CHAPTER – I

I. INTRODUCTION AND PREREQUISITES

Topology began in part out of necessity in the field of analysis, but has developed into a subject of its own that is very geometric in flavor. Topological ideas are present in almost all areas of today's mathematics. The subject of topology itself consists of several different branches, such as point set topology, algebraic topology and differential topology, which have relatively little in common. Topology is the study of sets on which one has a notion of "closeness" -- enough to decide which functions defined on it are continuous.

In quite diverse domains of Mathematics and their applications, certain limits or completion processes have to be performed. A fundamental tool for such constructions is that of closure or hull operators. The resulting collection of closed sets are stable under arbitrary intersections. If \( C \) is such a closure system on a set \( X \), the pair \( (X,C) \) is referred to as a closure space. The associated closure operator maps any subset of \( X \) to the least member of \( C \) containing it.

Easy is out of question in tracing back the development of this fundamental mathematical approximation to its roots. But for sure the following names have to be cited in connection with the origins of closure: Schroder, Dedekind, Cantor, Riesz, Hausdorff, Moore, Čech, Kuratowski, Sierpinski, Tarski, Birkhoff and Ore. Their legacy in the area of closure structures cannot be sketched in a few lines in a thesis form. So we commence with Moore, Kuratowski and advance to E. Čech and compile our findings in this thesis.

Closure operators are determined by their closed sets ie. by the sets of the form \( cl(X) \), which denotes the smallest closed set containing \( X \). Such families of "closed sets" are sometimes called “Moore families”, in honour of E H. Moore who studied closure operators in 1911 [81]. Closure operators are also called “hull operators” to prevent confusion with the “closure operators” studied in topology. A set together with a closure operators on it is called a closure systems. Nonetheless, it was the formulation of the axioms for topological space in terms of a closure operator by Kazimierz Kuratowski [66] in 1922 which introduced this concept into common use.
Closure operators have many applications. In topology, the closure operators are topological closure operators, which must satisfy \( \text{cl}(X_1 \cup X_2 \ldots \cup X_n) = \text{cl}(X_1) \cup \ldots \cup \text{cl}(X_n) \) of all \( n \in \mathbb{N} \).

A topological space \((X, \text{cl})\) is set \( X \) with a function \( \text{cl}: P(X) \rightarrow P(X) \) called the closure operator where \( P(X) \) is the power set of \( X \).

The closure operator has to satisfy the following properties for all \( A, B, \in P(X) \)

\( i) \ A \subseteq \text{cl}(A) \) (Extensivity)

\( ii) \ \text{cl}(\text{cl}(A)) = \text{cl}(A) \) (Idempotence)

\( iii) \ \text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B) \) (Preservation of binary Union)

\( iv) \ \text{cl}(\emptyset) = \emptyset \) (Preservation of nullary Unions)

If the second axiom that of idempotence is relaxed, then the axioms define a preclosure operator or Čech closure operator which will be dealt in detail in the later pages.

Generalized closed sets, briefly g-closed sets in a topological space were introduced by N.Levine [67] in order to extend some important properties of closed sets to a larger family of sets. The study of g-closed sets has given possible applications in Computer graphics [63], Computer Science and digital topology [65]. In Topology [36][37][46][61][94][99] define connected spaces by means of open and closed sets. The concept of generalized closed sets and generalized continuous maps [4] of a topological space were extended to a closure space in [8] and [10]. Palaniappan and Rao K.C [87] [88] introduced rg-closed sets. Later \( \pi \)-open, \( \pi \) closed sets and Quasi-normal spaces were initiated by Zaitsev [101]. Dontchev and Noirir [39] defined the notion of \( \pi g \)-closed sets. \( \pi g \)-closed sets are weak form of g-closed set due to Levine et.al. [58][67][84]. \( \beta \)-open and b-open sets were introduced by Andrijevic [2] [3]. Since the advent of these sets several research papers with interesting results came to existence [1][33][43][44][38][68][70][75][74][82][83][89]. J.C. Kelly [62] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Kelly et.al. [77][78] further extended some of the standard results of separation axioms in a topological space. With all the above predefined sets and maps we have studied Closure spaces.

