarly the notions of permuting left $*$-$n$-multiplier, permuting right $*$-$n$-multiplier and permuting $*$-$n$-multiplier can be defined. It is obvious to observe that the map $D$ just discussed above is also a $*$-$n$-multiplier of $*$-ring $R$ but not a permuting $*$-$n$-multiplier of $R$.

In 1989 Brešar and Vukman (see Proposition 1 of [32]) proved that if a prime $*$-ring $R$ admits a nonzero $*$-derivation (resp. reverse $*$-derivation), then $R$ is commutative. We have proved its analogue in the setting of $*$-$n$-derivation for prime $*$-rings.

Some properties of $*$-$n$-derivation in semiprime $*$-rings have also been discussed. Some results related with $*$-$n$-multipliers in prime $*$-rings and semiprime $*$-rings have also been obtained. In the beginning of this section, we have shown that $*$-derivations generate different $*$-$n$-derivations in prime $*$-rings or commutative $*$-rings. In the end of present section a structure theorem for $*$-$n$-derivation in commutative $*$-rings has also been investigated.

We facilitate our discussion with the following lemmas;

**Lemma 4.2.1.** Let $R$ be a prime $*$-ring having $*$-derivations $d_1, d_2, \cdots, d_n$. If $D : R \times R \times \cdots \times R \to R$ such that $D(x_1, x_2, \cdots, x_n) = \{d_1(x_1)\}^*\{d_2(x_2)\}^*\cdots\{d_n(x_n)\}^*$, then $D$ is a $*$-$n$-derivation of $R$.

**Proof.** If at least one among $d_1, d_2, \cdots, d_n$ is a zero map then we are done. Now suppose that none of the given $*$-derivations of $R$ is zero. Then by Proposition 1 of [32] one
conclude that $R$ is commutative. Consider

$$D(x_1 + x'_1, x_2, \ldots, x_n) = \{d_1(x_1 + x'_1)\}^* \{d_2(x_2)\}^* \cdots \{d_n(x_n)\}^*$$

$$= \{d_1(x_1)\}^* \{d_2(x_2)\}^* \cdots \{d_n(x_n)\}^*$$

$$+ \{d_1(x'_1)\}^* \{d_2(x_2)\}^* \cdots \{d_n(x_n)\}^*$$

$$= D(x_1, x_2, \ldots, x_n) + D(x'_1, x_2, \ldots, x_n).$$

Thus $D$ is additive in the first argument. Similarly we can prove that it is additive in all arguments. Therefore $D$ is an $n$-additive map.

Consider

$$D(x_1 x'_1, x_2, \ldots, x_n) = \{d_1(x'_1 x_1)\}^* \{d_2(x_2)\}^* \cdots \{d_n(x_n)\}^*$$

$$= \{d_1(x'_1) x_1^* + x_1^* d_1(x_1)\}^* \{d_2(x_2)\}^* \cdots \{d_n(x_n)\}^* x_1$$

$$+ \{d_1(x'_1)\}^* \{d_2(x_2)\}^* \cdots \{d_n(x_n)\}^* (x'_1)^*$$

$$= D(x_1, x_2, \ldots, x_n) (x'_1)^* + x_1 D(x'_1, x_2, \ldots, x_n).$$

Similarly we can prove that the above property holds in all arguments. Therefore, $D$ is a $*-n$-derivation of $R$.

\[ \square \]

**Lemma 4.2.2.** Let $R$ be a prime $*$-ring having $*$-derivations $d_1, d_2, \ldots, d_n$. If $D : R \times R \times \cdots \times R \rightarrow R$ such that $D(x_1, x_2, \ldots, x_n) = d_1(x_1) d_2(x_2) \cdots d_n(x_n)$, then $D$ is a $*-n$-derivation of $R$.

**Proof.** If at least one among $d_1, d_2, \ldots, d_n$ is a zero map then we are done. Now suppose that none of the given $*$-derivations of $R$ is zero. Then by Proposition 1 of [32] one conclude that $R$ is commutative. It can be seen that $D$ is an $n$-additive map, and

$$D(x_1 x'_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n) (x'_1)^* + x_1 D(x'_1, x_2, \cdots, x_n).$$

Similarly we can prove that the above property holds in all arguments. Therefore $D$ is a $*-n$-derivation of $R$. \[ \square \]

**Lemma 4.2.3.** Let $R$ be a prime $*$-ring having $*-n$-derivations $D_1$ and $D_2$. Further assume that $I_1, I_2, \cdots, I_n$ are nonzero right ideals of $R$ such that $D_1(i_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n)$ for all $i_r \in I_r, 1 \leq r \leq n$. Then $D_1 = D_2$.

**Proof.** We have

$$D_1(i_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n) \quad (4.2.1)$$

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for all $i_r \in I_r, 1 \leq r \leq n$. Now putting $i_1r_1$, where $r_1 \in R$, for $i_1$ in the relation (4.2.1) we have $D_1(i_1r_1, i_2, \cdots, i_n) = D_2(i_1r_1, i_2, \cdots, i_n)$ i.e.; $D_1(i_1r_1, i_2, \cdots, i_n)\, r_1^* + i_1D_1(r_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n)\, r_1^* + i_1D_2(r_1, i_2, \cdots, i_n)$. Using the relation (4.2.1) we get $i_1D_1(r_1, i_2, \cdots, i_n) = i_1D_2(r_1, i_2, \cdots, i_n)$ i.e.; $i_1\{D_1(r_1, i_2, \cdots, i_n) - D_2(r_1, i_2, \cdots, i_n)\} = 0$. This shows that $i_1R\{D_1(r_1, i_2, \cdots, i_n) - D_2(r_1, i_2, \cdots, i_n)\} = \{0\}$. Since $I_1 \neq \{0\}$, primeness of $R$ implies that

$$D_1(r_1, i_2, \cdots, i_n) = D_2(r_1, i_2, \cdots, i_n) \quad (4.2.2)$$

for all $r_1 \in R, i_r \in I_r, 2 \leq r \leq n$. Now putting $i_2r_2$, where $r_2 \in R$, for $i_2$ in the relation (4.2.2) and using the similar arguments we find that $D_1(r_1, r_2, i_3, \cdots, i_n) = D_2(r_1, r_2, i_3, \cdots, i_n)$. Now proceeding inductively in the same way as above we conclude that $D_1 = D_2$. \hfill \Box

