CHAPTER 8

b-Colouring of Some Graph Operations

In this Chapter, Graph Operations like addition and deletion of a vertex, an edge in a Cycle, Union and Complement of Path with Cycle, Strong Product of Path with Cycle, Cartesian Product of Cycles, Corona Product of Path graph with Cycle, Path graph with Complete graph, Cycle with Complete graph, Star graph with Complete graph are obtained. The b-Chromatic number and Structural properties for above operations on graphs are determined.

8.1 Introduction [57, 68,72, 76]

Operations on graphs produce new ones from older ones. There are several operations on Graph Theory that produce new graphs from old ones, which is mentioned in the following major categories.

- Unary Operations
- Binary Operations

Unary Operations

Unary operations create a new graph from the old one. Elementary operations are sometimes called "editing operations" on graphs. They create a new graph from the original one by a simple or a local change, such as addition or deletion of a vertex or an edge, merging and splitting of vertices, edge contraction, etc. Advanced operations such as Line graph, Dual graph, Complement graph, Graph minor, Power of graph, etc. also creates new graphs from the old ones.

Binary Operations

Binary operations creates a new graph from two initial graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$:

- The disjoint union of graphs, sometimes referred as simply graph union, which is defined as: For two graphs with disjoint vertex sets $V_1$ and $V_2$ and disjoint edge sets $E_1$ and $E_2$, their disjoint union is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.
- The graph join of two graphs is their graph union with all the edges that connect the vertices of the first graph with the vertices of the second graph.
Graph products based on the Cartesian product of the vertex sets are:

- Cartesian product of graphs
- Lexicographic product of graphs also called Graph composition
- Strong product of graphs
- Tensor product of graphs, also called as direct product, categorical product, cardinal product or Kronecker product.
- Zig-zag product of graphs

Other graph operations called "products" are:

- Rooted product of graphs.
- Corona product or simply corona of $G_1$ and $G_2$

### 8.2 b-Chromatic Number of a Graph in Addition of Parallel Chords

#### 8.2.1 Theorems

For any Cycle $C_n$, addition of parallel chords between non adjacent vertices holds the following statements:

- When $n$ is odd, there exists a unique 3 cycle and $\left\lceil \frac{n}{3} \right\rceil$ times 4 cycle.
- When $n$ is even, there exists more than one 3 cycle and $\left\lceil \frac{n}{2} \right\rceil$ - 2 times 4 cycle.

**Proof**

Let $C_n$ be a Cycle with $n$ vertices. Let $v_1,v_2,...,v_n$ be the vertices and $e_1,e_2,...,e_k$ be the edges of the Cycle $C_n$ with parallel chords, where $k$ is defined as

$$ k = \begin{cases} 
\left\lceil \frac{4n}{3} \right\rceil & \text{if } n \text{ is odd} \\
\left\lceil \frac{4n}{3} \right\rceil + 1 & \text{if } n \text{ is even} 
\end{cases} $$

**Case 1**

When $n$ is an odd cycle, it is clear that the edges $e_1,e_n,e_{n+1}$ are mutually adjacent to each other, so it obviously forms a 3 cycle. Also we see that the remaining number of vertices in the Cycle is even. Under observation, it forms $\left\lceil \frac{n}{3} \right\rceil$ times 4 cycles. Thus, when $n$ is odd, there exist a unique 3 cycle and $\left\lceil \frac{n}{3} \right\rceil$ times 4 cycle.
Case 2

When \( n \) is an even cycle, the vertices with minimum degree (i.e. \( \delta = 2 \)) forms a 3 cycle and vertices with maximum degree (i.e. \( \Delta = 3 \)) forms a 4 cycle. Thus for every even cycle there exist more than one 3 cycle and \( \left\lceil \frac{n}{2} \right\rceil - 2 \) times 4 cycle.

Example
8.2.2 Theorem

The Cycle $C_n$ with arallel chords has the b-Chromatic number four for every $n \geq 8$.