In Chapter I the origin and history of closure spaces and contributions by various authors are mentioned. Section 1 begins with the discussion of the isotonic
spaces. Section 2 deals with Čech closure spaces. Section 3 describes Bi-Čech closure spaces. Section 4 explains the concept of fuzzy Biclosure spaces. Section 5 outlines the contribution of the author.

In Chapter II of the thesis, we initiate the new concept of irg-closed sets in isotonic space. We define a new class of space namely $T_{\text{irg}}$-space and study their properties along with the characterizations of irg-continuous maps and irg-closed maps. We propose a new notion of $\beta$ connectedness and $\beta$ separated sets in generalised $\beta$ closure space. The last section deals with lower separation axioms in Isotonic spaces.

In Chapter III we deal with the concepts of $\pi g\beta$- closed sets, $\pi g\beta$- open sets, $\pi g\beta$- continuous maps, $\pi g\beta$- irresolute maps in Čech spaces. Utilizing this concept we also define Contra-$\pi g\beta$-continuous maps and Contra $\pi g\beta$-irresolute maps. As a step further we define the notion of $f$-compactness and $f$-connectedness in $\pi g\beta$-closed sets and investigate their properties.

In Chapter IV, we have formulated three types of different sets in Čech Space, BiČech Space and in Isotonic Space. We have introduced a new class of set namely D open set in Čech closure space and investigate their characterizations. We initiate the concept of pairwise $\pi$-closed set in BiČech closure space and their preservations are discussed. We also intend to coin the idea of generalized $\beta$-closure spaces and separations in isotonic spaces.

Chapter V pioneers the idea of $\pi g\beta$–closed sets and $\pi g\beta$–open sets in BiČech Closure Spaces. We also introduce $\beta$–normal biclosure space and study their relations. Further the next section poses a new form of $\pi$-Bicontinuous maps, pairwise $\pi$-Bicontinuous maps in biclosure spaces and discuss their characterizations.

Chapter VI proposes a concept of $\Delta$ Čech sets in Fuzzy closure Spaces. We have elucidated a new type of continuity namely $\Delta_f$ continuity and $\Delta_f$ irresolute mappings in Fuzzy Closure Spaces and their properties are studied. We also define $\Delta_f$ normal and $\Delta_f$ Hausdorff space in fuzzy biclosure spaces.

### 1.1 ISOTONIC SPACES

The generalization of a closure space is an isotonic space developed by many researchers[19][56][76]. It is also that not every property that hold in topological space
must hold in isotonic space. But every topological space is an isotonic space and if a property does not hold in topological space, must not hold in any isotonic space either.

An operator $\mu : P(X) \rightarrow P(X)$ is called

- **grounded** if $\mu\emptyset = \emptyset$,
- **isotonic** if $A \subseteq B \subseteq X$ implies $\mu A \subseteq \mu B$,
- **expansive** if $A \subseteq \mu A$ for every $A \subseteq X$,
- **idempotent** if $\mu (\mu A) = \mu A$ for every $A \subseteq X$ and
- **additive** if $\mu (A \cup B ) \subseteq \mu A \cup \mu B$ for all subsets $A$ and $B$ of $X$.

**Definition: 1.1.1**

1. The space $( X, \mu)$ is said to be **isotonic** if $\mu$ is grounded and isotonic.
2. The space $( X, \mu)$ is said to be a **neighbourhood space** if $\mu$ is grounded, expansive and isotonic.
3. The space $( X, \mu)$ is said to be a **closure space** if $\mu$ is grounded, expansive, isotonic and idempotent.
4. The space $( X, \mu)$ is said to be a **Čech closure space** if $\mu$ is grounded, expansive, isotonic and additive.

1.2 **ČECH CLOSURE SPACES**

Closure spaces were introduced by Eduard Čech in 1966 [22](i.e sets endowed with a grounded, extensive and additive closure operator) and then studied by many authors [7][21][23][30][31][59][71][91][92]. Structure of a closure space is more general than that of a topological space. Hammer[52][53][54][55] studied closure spaces more generally Isotonic Spaces and a recent study on these spaces can be found in Gnilka[47][48][49], Harris[56], Habil and Elzenati[50][51], Stadler[95][96], Day[34] and by Hausdorff [57]. Stadler[95][96] studied separation axioms on a generalized closure space. The definition of generalized closure space is given in [50][51][96]. The notions of closure system and closure operator are very useful tools in several areas of Mathematics playing an important role in the study of topological spaces, Boolean algebra and convex sets.
**Definition: 1.2.1**

A map \( k: \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) defined on the power set \( \mathcal{P}(X) \) of a set \( X \) is called a closure operator on \( X \) and the pair \( (X, k) \) is called a closure space if the following axioms are satisfied.