**Remark 4.2.1.** In 1989 Brešar and Vukman [32, Proposition 1] proved that if a prime $\ast$-ring $R$ admits a nonzero $\ast$-derivation (resp. reverse $\ast$-derivation) then $R$ is commutative. Recently Ali [4, Theorems 3.3 & 3.4] proved its analogue in the setting of symmetric $\ast$-bi-derivation for prime $\ast$-rings. We have shown that the restriction of symmetry of $\ast$-bi-derivation used by Ali is redundant. In fact, for $\ast$-$n$-derivation in a prime $\ast$-ring, we have obtained the following.

**Theorem 4.2.1.** Let $R$ be a prime $\ast$-ring. If it admits a nonzero $\ast$-$n$-derivation (resp. reverse $\ast$-$n$-derivation) $D$, then $R$ is commutative.

**Proof.** By hypothesis we have, for all $x_1, y, z, x_2, \cdots, x_n \in R$

$$D((x_1y)z, x_2, \cdots, x_n) = D(x_1y, x_2, \cdots, x_n)\, z^* + x_1yD(z, x_2, \cdots, x_n)$$

$$= \{D(x_1, x_2, \cdots, x_n)\, y^* + x_1D(y, x_2, \cdots, x_n)\}\, z^*$$

$$+ x_1yD(z, x_2, \cdots, x_n)$$

$$= D(x_1, x_2, \cdots, x_n)\, y^*z^* + x_1D(y, x_2, \cdots, x_n)\, z^*$$

$$+ x_1yD(z, x_2, \cdots, x_n).$$

Also

$$D(x_1(zy), x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)\, (yz)^* + x_1D(yz, x_2, \cdots, x_n)$$

$$= D(x_1, x_2, \cdots, x_n)\, z^*y^* + x_1\{D(yz, x_2, \cdots, x_n)\, z^*$$

$$+ yD(z, x_2, \cdots, x_n)\}$$

$$= D(x_1, x_2, \cdots, x_n)\, z^*y^* + x_1D(y, x_2, \cdots, x_n)\, z^*$$

$$+ x_1yD(z, x_2, \cdots, x_n).$$

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Combining the above two relations, we get
\[ D(x_1, x_2, \cdots, x_n)y^∗z^∗ = D(x_1, x_2, \cdots, x_n)z^∗y^∗ \text{ for all } x_1, x_2, \cdots, x_n, y, z \in R. \]
Putting \( y^∗ \) and \( z^∗ \) in the places of \( y \) and \( z \) respectively, we find that
\[ D(x_1, x_2, \cdots, x_n)yz = D(x_1, x_2, \cdots, x_n)zy. \tag{4.2.3} \]

Now replacing \( y \) by \( yr \) where \( r \in R \), in the relation \( (4.2.3) \) and using it again we arrive at
\[ D(x_1, x_2, \cdots, x_n)yrz = D(x_1, x_2, \cdots, x_n)yzr \text{ i.e.; } D(x_1, x_2, \cdots, x_n)R[r, z] = \{0\}. \]
Since \( D \neq 0 \), primeness of \( R \) implies that \( rz = zr \) for all \( z, r \in R \). Therefore, we conclude that \( R \) is commutative. \( \Box \)

**Corollary 4.2.1.** Let \( R \) be a noncommutative prime ring with involution ‘∗’. If it admits a ∗-n-derivation (resp. reverse ∗-n-derivation) \( D \), then \( D = 0 \).

Following example demonstrates that the primeness in the hypothesis of the above theorem cannot be omitted.

**Example 4.2.1.** Let \( Q \) and \( \mathbb{C} \) be the ring of real quaternions and complex numbers respectively. Assume \( R = Q \times \mathbb{C} \) is the ring of cartesian product of \( Q \) and \( \mathbb{C} \) with regard to componentwise addition and multiplication. Let ∗₁, ∗₂ and ∗ denote the involutions of rings \( Q, \mathbb{C} \) and \( R \) respectively, defined by \( q^∗₁ = α − βi − γj − δk \), where \( q = α + βi + γj + δk \in Q \); \( z^∗₂ = x − iy \), where \( z = x + iy \in \mathbb{C} \) and \((q, z)^* = (q^∗₁, z^∗₂)\) for all \((q, z) \in R \). Let \( d \) be ∗₂-derivation of \( \mathbb{C} \) defined by \( d(z) = η(z − z^∗₂) \) where \( η \) is any fixed complex number. Define \( D : R \times R \times ⋯ \times R \rightarrow R \) such that
\[ D((q_1, z_1), (q_2, z_2), \cdots, (q_n, z_n)) = (0, d(z_1)d(z_2)⋯d(z_n)). \]
It can be easily verified that \( R \) is a semiprime ring but not a prime ring and \( D \) is a nonzero ∗-n-derivation of \( R \). However, \( R \) is not commutative.

**Remark 4.2.2.** Lemma 4.2.3 also holds good for left ideals. In fact, in Lemma 4.2.3, if both \( D_1 \) and \( D_2 \) are zero then result holds trivially. On the other hand if at least one out of \( D_1 \) and \( D_2 \) is nonzero, then by Theorem 4.2.1, \( R \) is commutative and therefore each right ideal is also a left ideal.

**Theorem 4.2.2.** Let \( R \) be a prime ring with involution ‘∗’. If \( F : R^n \rightarrow R \) is a nonzero n-additive mapping such that \( F(x_1y, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)y^∗ \) for all \( x_1, y, x_2, \cdots, x_n \in R \), then \( R \) is commutative.