Example

![Figure 3: $C_8$ with parallel chords](image)

8.3 b-Chromatic Number of a Graph when an Edge is removed

8.3.1 Theorem

For any Cycle $(n \geq 5)$ with an edge $e \in V(C_n)$, $\phi(C_n) = \phi(C_n-e)$

Proof

Let $C_n$ be the Cycle of length $n$. Let $v_1, v_2, ..., v_n$ be the vertices arranged in anticlockwise direction i.e. $V(C_n) = \{v_1, v_2, ..., v_n\}$ and the edge set be denoted as $E(C_n) = \{e_1, e_2, e_3, ..., e_n\}$. Here the vertex $v_i$ is adjacent with the vertices $v_{i-1}$ and $v_{i+1}$ for $i=2, 3, ..., n-1$, $v_1$ is adjacent with $v_2$, $v_n$ and the vertex $v_n$ is adjacent with $v_{n-1}$ and $v_1$. We know that every Cycle is a connected graph with $n$ vertices. From the Theorem 3.2.1, it is clear that b-chromatic number of Cycle of length $n$ for $n \geq 5$ is tricolourable i.e. $\phi(C_n) = 3$. Suppose if we delete any edge from the Cycle, we obtain a Path graph of length $n-1$ which is again a tricolourable graph under observation.

Therefore $\phi(C_n) = \phi(C_n-e)$ for every $n \geq 5$.

Example

![Figure 4(a): $\phi(C_8) = 3$](image)  
![Figure 4(b): $\phi(C_8-e) = 3$](image)
8.3.2 Corollary

\( \phi(C_n) \neq \phi(C_n-e) \) for every \( n \leq 3 \).

Example

![Figure 5(a): C_3](image)

![Figure 5(b): C_3-e](image)

8.3.3 Corollary

For any Path for \( n \geq 5 \), \( e \in V(P_n) \), \( \phi(P_n) = \phi(P_n-e) \)

8.4 b-Colouring of Adding a Pendant Vertex to each Vertex of a Cycle

8.4.1 Theorem

For any \( n \geq 6 \), \( \phi(C_n \cdot K_1) = \phi(W_n) \)

Proof

Let \( v_i \) for \( 1 \leq i \leq n \) are the vertices taken in the anticlockwise direction in the wheel graph \( W_n \), where \( v_n \) is the hub. It is clear that the vertex \( v_i \) is adjacent with the vertices \( v_{i-1} \) and \( v_{i+1} \) for \( i=2,3,...,n-1 \), the vertex \( v_1 \) is adjacent with \( v_2 \) and \( v_{n-1} \), the vertex \( v_n \) is adjacent with all the vertices. Here every vertex except the hub is incident with three edges, so we assign four colours, which produces a maximum and b-chromatic colouring by the colouring procedure. Also we know that the b-chromatic number of any Cycle has three colours for \( n \geq 5 \). If we attach a pendant vertex to every vertex of Cycle \( C_n \), it is obvious that it has four colours under observation.

Therefore \( \phi(C_n \cdot K_1) = \phi(W_n) \)
Example

Figure 6(a): $\phi(C_6 \cdot K_1) = 4$

Figure 6(b): $\phi(W_6) = 4$

8.4.2 Results obtained by Removing Edges from the Complete Graph

- $\phi(K_4 - e) = \phi(C_5) = \phi(K_{1,n,n}), (n \geq 2)$
- $\phi(K_3 - e) = \phi(C_2) = \phi(K_{1,n}), (n \geq 2)$
- $\phi(K_4 - 2e) = \phi(C_3)$
- $\phi(K_5 - 2e) = \phi(W_n), (n \geq 6)$
- $\phi(K_5 - 3e) = \phi(C_n), (n \geq 5)$

8.5 b-Chromatic Number of Union of Path with Cycle

The Disjoint union of graphs [31,45] sometimes referred as simply graph union, which is defined as follows. Given two graphs $G_1$ and $G_2$, their union will be a graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

The Complement $G'$ [31,82] of a graph $G$ is defined as a simple graph with the same vertex set as $G$ and where two vertices $u$ and $v$ are adjacent only when they are not adjacent in $G$.

8.5.1 Theorem

For any Path $P_n$ and the Cycle $C_n$ with $n$ vertices, the b-chromatic number of $P_n \cup C_n$ is given by $\phi(P_n \cup C_n) = n-1$ for $n \geq 2$. 