(i) \( k(\emptyset) = \emptyset \)

(ii) \( A \subseteq k(A) \) for every \( A \subseteq X \)

(iii) \( k(A \cup B) = k(A) \cup k(B) \) for all \( A, B \subseteq X \)

A closure operator \( k \) on a set \( X \) is called idempotent if \( k(A) = k[k(A)] \) for all \( A \subseteq X \).

**Definition: 1.2.2 [22]**

A subset \( A \) of a Čech closure space \( (X, k) \) is said to be

(i) Čech closed if \( k(A) = A \)

(ii) Čech open if \( k(X-A) = X-A \)

(iii) Čech semi-open if \( A \subseteq k(\text{int}(A)) \)

(iv) Čech pre-open if \( A \subseteq \text{int}[k(A)] \)

(v) Čech pre-closed if \( k[\text{int}(A)] \subseteq A \)

(vi) Čech \( \alpha \)-open if \( A \subseteq \text{int}[k(\text{int}(A))] \)

(vii) Čech \( \alpha \)-closed if \( k[\text{int}(k(A))] \subseteq A \)

(viii) Čech regular closed if \( A = k(A) \).

(ix) Čech \( \beta \)-open if \( A \subseteq k(\text{int}(k(A)) \)

(x) Čech \( \beta \)-closed if \( \text{int}(k(\text{int}(A))) \subseteq A \).

(xi) Čech \( g \)-closed if \( k(A) \subseteq G \) whenever \( G \) is an open subset with \( A \subseteq G \).

**Definition: 1.2.3 [28]**

Let \( f: (X, k_1) \rightarrow (Y, k_2) \) be a function from Čech closure space \( (X, k_1) \) into \( (Y, k_2) \). The function \( f \) is called

(i) Čech open if \( f(A) \) is Čech open set in \( Y \) for each Čech open set \( A \) of \( X \).

(ii) Čech \( \beta \)-open if \( f(A) \) is a Čech \( \beta \) – open set in \( Y \), for each Čech open set \( A \) of \( X \).

(iii) Čech preclosed if \( f(A) \) is a Čech preclosed set of \( Y \),for each Čech closed set \( A \) of \( X \)

(iv) Almost Čech open if \( f(A) \) is open set in \( Y \), for each Čech regular open set \( A \) of \( X \).
(v) Weekly Čech open if \( f(A) \subseteq \text{int}(f(k(A))) \) for each Čech open set \( A \) of \( X \).

(vi) Čech \( \delta \)–open if \( f(A) \) is Čech \( \delta \)–open set of \( Y \) for each Čech open set \( A \) of \( X \).

**Definition: 1.2.4 [14]**

A space \((X,k)\) is said to be connected if it cannot written as two disjoint nonempty subsets \( A \) and \( B \) such that \( k(A) \cup k(B) = X \), \( k(A) \cap k(B) = \emptyset \) and \( k(A) \) and \( k(B) \) are non empty.

**Definition: 1.2.5 [14]**

A closure operator \( k \) on a set \( X \) is called idempotent if \( kA = k(kA) \) for all \( A \subseteq X \).

**Definition: 1.2.6 [14]**

A subset \( A \) is closed in the Čech closure space \((X,k)\) if \( kA = A \) and it is open if its complement is closed. The empty set and the whole space are both open and closed.

**Definition: 1.2.7 [64]**

A subset \( B \) of a closure space is said to be r-open if there exists a open set \( G \) such that \( G \subseteq B \) and \( kG = kB \). It is r-closed if its complement is r-open.

**Definition: 1.2.8 [64]**

A subset \( B \) of a closure space is said to be b-open if there exists a open set \( G \) such that \( B \subseteq k(\text{int}(B)) \cup \text{int}(k(B)) \). It is b-closed if its complement is b-open.