**Proof.** By hypothesis, for all \( x_1, y, z, x_2, \cdots, x_n \in R \) we have \( F(x_1yz, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)z^∗y^∗ \). On the other hand we also have \( F((x_1y)z, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)z^∗y^∗ \).
Let Corollary 4.2.2. Then

\[ F(x_1, x_2, \cdots, x_n)yz = F(x_1, x_2, \cdots, x_n)yz. \quad (4.2.4) \]

Now putting \( rz \) for \( z \) where \( r \in R \), in the relation \((4.2.4)\) and using it again we have

\[ F(x_1, x_2, \cdots, x_n)z[r, y] = 0 \] i.e.; \( F(x_1, x_2, \cdots, x_n)R[r, y] = \{0\} \). Since \( F \neq 0 \), primeness of \( R \) yields \( [r, y] = 0 \) for all \( r, y \in R \). Finally, we conclude that the ring \( R \) is commutative.

\[ \square \]

**Corollary 4.2.2.** Let \( R \) be a prime \(*\)-ring. If it admits a nonzero left \(*\)-n-multiplier \( T \), then \( R \) is commutative.

**Corollary 4.2.3** (\[4, \text{Theorem 2.4}]\). Let \( R \) be a prime \(*\)-ring. If \( M : R \times R \rightarrow R \) is a nonzero biadditive mapping such that \( M(xy, z) = M(x, z)y^* \) for all \( x, y, z \in R \), then \( R \) is commutative.

**Theorem 4.2.3.** Let \( R \) be a prime \(*\)-ring. If \( F : R^n \rightarrow R \) is a nonzero \( n \)-additive mapping such that \( F(yx_1, x_2, \cdots, x_n) = y^*F(x_1, x_2, \cdots, x_n) \) for all \( x_1, y, x_2, \cdots, x_n \in R \), then \( R \) is commutative.

**Proof.** Computing \( F((yz)x_1, x_2, \cdots, x_n) \) and \( F(y(xz_1), x_2, \cdots, x_n) \) where \( x_1, y, z, x_2, \cdots, x_n \in R \), in two different ways according to the given hypothesis and using the similar techniques as used to prove Theorem 4.2.2, one can easily get the required result. \( \square \)

**Corollary 4.2.4.** Let \( R \) be a prime \(*\)-ring. If \( R \) admits a nonzero right \(*\)-n-multiplier \( T \), then \( R \) is commutative.

**Corollary 4.2.5** (\[4, \text{Theorem 2.5}]\). Let \( R \) be a prime \(*\)-ring. If \( M : R \times R \rightarrow R \) is a nonzero bi-additive mapping such that \( M(xy, z) = x^*M(y, z) \) for all \( x, y, z \in R \), then \( R \) is commutative.

The following example shows that the restriction of primeness imposed on the hypotheses of Theorems 4.2.2 and 4.2.3 is not superfluous.

**Example 4.2.2.** Consider the \(*\)-ring \( R \) given in Example 4.2.1. Now define

\[ D_1 : R \times R \times \cdots \times R \rightarrow R \text{ such that } D_1((q_1, z_1), (q_2, z_2), \cdots, (q_n, z_n)) = \left(0, z_1^{n}z_2^{n} \cdots z_n^{n}\right). \]

It can be easily verified that \( R \) is a semiprime ring but not a prime ring and \( D_1 \) is a nonzero \( n \)-additive mapping of \( R \) such that \( D_1((q_1, z_1)(q_1', z_1'), (q_2, z_2), \cdots, (q_n, z_n)) = D_1((q_1, z_1), (q_2, z_2), \cdots, (q_n, z_n))(q_1', z_1')^* \) and \( D_1((q_1, z_1)(q_1', z_1'), (q_2, z_2), \cdots, (q_n, z_n)) = \left(0, z_1^{n}z_2^{n} \cdots z_n^{n}\right). \]
Corollary 4.2.6. Let \((q_1, z_1)^* D_1((q'_1, z'_1), (q_2, z_2), \ldots, (q_n, z_n))\) hold for all \((q_1, z_1), (q'_1, z'_1), (q_2, z_2), \ldots, (q_n, z_n)\) in \(R\). However \(R\) is not commutative.

**Remark 4.2.3.** The above Theorems 4.2.2 & 4.2.3 also hold if the relations in the hypotheses hold in any argument.

**Theorem 4.2.4.** Let \(R\) be a prime ring with involution ‘\(*\)’. If there exists \(0 \neq a \in R\) such that \(a^* x = \pm x^* a\) for all \(x \in R\). Then \(R\) is commutative.

**Proof.** We have \(a^* x = \pm x^* a\) for all \(x \in R\). Putting \(xy\), where \(y \in R\) in place of \(x\) in the preceding relation and using it again we get \(a^* xy = y^* a^* x\) for all \(x, y \in R\). Now replacing \(x\) by \(xt\), where \(t \in R\) in previous relation and using it again we obtain \(a^* R[t, y] = \{0\}\) for all \(y, t \in R\). Primeness of \(R\) yields either \(a^* = 0\) or \([t, y] = 0\). If first case holds then we obtain \((a^*)^* = 0\) i.e.; \(a = 0\) which is contrary to our hypothesis. Thus we conclude that \([t, y] = 0\) for all \(t, y \in R\). Hence \(R\) is commutative. \(\square\)

**Corollary 4.2.6.** Let \(R\) be a prime ring with involution ‘\(*\)’. If \(x^* y = \pm y^* x\) for all \(x, y \in R\). Then \(R\) is commutative.

**Theorem 4.2.5.** Let \(R\) be a 2-torsion free prime \(*\)-ring possessing \(*\)-\(n\)-derivations \(D_1\) and \(D_2\). Then

\[
D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n) = 0
\]

for all \(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n \in R\) iff either \(D_1 = 0\) or \(D_2 = 0\).