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Proof

Let \( G_1 = P_n \) be a Path graph with \( n \) vertices and \( n-1 \) edges and \( G_2 = C_n \) be a Cycle with \( n \) vertices and \( n \) edges. Let \( G = G_1 \cup G_2 \) be the graph obtained by the union of subgraph \( P_n \) and \( C_n \) of a graph has the vertex set \( V(P_n) \cup V(C_n) \) and edge set \( E(P_n) \cup E(C_n) \).

Consider \( G = P_n \cup C_n \) whose vertex set \( V(G) = \{v_1, v_2, v_3, \ldots, v_{2n-2}\} \). Here in \( P_n \cup C_n \), we see that the vertex \( v_i \) is adjacent with the vertices \( v_{i+1} \) and \( v_{i-1} \) for \( i=2,3,\ldots,n-1,n+1,\ldots,2n-2 \), \( v_1 \) is adjacent with \( v_2, v_{2n-2} \) and the vertex \( v_n \) is adjacent with the vertices \( v_1, v_{n-1} \) and \( v_{n+1} \).

Now consider the graph \( G = \overline{G_1 \cup G_2} \). By the definition of Complement, for any graph \( G \), the non-adjacent vertices are adjacent in its complement. Here \( \overline{G_1 \cup G_2} \) contains \( 2n-2 \) vertices as in \( G_1 \cup G_2 \). Arrange the vertices of \( \overline{G_1 \cup G_2} \) namely \( v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n-2} \) in clockwise direction.

Assign a proper colouring to these vertices as follows. Consider the colour class \( C = \{c_1, c_2, c_3, \ldots, c_{n-1}\} \). First assign the colour \( c_i \) to the vertex \( v_i \) for \( i=1,2,\ldots,2n-2 \), it will not produce a \( b \)-chromatic colouring, due to the above mentioned non-adjacency condition.

Hence to make the colouring as \( b \)-chromatic one, assign the colour \( c_{\left\lfloor \frac{(i+1)}{2} \right\rfloor} \) to the vertices \( v_i \) and \( v_{i+1} \) for \( i=1,3,5,\ldots,2n-3 \). Now all the vertices \( v_i \) for \( i=1,2,\ldots,2n-2 \) realizes its own colour, which produces a \( b \)-chromatic colouring. Furthermore it is the maximum colouring possible.

Example

![Figure 7(a): P₅ \( \cup \) C₅](image-url)
8.5.2 Theorem

\[ \phi(P_n \cup C_n) = 3 \] for every \( n \geq 3 \)

Proof

The result is trivial from the above theorem.

8.5.3 Theorem

For any Path graph \( P_n \) and Cycle \( C_m \) with \( n \) and \( m \) vertices respectively, then

\[ \phi(P_n \cup C_m) = 3 \] for \( n \geq 2 \).

Example

Figure 8: \( P_4 \cup C_{10} = 3 \)
8.5.4 Result

For any integer \( n > 2 \), \( \varphi(C_n) = \varphi(C_n^c) \)

Example

Figure 9(a): \( \varphi(C_6) = 3 \)  

Figure 9(b): \( \varphi(C_6^c) = 3 \)

8.6 b-Chromatic Number of Strong Product of Path with Cycle

The strong product is also called the [3, 66, 89, 90] normal product and AND product. It was first introduced by Sabidussi in 1960. In Graph theory, the strong product \( G \otimes H \) of graphs \( G \) and \( H \) is a graph such that the vertex set of \( G \otimes H \) is the Cartesian product \( V(G) \times V(H) \); and any two vertices \((u,u') \) and \((v,v') \) are adjacent in \( G \otimes H \) if and only if \( u' \) is adjacent with \( v' \) or \( u' = v' \) and \( u \) is adjacent with \( v \) or \( u = v \).