**Definition: 1.2.9 [64]**

A subset \( B \) of a closure space is said to be gs-open if there exists a semi-open set \( G \) such that \( G \subseteq B \subseteq k(G) \). The complement of \( B \) is gs-closed.

**Definition: 1.2.10 [64]**

A space is said to be \( T_r \) space if every r-open set is open.
Definition: 1.2.11 [64]
A space is said to be $T_{sr}$ space if every semi-open set is $r$-open.

Definition: 1.2.12 [64]
A space is said to be $T_{gs}$ space if every g-semi-open set is semi-open.

Definition: 1.2.13 [64]
A space is said to be $T_s$ space if every semi-open set is open.

Definition: 1.2.14 [41][42]
A space is said to be $T_\pi$ space if every closed set is $\pi$-closed and (resp open set is $\pi$-open).

Definition: 1.2.15 [14]
Let $(Y, v)$ be a Čech closed subspace of $(X,u)$. If $F$ is a closed subset of $(Y,v)$, then $F$ is a closed subset of $(X,u)$.

Definition: 1.2.16 [14]
If $f:(X,u)\to(Y,v)$ is continuous, then $f^{-1}(F)$ is a closed subset $(X,u)$ for every closed subset $F$ of $(Y,v)$.

Definition: 1.2.17 [8]
A Čech closure space $(Y, v)$ is said to be a subspace of $(X, u)$ if $Y \subseteq X$ and $k(A) = k(A) \cap Y$ for each subset $A \subseteq Y$. If $Y$ is closed in $(X, u)$ then the subspace $(Y, v)$ of $(X, u)$ is said to be closed too.

Definition: 1.2.18 [8]
Let $(X, u)$ and $(Y, v)$ be Čech closure spaces. A map $f: (X,u) \to(Y, v)$ is said to be continuous, if $f(uA) \subseteq v(f(A)$ for every subset $A \subseteq F$. 


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Definition: 1.2.19 [64]
Let (X, u) and (Y, v) be Čech closure spaces. A map f: (X, u) \rightarrow (Y, v) is said to be r-continuous if f^{-1}(V) is a r-open subset of (X, u) for every open subset V of (Y, v).

Definition: 1.2.20 [8]
Let (X, u) and (Y, v) be Čech closure spaces. A map f: (X, u) \rightarrow (Y, v) is said to be closed (resp. open) if f(F) is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, k).

Definition: 1.2.21 [8]
The product of a family \{(X_\alpha, k_\alpha) ; \alpha \in I\} of closure spaces denoted by \prod_{\alpha \in I} (X_\alpha, k_\alpha) is the closure space \prod_{\alpha \in I} (X_\alpha, k) where \prod_{\alpha \in I} X_\alpha denotes the Cartesian product of sets X_\alpha, \alpha \in I and k is Čech closure operator generated by the projections \pi_\alpha : \prod_{\alpha \in I} (X_\alpha, k_\alpha) \rightarrow (X_\alpha, k_\alpha), \alpha \in I i.e defined by k(A) = \prod_{\alpha \in I} k_\alpha \pi_\alpha (A) for each A \subseteq \prod_{\alpha \in I} X_\alpha Clearly, if \{(X_\alpha, k_\alpha) ; \alpha \in I\} is a family of closure spaces, then the projection map \pi_\beta : \prod_{\alpha \in I} (X_\alpha, k_\alpha) \rightarrow (X_\beta, k_\beta) is closed and continuous for every \beta \in I.

Definition: 1.2.22 [8]
Let \{(X_\alpha, k_\alpha) ; \alpha \in I\} be a family of closure spaces, let \beta \in I and F \subseteq X_\beta. Then F is a closed subset of (X_\beta, k_\beta) if and only if F \times \prod_{\alpha \neq \beta} X_\alpha is a closed subset of \prod_{\alpha \in I} (X_\alpha, k_\alpha).

Definition: 1.2.23 [8]
Let \{(X_\alpha, k_\alpha) ; \alpha \in I\} be a family of closure spaces, let \beta \in I and G \subseteq X_\beta. Then G is a open subset of (X_\beta, k_\beta) if and only if G \times \prod_{\alpha \neq \beta} X_\alpha is an open subset of \prod_{\alpha \in I} (X_\alpha, k_\alpha).
\[ \prod_{\alpha \in I} (X_\alpha, k_\alpha). \]

**Definition: 1.2.24**

A topological property is a property which, if possessed by a space \( X \), then it is possessed by all spaces homeomorphic to \( X \).