**Proof.** Given that

\[
D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n) = 0 \quad (4.2.5)
\]

for all \(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n \in R\). Then we have to show that either \(D_1 = 0\) or \(D_2 = 0\). Now putting \(y_1 z\) where \(z \in R\) in place of \(y_1\) in identity (4.2.5) we arrive at

\[
D_1(x_1, x_2, \cdots, x_n)D_2(y_1 z, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)D_1(y_1 z, y_2, \cdots, y_n) = 0
\]

for all \(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n \in R\) i.e.; \(D_1(x_1, x_2, \cdots, x_n)\{D_2(y_1, y_2, \cdots, y_n)z^* + y_1 D_2(z, y_2, \cdots, y_n)\} + D_2(x_1, x_2, \cdots, x_n)\{D_1(y_1, y_2, \cdots, y_n)z^* + y_1 D_1(z, y_2, \cdots, y_n)\} = 0\).

Now the previous relation takes the form

\[
\{D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n)\}z^* + D_1(x_1, x_2, \cdots, x_n)y_1 D_2(z, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)y_1 D_1(z, y_2, \cdots, y_n) = 0.
\]

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Using relation (4.2.5) we have,

\[ D_1(x_1, x_2, \cdots, x_n)y_1D_2(z, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)y_1D_1(z, y_2, \cdots, y_n) = 0. \]  

(4.2.6)

Multiplying by \( pD_1(r_1, r_2, \cdots, r_n) \) where \( r_1, r_2, \cdots, r_n; p \in R \) from right in the relation (4.2.6) we arrive at

\[ D_1(x_1, x_2, \cdots, x_n)y_1D_2(z, y_2, \cdots, y_n)pD_1(r_1, r_2, \cdots, r_n) + D_2(x_1, x_2, \cdots, x_n)(y_1D_1(z, y_2, \cdots, y_n)p)D_1(r_1, r_2, \cdots, r_n) = 0. \]

Relation (4.2.6) and 2-torsion freeness of \( R \) provide us

\[ D_1(x_1, x_2, \cdots, x_n)y_1D_1(z, y_2, \cdots, y_n)pD_2(r_1, r_2, \cdots, r_n) = 0 \]

i.e.,

\[ D_1(x_1, x_2, \cdots, x_n)y_1D_1(z, y_2, \cdots, y_n)RD_2(r_1, r_2, \cdots, r_n) = \{0\}. \]

Now primeness of \( R \) forces either

\[ D_1(x_1, x_2, \cdots, x_n)y_1D_1(z, y_2, \cdots, y_n) = 0 \]

or \( D_2 = 0 \). But in first case we have \( D_1(x_1, x_2, \cdots, x_n)RD_1(z, y_2, \cdots, y_n) = \{0\} \). Now again using the primeness of \( R \) we conclude that \( D_1 = 0 \). Finally we have shown that either \( D_1 = 0 \) or \( D_2 = 0 \). Converse is a trivial fact.

**Corollary 4.2.7.** Let \( R \) be a 2-torsion free prime \(*\)-ring, admitting \(*\)-derivations \( D_1 \) and \( D_2 \). Then \( D_1(x)D_2(y) + D_2(x)D_1(y) = 0 \) for all \( x, y \in R \) iff either \( D_1 = 0 \) or \( D_2 = 0 \).

**Theorem 4.2.6.** Let \( R \) be a semiprime \(*\)-ring, admitting a \(*\)-\( n \)-derivation \( D \). Then \( D(R, R, \cdots, R) \subseteq Z \).

**Proof.** Since \( R \) is a \(*\)-ring having a \(*\)-\( n \)-derivation \( D \), we have relation (4.2.3). Now putting \( yD(x_1, x_2, \cdots, x_n) \) in place of \( y \) in the relation (4.2.3) and using it again we get

\[ D(x_1, x_2, \cdots, x_n)y[D(x_1, x_2, \cdots, x_n), z] = 0 \]

for all \( x_1, x_2, \cdots, x_n; y, z \in R \). This relation provides us

\[ zD(x_1, x_2, \cdots, x_n)y[D(x_1, x_2, \cdots, x_n), z] = 0. \]  

(4.2.7)

Replacing \( y \) by \( zy \) in the relation \( D(x_1, x_2, \cdots, x_n)y[D(x_1, x_2, \cdots, x_n), z] = 0 \) we obtain that

\[ D(x_1, x_2, \cdots, x_n)zy[D(x_1, x_2, \cdots, x_n), z] = 0. \]  

(4.2.8)

Now comparing the identities (4.2.7) and (4.2.8) we arrive at

\[ D(x_1, x_2, \cdots, x_n)zy[D(x_1, x_2, \cdots, x_n), z] = zD(x_1, x_2, \cdots, x_n)y[D(x_1, x_2, \cdots, x_n), z] \]

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i.e.; \([D(x_1, x_2, \ldots, x_n), z]y[D(x_1, x_2, \ldots, x_n), z] = 0\). This relation provides us

\[
[D(x_1, x_2, \ldots, x_n), z] R[D(x_1, x_2, \ldots, x_n), z] = \{0\}.
\]

Now semiprimeness of \(R\) yields that \([D(x_1, x_2, \ldots, x_n), z] = 0\) i.e; \(D(R, R, \ldots, R) \subseteq Z\).

**Corollary 4.2.8.** Let \(R\) be a semiprime \(*\)-ring. If \(R\) admits a \(*\)-derivation \(d\), then \(d\) maps \(R\) in to \(Z\).

**Theorem 4.2.7.** Let \(R\) be a semiprime ring with involution ‘\(*\)’. If \(R\) admits an \(n\)-additive mapping \(F : R^n \to R\) such that \(F(x_1y, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)y^*\) for all \(x_1, x_2, \ldots, x_n, y \in R\). Then \(F(R, R, \ldots, R) \subseteq Z\).