8.6.1 Theorem

If \( P_m \) is a Path graph on \( m \) vertices and \( C_n \) be a Cycle on \( n \) vertices respectively. Then

\[ \varphi(P_m \otimes C_n) = 6, \quad n \geq 3 \text{ and } m=2 \]

Proof

By observation, the strong product of \( P_m \otimes C_n \) is a 5-regular graph. Therefore we say that \( \varphi(P_m \otimes C_n) \geq 5 \). By colouring procedure we assign six colours to every \( P_m \otimes C_n \), which produces a b-chromatic colouring. Suppose if we assign more than six colours, it contradicts the definition of b-colouring. Hence the b-chromatic number of Strong product of \( P_m \otimes C_n \) is six.
Example

\[ \phi(P_2 \otimes C_n) = 6 \]

8.6.2 Structural Properties of \( P_2 \otimes C_n \)

- Number of vertices in \( P_2 \otimes C_n \) is two times the number of vertices in Cycle \( C_n \).
- Number of edges in \( P_2 \otimes C_n \) is five times the number of edges in Cycle \( C_n \).
- Every \( P_2 \otimes C_n \) is a 5 regular graph.

8.6.3 Theorem

If \( P_n \) is a Path graph of length \( n-1 \) and \( K_m \) be a Complete graph on \( m \) vertices respectively. Then

\[ \phi(P_n \otimes K_2) \]

\[ \begin{cases} 
4 & \text{for } n \leq 3 \\
6 & \text{for every } n \geq 4 
\end{cases} \]

Example

\[ \phi(P_2 \otimes K_2) = 4 \]

\[ \phi(P_3 \otimes K_2) = 4 \]

\[ \phi(P_5 \otimes K_2) = 6 \]
8.6.4 Theorem

Let $P_n$ and $P_m$ be the Path graph on $n$ and $m$ vertices respectively, then

$$\phi(P_n \otimes P_m) = \begin{cases} n + 2 & \text{for } 2 \leq n \leq 4 \text{ also } m = n \\ 9 & \text{for every } n \geq 5 \end{cases}$$

Proof

The Proof of the Theorem 8.6.3 and 8.6.4 is immediate from the Theorem 8.6.1.

Example

\begin{align*}
\text{Figure 12 (a): } & \phi(P_2 \otimes P_2) = 4 \\
\text{Figure 12 (b): } & \phi(P_3 \otimes P_3) = 5 \\
\text{Figure 12 (c): } & \phi(P_5 \otimes P_5) = 9
\end{align*}
8.7. b-Chromatic Number of Cartesian product of $C_3$ with Cycle $C_n$

In Graph theory, the Cartesian product [75,76,89,90,91] $G \square H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \square H$ is the Cartesian product $V(G) \times V(H)$; and any two vertices $(u,u')$ and $(v,v')$ are adjacent in $G \square H$ if and only if either $u = v$ and $u'$ is adjacent with $v'$ in $H$, or $u' = v'$ and $u$ is adjacent with $v$ in $G$.

8.7.1 Theorem

Let $C_n$ and $C_m$ be the Cycles on $n$ and $m$ vertices respectively, then $\phi(C_m \square C_n) = 5$ for every $n \geq 6$ and $m=3$.

Proof

By the colouring procedure under observation, we assign five colours to every $C_m \square C_n$, for producing a b-chromatic colouring. Suppose if we assign more than five colours, it contradicts the definition of b-colouring. Thus by the colouring procedure, the b-chromatic number of Cartesian product of $C_m \square C_n$ is five.

Example

\[ \text{Figure 13: } \phi(C_3 \square C_6) = 5 \]

8.7.2 Corollary

$\phi(C_m \square C_n) = n$ for every $n \leq 5, m=3$

Example

\[ \text{Figure 14: } \phi(C_3 \square C_4) = 4 \]
8.7.3 Theorem
Let $P_n$ be a Path graph of length $n$ and a Complete graph $K_2$ respectively, then

$$\phi(P_n \Box K_2) = \begin{cases} n & \text{for } 2 \leq n \leq 4 \\ 4 & \text{for every } n \geq 5 \end{cases}$$

Example

Figure 15(a): $\phi(P_3 \Box K_2) = 3$

Figure 15(b): $\phi(P_5 \Box K_2) = 4$

8.7.4 Theorem
Let $G$ be a Cartesian product of $P_n \Box P_n$ for every $n > 1$,

$$\phi(P_n \Box P_n) = \begin{cases} n & \text{for } n = 2 \\ 4 & \text{for } n = 3, 4 \\ 5 & \text{for } n \geq 5 \end{cases}$$

8.8 $b$-Chromatic Number of Corona Product of Graphs

In Graph Theory, Corona product[86,90] or simply Corona of $G_1$ and $G_2$, defined by Frucht and Harary is the graph which is the disjoint union of one copy of $G_1$ and $|V_1|$ copies of $G_2$ ($|V_1|$ is the number of vertices of $G_1$) in which each vertex of the copy of $G_1$ is connected to all vertices of a separate copy of $G_2$.