### 1.3 BIČECH CLOSURE SPACE

C. Boonpok [20] introduced the notion of biclosure spaces. Such spaces are equipped with two arbitrary closure operators. He extended some of the standard results of separation axioms in closure space to biclosure space. Thereafter a large number of papers have been developed to generalize the concept of closure space to biclosure space [11][12][15][18][24][25]. BiČech Closure spaces were introduced by Chandrasekhara Rao et.al. [26][27].

**Definition: 1.3.1 [27]**

Two maps \( k_1 \) and \( k_2 \) from power set \( X \) to itself are called BiČech closure operator (simply biclosure operator) for \( X \) if they satisfy the following properties:

1. \( k_1 \{ \varnothing \} = \varnothing \) & \( k_2 \{ \varnothing \} = \varnothing \)
2. \( A \subseteq k_1(A) \& A \subseteq k_2(A) \forall A \subseteq X \)
3. \( k_1(A \cup B) = k_1(A) \cup k_1(B); k_2(A \cup B) = k_2(A) \cup k_2(B) \)

A structure \( (X,k_1,k_2) \) is called a BiČech closure spaces.

**Definition: 1.3.2 [20]**

1. A biclosure space is a triplet \( (X,k_1,k_2) \) where \( X \) is a set and \( k_1, k_2 \) are two closure operators on \( X \).
2. A subset \( A \) of biclosure space \( (X,k_1,k_2) \) is called closed if \( k_1k_2(A) = A \). The complement of the closed set is called open.
3. \( A \) is a closed subset of a biclosure space \( (X,k_1,k_2) \) if and only if \( A \) is both closed subset of \( (X,k_1) \) and \( (X,k_2) \).
4. If $A$ is a closed subset of a biclosure space $(X,k_1,k_2)$. The following conditions are equivalent.
   
   i. $k_1 k_2 (A) = A$
   
   ii. $k_1 (A) = k_2 (A) = A$

5. If $(X,u_1,u_2)$ and $(Y,v_1,v_2)$ are biclosure spaces, then a biclosure space $(Y,v_1,v_2)$ is called a subspace of $(X,u_1,u_2)$ if $Y \subseteq X$ and $v_i (A) = u_i (A) \cap Y$ for each $i \in \{1,2\}$ and each subset $A \subseteq Y$.

**Definition: 1.3.3 [13]**

Let $(X,u_1,u_2)$ and $(Y,v_1,v_2)$ be biclosure spaces and let $i \in \{1,2\}$ and map $f: (X,u_1,u_2) \to (Y,v_1,v_2)$ is called $i$-closed (resp. $i$-open) if the map $f: (X,u_i) \to (Y,v_i)$ is closed (resp. open). A map $f$ is called closed (resp. open) if $f$ is $i$-closed (resp. $i$-open) for each $i \in \{1,2\}$.

**Definition: 1.3.4 [17]**

Let $(X,u_1,u_2)$ and $(Y,v_1,v_2)$ be biclosure spaces and let $i \in \{1,2\}$ and map $f: (X,u_1,u_2) \to (Y,v_1,v_2)$ is called $i$-continuous if the map $f: (X,u_i) \to (Y,v_i)$ is continuous. A map $f$ is called continuous if $f$ is $i$-continuous for each $i \in \{1,2\}$.

**Definition: 1.3.5 [16]**

A biclosure space $(X,u_1,u_2)$ is said to be a normal biclosure space if, for every disjoint closed subset $H$ of $(X,u_1)$ and closed subset $K$ of $(X,u_2)$ there exists a disjoint open subset $U$ of $(X,u_1)$ and an open subset $V$ of $(X,u_2)$ such that $H \subseteq U$ and $K \subseteq V$.