**Proof.** By hypothesis \(R\) is a \(*\)-ring having an \(n\)-additive mapping \(F : R^n \to R\) such that \(F(x_1y, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)y^*\) for all \(x_1, x_2, \ldots, x_n, y \in R\), hence we have relation (4.2.4). Now substituting \(yF(x_1, x_2, \ldots, x_n)\) in place of \(y\) in the relation (4.2.4) and using it again we arrive at \(F(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0\) for all \(x_1, x_2, \ldots, x_n, y, z \in R\). This relation provides us

\[
zF(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0. \tag{4.2.9}
\]

Replacing \(y\) by \(zy\) in the relation \(F(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0\) we obtain that

\[
F(x_1, x_2, \ldots, x_n)zy[F(x_1, x_2, \ldots, x_n), z] = 0. \tag{4.2.10}
\]

Now comparing the identities (4.2.9) and (4.2.10) we arrive at \([F(x_1, x_2, \ldots, x_n), z]y[F(x_1, x_2, \ldots, x_n), z] = 0\) for all \(x_1, x_2, \ldots, x_n; y, z \in R\). This implies that \([F(x_1, x_2, \ldots, x_n), z] R[F(x_1, x_2, \ldots, x_n), z] = \{0\}\). Now semiprimeness of \(R\) yields that \([F(x_1, x_2, \ldots, x_n), z] = 0\) i.e; \(F(R, R, \ldots, R) \subseteq Z\).

**Corollary 4.2.9.** Let \(R\) be a semiprime \(*\)-ring. If it admits a left \(*\)-\(n\)-multiplier \(T\), then \(T(R, R, \ldots, R) \subseteq Z\).

**Corollary 4.2.10 ([4, Theorem 2.1]).** Let \(R\) be a semiprime \(*\)-ring. If \(M : R \times R \to R\) is a nonzero biadditive mapping such that \(M(xy, z) = M(x, z)y^*\) for all \(x, y, z \in R\), then \(M\) maps \(R \times R\) in to \(Z\).

**Corollary 4.2.11 ([3, Theorem 2.2]).** Let \(R\) be a semiprime \(*\)-ring. If \(T : R \to R\) is an additive mapping such that \(T(xy) = T(x)y^*\) for all \(x, y \in R\), then \(T\) maps \(R\) in to \(Z\).
**Theorem 4.2.8.** Let $R$ be a semiprime $\ast$-ring. If $R$ admits an $n$-additive mapping $F : R^n \rightarrow R$ such that $F(yx_1, x_2, \cdots, x_n) = y^*F(x_1, x_2, \cdots, x_n)$ for all $x_1, x_2, \cdots, x_n; y \in R$, then $F(R, R, \cdots, R) \subseteq Z$.

**Proof.** Using similar arguments with necessary variations as used to prove Theorem 4.2.7, one can easily obtain the required result. \hfill \Box

**Remark 4.2.4.** Focussing on the examples of left (resp. right) $\ast$-$n$-multipliers given in the beginning of this section, it is obvious to see that the restriction of semiprimeness imposed on the hypotheses of Theorems 4.2.7 & 4.2.8 is not superfluous.

**Corollary 4.2.12.** Let $R$ be a semiprime $\ast$-ring. If $R$ admits a right $\ast$-$n$-multiplier $T$, then $T(R, R, \cdots, R) \subseteq Z$.

**Corollary 4.2.13.** ([4, Theorem 2.2]). Let $R$ be a semiprime $\ast$-ring. If $M : R \times R \rightarrow R$ is a nonzero biadditive mapping such that $M(xy, z) = x^*M(y, z)$ for all $x, y, z \in R$, then $M$ maps $R \times R$ in to $Z$.

**Remark 4.2.5.** The above Theorems 4.2.7 & 4.2.8 also hold if the relations in the hypotheses hold in any argument.

**Theorem 4.2.9.** Let $R$ be a semiprime ring with involution $\ast$. If $D$ is a $\ast$-$n$-derivation of $R$ such that $D(x_1, x_2, \cdots, x_n)y_1 = x_1D(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n \in R$, then $D = 0$.

**Proof.** By hypothesis we have $D(x_1, x_2, \cdots, x_n)y_1 = x_1D(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n \in R$. Substituting $x_1z$ where $z \in R$ in the place of $x_1$ in the previous relation we obtain

$$D(x_1, x_2, \cdots, x_n)z^*y_1 + x_1D(z, x_2, \cdots, x_n)y_1 = x_1zD(y_1, y_2, \cdots, y_n).$$

Using hypothesis again we have

$$D(x_1, x_2, \cdots, x_n)z^*y_1 + x_1zD(y_1, y_2, \cdots, y_n) = x_1zD(y_1, y_2, \cdots, y_n)$$

i.e.; $D(x_1, x_2, \cdots, x_n)z^*y_1 = 0$. Now replacing $z$ in the preceding relation by $z^*$ we get $D(x_1, x_2, \cdots, x_n)zy_1 = 0$ for all $x_1, x_2, \cdots, x_n, y_1, z \in R$. As $y_1$ is an arbitrary element of $R$, using $D(x_1, x_2, \cdots, x_n)$ for $y_1$ in the relation $D(x_1, x_2, \cdots, x_n)zy_1 = 0$ we have $D(x_1, x_2, \cdots, x_n)D(x_1, x_2, \cdots, x_n) = 0$ i.e.; $D(x_1, x_2, \cdots, x_n)RD(x_1, x_2, \cdots, x_n) = \{0\}$. Finally semiprimeness of $R$ forces $D = 0$. \hfill \Box

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Corollary 4.2.14. Let \( R \) be a semiprime ring with involution ‘\(*\)’. If \( D \) is a \(*\)-derivation of \( R \) such that \( D(x)y = xD(y) \) for all \( x, y \in R \), then \( D = 0 \).

The following example shows that the restriction of semiprimeness imposed in the hypothesis of Theorem 4.2.9 is not superfluous.