**Definition: 1.3.6 [18]**

A biclosure space $(X,u_1,u_2)$ is said to be a generalized normal biclosure space briefly, g normal biclosure space if, for every disjoint closed subset $H$ of $(X,u_1)$ and closed subset $K$ of $(X,u_2)$ there exists a disjoint g-open subset $U$ of $(X,u_1)$ and an g-open subset $V$ of $(X,u_2)$ such that $H \subseteq U$ and $K \subseteq V$.

**Definition: 1.3.7 [11]**

Let $(X,u_1,u_2)$ be a biclosure space and let $(Y,v_1,v_2)$ be a closed subspace of $(X,u_1,u_2)$. If $F$ is a closed subset of $(Y,v_1,v_2)$ then $F$ is a closed subset $(X,u_1,u_2)$.
1.4 FUZZY BICLOSURE SPACE

In [72], Mashhour and Ghanim have introduced the concept of fuzzy closure spaces as an extension of Čech closure spaces [22]. They have extended the notions of subspaces, sums, and products of fuzzy closure spaces. Later, Srivastava and Srivastava [93] have studied subspaces, sums, and products of the modification fuzzy closure spaces. In 2008, Boonpok and Khampakdee [8] have introduced and studied two notions of closed sets in closure spaces. Later, Boonpok [13] has introduced the notion of biclosure spaces.

Gerla et al. [6] studied fuzzy closure operator and fuzzy closure system as an extension of closure operator and closure system. With the introduction of fuzzy sets by Zadeh [100] and fuzzy topology by Chang [29], the theory of fuzzy topological spaces was subsequently developed by several authors [40] by considering the basic concepts of general topology. Balasubramaniam and Sundaram [5] defined generalized fuzzy sets in fuzzy topological spaces in order to extend some important properties of fuzzy closed sets to a larger family of sets.

Let $X$ be a non-empty set and $I = [0,1]$. The set of all functions from $X$ into $I$ is denoted by $I^X$. A member of $I^X$ is called a fuzzy set in $X$. The support of a fuzzy set $\mu$ is denoted by $\text{supp}(\mu) = \{x \in X: \mu(x) > 0\}$. Let $\mu$ and $\nu$ be a fuzzy sets in $X$. $\mu$ contained in $\nu$, denoted by $\mu \leq \nu$, if $\mu(x) \leq \nu(x)$ for all $x \in X$. $\mu$ and $\nu$ are equal (ie $\mu = \nu$) if and only if $\mu \leq \nu$ and $\nu \leq \mu$. The union of $\mu$ and $\nu$ denoted by $\mu \vee \nu$, is defined as the fuzzy set that $(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in X$. The intersection of $\mu$ and $\nu$ denoted by $\mu \wedge \nu$, is defined as the fuzzy set that $(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$. The complement of $\mu$, denoted by $1 - \mu$ is defined as the fuzzy set that $(1 - \mu)(x)$ for every $x \in X$.

**Definition: 1.4.1 [72][90]**

A function $u : I^X \rightarrow I^X$ defined on the family of all fuzzy sets of $X$ is called a fuzzy closure operator on $X$ and the pair $(X, u)$ is called fuzzy closure space, if the following conditions are satisfied.

1) $u \phi = \phi$
2) $A \leq u(A)$ for all $A \in I^X$
3) \( u(A \lor B) = u(A) \lor u(B) \) for all \( A, B \in I^X \).

**Definition: 1.4.2 [45]**

A fuzzy subset \( A \) of a fuzzy closure space (fcs), \( (X, u) \) is said to be fuzzy closed, if \( uA = A \) and it is fuzzy open if its complement \( X - A \) is fuzzy closed. The empty set and the whole set are both fuzzy open and fuzzy closed.

**Definition: 1.4.3 [45]**

A fuzzy closure space \( (Y, v) \) is said to be a fuzzy subspace of \( (X, u) \) if \( Y \subseteq X \) and \( vA = uA \land Y \) for each fuzzy subset \( A \subseteq Y \). If \( Y \) is fuzzy closed in \( (X, u) \) then the fuzzy subspace \( (Y, v) \) of \( (X, u) \) is also said to be fuzzy closed.

**Definition: 1.4.4 [86]**

Let \( (X, u) \) and \( (Y, v) \) be fuzzy closure spaces. A map \( f: (X, u) \rightarrow (Y, v) \) is said to be fuzzy continuous if \( f(uA) \leq v(f(A)) \) for every fuzzy subset \( A \subseteq X \). In other words a map \( f: (X, u) \rightarrow (Y, v) \) is fuzzy continuous if and only if \( u f^{-1}(B) \leq f^{-1}(vB) \) for every fuzzy subset \( B \subseteq Y \). Clearly, if map \( f: (X, u) \rightarrow (Y, v) \) is fuzzy continuous, then \( f^{-1}(F) \) is a fuzzy closed subset of \( (X, u) \) for every fuzzy closed subset \( F \) of \( (Y, v) \).

**Definition: 1.4.5 [86]**

Let \( (X, u) \) and \( (Y, v) \) be fuzzy closure spaces. A map \( f: (X, u) \rightarrow (Y, v) \) is said to be fuzzy closed (resp. fuzzy open) if \( f(F) \) is a fuzzy closed (resp. fuzzy open) subset of \( (Y, v) \) whenever \( F \) is a fuzzy closed (resp. fuzzy open) subset of \( (X, u) \).

**Definition: 1.4.6 [72]**

The product of a family \( \{ (X_\alpha, u_\alpha): \alpha \in I \} \) of fuzzy closure spaces denoted by \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \), is the fuzzy closure space \( (\prod_{\alpha \in I} X_\alpha, u) \) where \( \prod_{\alpha \in I} X_\alpha \) denotes the Cartesian product of fuzzy sets \( X_\alpha, \alpha \in J \), and \( u \) is the fuzzy closure operator generated
by the projections $\pi_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha, \alpha \in J$, ie. is defined by $uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \leq \prod_{\alpha \in I} X_\alpha$.

Clearly if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of fuzzy closure spaces, then the projection map $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is fuzzy closed and fuzzy continuous for every $\beta \in J$.

**Definition: 1.4.7 [72]**

Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of fuzzy closure spaces and $\beta \in J$. Then $F$ is a fuzzy closed subset of $(X_\beta, u_\beta)$ if and only if $F \times \prod_{\alpha \in J \setminus \beta} X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

**Definition: 1.4.8 [72]**

Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of fuzzy closure spaces and $\beta \in J$. Then $G$ is a fuzzy open subset of $(X_\beta, u_\beta)$ if and only if $G \times \prod_{\alpha \in J \setminus \beta} X_\alpha$ is a fuzzy open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

**Definition: 1.4.9 [98]**

A fuzzy biclosure space is a triple $(X, u_1, u_2)$ where $X$ is a set and $u_1, u_2$ are two fuzzy closure operators on $X$ which satisfies the following properties:

(i) $u_1(\phi) = \phi$ and $u_2(\phi) = \phi$

(ii) $A \leq u_1(A)$ and $A \leq u_2(A)$ for all $A \in I^X$

(iii) $u_1(A \lor B) = u_1(A) \lor u_1(B)$ and $u_2(A \lor B) = u_2(A) \lor u_2(B)$ for all $A, B \leq I^X$.

**Definition: 1.4.10 [98]**

A subset $A$ of a fuzzy biclosure space $(X, u_1, u_2)$ is called fuzzy closed if $u_1 u_2 A = A$. The complement of fuzzy closed set is called fuzzy open.

Clearly, $A$ is a fuzzy closed subset of fuzzy biclosure space $(X, u_1, u_2)$ if and only if $A$ is both a fuzzy closed subset of $(X, u_1)$ and $(X, u_2)$. 
Let $A$ be a fuzzy closed subset of a fuzzy biclosure space $(X, u_1, u_2)$. The following conditions are equivalent.

1) $u_1u_2A = A$.
2) $u_1A = A$, $u_2A = A$.

**Definition: 1.4.11 [98]**

Let $(X, u_1, u_2)$ be a fuzzy biclosure space and let $A \subseteq X$. Then

(i) $A$ is open if and only if $A = X - u_1u_2(X - A)$.

(ii) If $G$ is open and $G \subseteq A$, then $G \subseteq X - u_1u_2(X - A)$.

**Definition: 1.4.12 [98]**

Let $(X, u_1, u_2)$ be a fuzzy biclosure space. A fuzzy biclosure space $(Y, v_1, v_2)$ is called a fuzzy subspace of $(X, u_1, u_2)$ if $Y \subseteq X$ and $v_iA = u_iA \wedge Y$ for each $i \in \{1, 2\}$ and each subset $A \subseteq Y$.