Example 4.2.3. Consider the \(*\)-ring \( R \) used in the beginning of this section while constructing the examples of left (resp. right) \(*\)-\( n \)-multiplier. Define \( D : R \times R \times \cdots \times R \to R \) such that

\[
D \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

It is straightforward to check that \( R \) is not a semiprime \(*\)-ring and \( D \) is a \(*\)-\( n \)-derivation of \( R \) such that \( D(r_1, r_2, \cdots, r_n) = r_1 D(r'_1, r'_2, \cdots, r'_n) \) for all \( r_1, r_2, \cdots, r_n; r'_1, r'_2, \cdots, r'_n \in R \). However \( D \neq 0 \).

Theorem 4.2.10. Let \( R \) be a commutative \(*\)-ring admitting a \(*\)-derivation \( d \). Suppose \( I \) is a nonzero ideal of \( R \) such that it is invariant under both \(*\) and \( d \) i.e.; \( I^* \subseteq I \) and \( d(I) \subseteq I \). Then \( d \) induces an \( \hat{*} \)-\( n \)-derivation \( D \) on the quotient ring \( R/I \) where \( \hat{*} \) is an involution on quotient ring \( R/I \) induced by the involution \(*\) of \( R \).

Proof. Define a map \( x + I \mapsto (x + I)^* \) of \( R/I \) into itself such that \((x + I)^* = x^* + I \) for all \((x + I) \in R/I \). Let \( x + I = y + I \). This implies that \( x - y \in I \). Hence by hypothesis \((x - y)^* \in I \) i.e.; \( x^* - y^* \in I \). Therefore \( x^* + I = y^* + I \) i.e.; 

\[
(x + I)^* = \hat{*}(x + I)^* = ((x + I)^*)^* = (x^* + I)^* = (x^*)^* + I = x + I.
\]

Thus \( \hat{*} \) is a well defined map on quotient ring \( R/I \). By using addition and product of quotient ring \( R/I \) we see that \( i) \ (\{x + I + (y + I)\}^* = (x + y) + I = (x + I)^* + I = (x + I)^* + I = (x^* + I) = (x + I)^* + I = (x + I)^* = (x^*)^* + I = x + I \) and \( ii) \ (\{x + I\}^* = (x + I)^* = (x^*)^* + I = (x^*)^* + I = (y + I)^* \) for all \((x + I), (y + I) \in R/I \). All previous facts \( i), ii) \) and \( iii) \) insure that \( \hat{*} \) is an involution on quotient ring \( R/I \). Now define \( D : R/I \times R/I \times \cdots \times R/I \to R/I \) as below \( D((x_1 + I), (x_2 + I), \cdots, (x_n + I)) = d(x_1) d(x_2) \cdots d(x_n) + I \).

Let \((x_1 + I), (x_2 + I), \cdots, (x_n + I) = ((y_1 + I), (y_2 + I), \cdots, (y_n + I) \). This implies that \((x_1 - y_1) \in I, (x_2 - y_2) \in I, \cdots, (x_n - y_n) \in I \). By hypothesis \( d(x_1) + I = d(y_1) + I, d(x_2) + I = d(y_2) + I, \cdots, d(x_n) + I = d(y_n) + I \) i.e.; \((d(x_1) + I) (d(x_2) + I) \cdots (d(x_n) + I) = (d(y_1) + I) (d(y_2) + I) \cdots (d(y_n) + I) \). Now we obtain that \( d(x_1) d(x_2) \cdots d(x_n) + I = d(y_1) d(y_2) \cdots d(y_n) + I \) i.e.; \( D((x_1 + I), (x_2 + I), \cdots, (x_n + I)) = D((y_1 + I), (y_2 + I), \cdots, (y_n + I)) \).
Remark 4.2.6. By above proof it is clear that for $I$, $d$ is a well defined map. Consider $D((x_1 + I) + (x'_1 + I), (x_2 + I), \cdots, (x_n + I)) = D((x_1 + x'_1 + I), (x_2 + I), \cdots, (x_n + I)) = d(x_1 + x'_1)d(x_2) \cdots d(x_n) + I = (d(x_1)d(x_2) \cdots d(x_n) + I) + (d(x'_1)d(x_2) \cdots d(x_n) + I) = D((x_1 + I), (x_2 + I), \cdots, (x_n + I)) + D((x'_1 + I), (x_2 + I), \cdots, (x_n + I))$. The previous relation insures that $D$ is additive in the first argument. Similarly one can show that $D$ is additive in all arguments. Thus $D$ is an $n$-additive map. Consider $D((x_1 + I)(x'_1 + I), (x_2 + I), \cdots, (x_n + I)) = D((x_1x_1' + I), (x_2 + I), \cdots, (x_n + I)) = d(x_1x_1')d(x_2) \cdots d(x_n) + I = \{(d(x_1)(x_1')^* + x_1d(x_1'))d(x_2) \cdots d(x_n)\} + I = \{(d(x_1)d(x_2) \cdots d(x_n)(x_1')^* + x_1d(x_1')d(x_2) \cdots d(x_n))\} + I = \{(d(x_1)d(x_2) \cdots d(x_n) + I)((x_1')^* + I)\} + \{(x_1 + I)(d(x_1')d(x_2) \cdots d(x_n) + I)\} = D((x_1 + I), (x_2 + I), \cdots, (x_n + I))(x_1' + I)^* + (x_1 + I)D((x_1' + I), (x_2 + I), \cdots, (x_n + I))$. Similarly we can prove that the previous relation holds in each argument. Finally we conclude that $D$ is a $\hat{*}$-n-derivation on quotient ring $R/I$.

\begin{itemize}
  \item Remark 4.2.6. By above proof it is clear that for $n = 1$ the commutativity in the hypothesis of Theorem 4.2.10 becomes redundant. Thus we can say, if $R$ is a $*$-ring having $*$-derivation $d$ and $I$ is a nonzero ideal of $R$ such that it is invariant under both $*$ and $d$ i.e.; $I^* \subseteq I$ and $d(I) \subseteq I$, then $d$ induces an $\hat{*}$-derivation $D$ on the quotient ring $R/I$ where $\hat{*}$ is an involution on quotient ring $R/I$ induced by the involution $*$ of $R$.
\end{itemize}