**Definition: 1.4.13 [98]**

Let $(X, u_1, u_2)$ be a fuzzy biclosure space and let $(Y, v_1, v_2)$ be a fuzzy closed subspace of $(X, u_1, u_2)$. If $F$ is a fuzzy closed subset of $(Y, v_1, v_2)$ then $F$ is a fuzzy closed subset of $(X, u_1, u_2)$.

**Definition: 1.4.14 [98]**

Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha): \alpha \in I\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then $F$ is a fuzzy closed subset of $(X_\alpha, u^1_\alpha, u^2_\alpha)$ if and only if $F \times \prod_{\alpha \in I \setminus \beta} X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

**Definition: 1.4.15 [98]**

Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha): \alpha \in I\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then $G$ is a fuzzy open subset of $(X_\alpha, u^1_\alpha, u^2_\alpha)$ if and only if $G \times \prod_{\alpha \in I \setminus \beta} X_\alpha$ is a fuzzy open subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.
Definition: 1.4.16 [98]

Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be fuzzy biclosure spaces and let \(i \in \{1, 2\}\). A map \(f: (X, u_i) \rightarrow (Y, v_i)\) is said to be fuzzy \(i\)-closed (resp. fuzzy \(i\)-open) if the map \(f: (X, u_i) \rightarrow (Y, v_i)\) is fuzzy \(i\)-closed (resp. fuzzy \(i\)-open). A map \(f\) is called fuzzy closed (resp. fuzzy open) if \(f\) is fuzzy \(i\)-closed (resp. fuzzy \(i\)-open) for each \(i \in \{1, 2\}\).

Definition: 1.4.17 [98]

Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be fuzzy biclosure spaces and let \(i \in \{1, 2\}\). A map \(f: (X, u_i) \rightarrow (Y, v_i)\) is said to be fuzzy \(i\)-continuous if the map \(f: (X, u_i) \rightarrow (Y, v_i)\) is fuzzy continuous. A map \(f\) is called fuzzy continuous if \(f\) is fuzzy \(i\)-continuous for each \(i \in \{1, 2\}\).

Definition: 1.4.18 [79]

A fuzzy biclosure space \((X, u_1, u_2)\) is said to be \(\partial\)-Hausdorff fuzzy biclosure space if for any two distinct fuzzy points \(x_r, y_s\) in \(X\) with \(x \neq y\), there exists a \(\partial\)-fuzzy open set \(\mu\) in \((X, u_1)\) and a \(\partial\)-fuzzy open set \(\nu\) in \((X, u_2)\) such that \(x_r \in \mu\) and \(y_s \in \nu\) and \(\mu \land \nu = 0_X\).

Definition: 1.4.19 [80]

A fuzzy biclosure space \((X, u_1, u_2)\) is said to be \(\partial\)-normal fuzzy biclosure space if any \(\partial\) fuzzy closed set \(\eta\) in \((X, u_1)\) and any \(\partial\) fuzzy closed set \(\gamma\) in \((X, u_2)\) with \(\eta \land \gamma = 0_X\), there exist a \(\partial\) fuzzy open set \(\mu\) in \((X, u_1)\) and \(\partial\) fuzzy open set \(\nu\) in \((X, u_2)\) such that \(\eta \leq \mu\), \(\gamma \leq \nu\) and \(\mu \land \nu = 0_X\).

Definition: 1.4.20 [93]

A subset \(A\) of a fuzzy biclosure space \((X, u_1, u_2)\) is called generalized fuzzy closed briefly, g-fuzzy closed, if \(u_i A \leq G\) whenever \(G\) is a fuzzy open subset of \((X, u_2)\) and \(A \leq G\). The complement of a fuzzy g-closed set is called fuzzy g-open.
1.5 CONTRIBUTIONS OF THE AUTHOR

The author has obtained some interesting generalizations on the following topics:


2. Functions on Čech $\pi g\beta$ – closed sets.

3. A new class of sets in closure spaces.

4. $\pi$ - Continuous mappings in biclosure space.

5. $\Delta$ - Čech sets in fuzzy biclosure spaces.