4.3 Posner’s first theorem for $*$-prime rings

An additive mapping $d : R \to R$ is said to be a derivation on $R$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let $I$ be a nonzero ideal of $R$. Then an additive mapping $d : I \to R$ is called a derivation from $I$ to $R$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in I$. In the year 1957, E. C. Posner initiated the study of derivations in rings and proved two very striking theorems. These results have been generalized by several authors in different directions (see for reference [29], [61], & [64] for reference where further references can be found). Posner’s first theorem [75, Theorem 1] states that if $R$ is a prime ring of characteristic not 2 and the iterate of two derivations on $R$ is also a derivation, then at least one of them is zero. In this section we extend this result to $*$-prime rings of characteristic different from 2.

\begin{itemize}
  \item Theorem 4.3.1. Let $R$ be a $*$-prime ring of characteristic not 2, $I$ a nonzero $*$-ideal and $d_1, d_2 : I \to R$ are derivations such that the product map $d_1d_2 : I \to R$ is also a derivation. If at least one of $d_1$ and $d_2$ commutes with ‘$*$’, then $d_1 = 0$ or $d_2 = 0$.
\end{itemize}

For developing the proof of the above theorem we begin with the following lemmas:
Lemma 4.3.1. If $R$ is a \ast\-prime ring of characteristic different from 2, then $R$ is 2-torsion free.

Proof. Suppose that $x \in R$ such that $2x = 0$. This implies that $2xrs = 0$ for all $r, s \in R$ i.e.; $xR(2s) = \{0\}$ for all $s \in R$. Since characteristic of $R$ is different from 2 and $R \neq \{0\}$, this provides us a nonzero element $l \in R$ such that $2l \neq 0$. Now we conclude that $xR(2l) = \{0\} = xR(2l)^*$. Finally \ast\-primeness of $R$ provides us $x = 0$ and hence $R$ is 2-torsion free. 

Lemma 4.3.2. Let $R$ be a \ast\-prime ring and $I$ a nonzero \ast\-ideal of $R$. If $d : I \rightarrow R$ is a derivation such that $d$ commutes with \ast\. If $a$ is an element of $R$ and $ad(x) = 0$ (resp. $d(x)a = 0$) for all $x \in I$, then either $a = 0$ or $d = 0$.

Proof. Replacing $x$ by $xy$, where $y \in I$ in the relation $ad(x) = 0$, we obtain that $ad(x)y + axd(y) = 0$ i.e.; $axd(y) = 0$ for all $x, y \in I$. Replacing $x$ by $xs$ where $s \in R$ in the latter relation, we arrive at $axsd(y) = 0$ i.e.; $axRd(y) = \{0\}$ for all $x, y \in I$. Since $d$ commutes with \ast\ and $I$ is a \ast\-ideal, we obtain that $axRd(y) = \{0\} = axR\{d(y)\}^*$ for all $x, y \in I$. Now \ast\-primeness of $R$ provides us $d = 0$ or $ax = 0$ for all $x \in I$. Putting $tx$ where $t \in R$ for $x$ in the latter relation, we arrive at $atx = 0$ i.e.; $aRx = \{0\}$ for all $x \in I$. Since $I$ is a \ast\-ideal of $R$, we also have $aRx = aRx^* = \{0\}$. Now \ast\-primeness of $R$ and $I \neq \{0\}$ imply that $a = 0$. Similarly we can also show that $d(x)a = 0$ for all $x \in I$ implies that $a = 0$ or $d = 0$.

Proof of the Theorem 4.3.1. We divide the proof in following two cases:

Case I: Let us suppose that $d_1$ commutes with \ast\. Since the map $d_1d_2 : I \rightarrow R$ is a derivation, it is obvious that $d_2(I) \subseteq I$ and $d_1d_2(xy) = d_1d_2(x)y + xd_1d_2(y)$ for all $x, y \in I$. As $d_1, d_2 : I \rightarrow R$ are derivations, we obtain that

$$d_1d_2(xy) = d_1(d_2(xy)) = d_1d_2(x)y + d_2(x)d_1(y) + d_1(x)d_2(y) + xd_1d_2(y).$$

By above relations we conclude that

$$d_2(x)d_1(y) + d_1(x)d_2(y) = 0 \text{ for all } x, y \in I. \tag{4.3.1}$$

Now replacing $x$ by $xd_2(z)$, where $z \in I$ in the relation (4.3.1) we obtain that

$$d_2(xd_2(z))d_1(y) + d_1(xd_2(z))d_2(y) = 0$$

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for all \(x, y, z \in I\). This gives us 
\[d_2(x)d_2(z)d_1(y) + xd_2^2(z)d_1(y) + d_1(x)d_2(z)d_2(y) + xd_1d_2(z)d_2(y) = 0.\]
In view of equation (4.3.1) and using the fact that \(d_2(I) \subseteq I\), we find that 
\[(d_2(d_2(z)))d_1(y) + d_1(d_2(z))d_2(y)) = 0.\] Hence we arrive at
\[d_2(x)d_2(z)d_1(y) + d_1(x)d_2(z)d_2(y) = 0\] (4.3.2)
for all \(x, y, z \in I\). Using the relation (4.3.1) and Lemma 4.3.1, the relation (4.3.2) reduces to 
\[d_1(x)d_2(z)d_2(y) = 0\] for all \(x, y, z \in I\). Now Lemma 4.3.2 provides us either 
\(d_1 = 0\) or \(d_2(z)d_2(y) = 0\) for all \(y, z \in I\). If the first case holds then nothing to do, if not we have 
\(d_2(z)d_2(y) = 0\) for all \(y, z \in I\). Replacing \(y\) by \(yz\) in the latter relation and using 
the same again we arrive at 
\[d_2(z)y_2(z) = 0\] for all \(y, z \in I\). Replacing \(y\) by \(sy\) where 
\(s \in R\) in the latter relation we arrive at 
\[d_2(z)y_2d_2(z) = \{0\}\] i.e., 
\[y_2d_2(z)y_2d_2(z) = \{0\}\] for all \(y, z \in I\). Since \(R\) is a \(\ast\)-prime ring, it is semiprime also and hence we obtain that 
\[y_2d_2(z) = 0\] for all \(y, z \in I\). Replacing \(y\) by \(yt\) where \(t \in R\) in the latter relation we arrive at 
\[ytd_2(z) = 0\] i.e., 
\[yRd_2(z) = \{0\}\] for all \(y, z \in I\). But since \(I\) is a \(\ast\)-ideal of \(R\), also get 
\[y^Rd_2(z) = \{0\}\] for all \(y, z \in I\). Finally \(\ast\)-primeness of \(R\) and \(I \neq \{0\}\) imply that 
\(d_2 = 0\).

Case II: Let us suppose that \(d_2\) commutes with \(\ast\). From Case I, we have 
\[d_1(x)d_2(z)d_2(y) = 0\] for all \(x, y, z \in I\). Now Lemma 4.3.2 provides us either 
\(d_2 = 0\) or \(d_1(x)d_2(z) = 0\) for all \(x, z \in I\). If the first case holds then nothing to do, if not we have 
\(d_1(x)d_2(z) = 0\) for all \(x, z \in I\). Again using Lemma 4.3.2 we conclude that either 
\(d_1 = 0\) or \(d_2 = 0\).

The following example shows that the hypothesis of \(\ast\)-primeness is crucial in the above theorem.

**Example 4.3.1.** Let \(R = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mid x, y, z, 0 \in \mathbb{Z} \right\}\), where \(\mathbb{Z}\) is the set of integers.

Consider the map
\[
\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mapsto \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^* 
\]
of \(R\) into itself such that
\[
\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^* = \begin{pmatrix} z & 0 \\ -y & x \end{pmatrix}.
\]
It is easy to verify that \(\ast\) is an involution of the ring \(R\), where characteristic of \(R\) is different from 2. Further if we set 
\(I = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid y, 0 \in \mathbb{Z} \right\}\), then \(I\) is a nonzero
*-ideal of $R$. Now consider the maps $d_1, d_2, : I \rightarrow R$ defined by

$$d_1 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, \quad d_2 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -y & 0 \end{pmatrix}.$$  

Then it is obvious to observe that $d_1$ and $d_2$ are derivations and ‘*’ commutes with $d_1$. Further it can be also shown that the map $d_1d_2 : I \rightarrow R$ is a derivation and $R$ is not a *-prime ring. However neither $d_1 = 0$ nor $d_2 = 0$.

The following example shows that the hypothesis of ”characteristic different from 2” is crucial in the above theorem.

**Example 4.3.2.** Suppose that $R = \mathbb{Z}_2[x] \times \mathbb{Z}_2[x]$, where $\mathbb{Z}_2[x]$ is the polynomial ring over $\mathbb{Z}_2$. Let us consider the map $(f(x), g(x)) \mapsto (f(x), g(x))^*$ of $R$ into itself such that $(f(x), g(x))^* = (g(x), f(x))$. It is easy to check that ‘*’ is an involution of $R$, known as exchange involution denoted by $*_\text{ex}$ and $R$ is a $*_\text{ex}$-prime ring. Further assume that $I = [x^2]$ is the ideal of $\mathbb{Z}_2[x]$ generated by $x^2 \in \mathbb{Z}_2[x]$. Then it can be easily shown that $I = I \times I$ is a nonzero $*_\text{ex}$-ideal of $R$. Next consider $D_1, D_2 : I \rightarrow R$ such that $D_1(f(x), g(x)) = (d(f(x)), d(g(x)))$ and $D_2(f(x), g(x)) = (d(f(x)), 0)$, where $d$ is the usual differentiation in $\mathbb{Z}_2[x]$. It is obvious to see that $D_1$, $D_2$ and $D_1D_2 : I \rightarrow R$ are derivations. Moreover, $R$ is a ring of characteristic 2 and $D_1*_\text{ex} = *_\text{ex}D_1$. However $D_1 \neq 0$ and $D_2 \neq 0$.

Now taking $I = R$ in the above theorem we obtain the following:

**Corollary 4.3.1.** Let $R$ be a $*_\text{ex}$-prime ring of characteristic not 2 and $d_1, d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation of $R$. If at least one of $d_1$ and $d_2$ commutes with ‘*’, then $d_1 = 0$ or $d_2 = 0$.

Now using the above theorem we can obtain Posner’s first theorem.

**Corollary 4.3.2 ( [75, Theorem 1]).** Let $R$ be a prime ring of characteristic not 2 and $d_1, d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation, then one at least of $d_1, d_2$ is zero.

**Proof.** Since $R$ is a prime ring of characteristic not 2, $\mathcal{R} = R \times R^p$ is clearly a $*_\text{ex}$-prime ring of characteristic not 2. Set $I = \mathcal{R}$, which is a nonzero $*_\text{ex}$-ideal of $\mathcal{R}$. Now define $D_1, D_2 : I \rightarrow \mathcal{R}$ by $D_1(x, y) = (d_1(x), d_1(y))$ and $D_2(x, y) = (d_2(x), d_2(y))$. Using hypothesis it can be easily seen that $D_1, D_2 : I \rightarrow \mathcal{R}$ are derivations and the product map $D_1D_2 : I \rightarrow \mathcal{R}$ is also a derivation. Moreover $D_1*_\text{ex} = *_\text{ex}D_1$. In view of the
Theorem 4.3.1 we deduce that either $D_1 = 0$ or $D_2 = 0$, in turn we obtain that either $d_1 = 0$ or $d_2 = 0$. □