8.8.1 $b$-Chromatic Number of Corona Product of Path Graph with Cycle

8.8.2 Theorem
For any integer $n > 3$, $\phi (P_n \circ C_n) = n$
Proof

Let \( G_1 = P_n \) be a Path graph of length \( n-1 \) with vertices \( v_1,v_2,v_3,...,v_n \) and edges \( e_1,e_2,e_3,...,e_{n-1} \). Consider \( G_2 = C_n \) be a Cycle of length \( n \) whose vertices are \( v_1,v_2,v_3,...,v_n \) and edges are denoted as \( e_1,e_2,e_3,...,e_n \).

Now consider the Corona product of \( G_1 \) and \( G_2 \) i.e. \( G = P_n \circ C_n \) is obtained by taking unique copy of \( P_n \) with \( n \) vertices and \( n \) copies of \( C_n \) and joining the \( i^{th} \) vertex of \( P_n \) to every vertex in \( i^{th} \) copy of \( C_n \). The Vertex set is defined as follows:

\[
i.e. \quad V(G) = V(P_n) \cup V(C^1_n) \cup V(C^2_n) \cup V(C^3_n) \cup ... \cup V(C^n_n)
\]

where \( V(P_n) = \{v_1,v_2,v_3,...,v_n\} \) and \( V(C_i^n) = \{u^i_1: 1 \leq i \leq n, 1 \leq j \leq n\} \)

Now assign a proper colouring to these vertices as follows. Consider the colour class \( C= \{c_1,c_2,c_3,...,c_n\} \). First assign the colour \( c_i \) to the vertex \( v_i \) for \( i=1,2,3...,n \) and assign the colour to \( u^i_j \) as \( c_{i+j} \) when \( i+j \leq n \) and \( c_{i+j-n} \) when \( i+j > n \) for \( 1 \leq i \leq n, 1 \leq j \leq n-1 \). Next the only vertex remaining to be coloured is \( u^i_j \) for \( j=n \). Suppose if we assign any new colour to \( u^i_j \) for \( i=1,2,3...,n, j=n \) it will not produce a b-chromatic colouring, because \( u^i_j \) ( \( i=1,2,3...,n, j=n \)) is adjacent only with \( u^i_1 \) and \( u^i_{n-1} \). To make the above colouring as b-chromatic one, we assign the colour to \( u^i_j \) other than the colour which we assigned previously for \( u^i_1 \) and \( u^i_{n-1} \). Now the vertices \( \{v_i: 1 \leq i \leq n\} \) realize its own colours, which produce a b-chromatic colouring. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.

Example

\[\text{Figure 16: } \phi(P_4 \circ C_4) = 4\]
8.8.3 Structural Properties of \( P_n \circ C_n \)

The number of vertices in \( P_n \circ C_n \) \((n > 3)\) i.e. \( p(P_n \circ C_n) = n(n+1) \), number of edges in the \( P_n \circ C_n \) i.e. \( q(P_n \circ C_n) = 2n^2 + n - 1 \). The maximum and minimum degree of \( P_n \circ C_n \) is denoted as \( \Delta = n+2 \) and \( \delta = n-1 \) respectively. The number of vertices having maximum degree \( \Delta \) in \( P_n \circ C_n \) is denoted by \( n(p_\Delta) = n-2 \) and the number of vertices having minimum degree \( \delta \) in \( P_n \circ C_n \) is denoted by \( n(p_\delta) = n^2 \) respectively.

8.8.4 Theorem

For any Path \( P_n \) and Cycle \( C_n \), the number of edges in Corona product of \( P_n \) with Cycle \( C_n \) is \( 2n^2 + n - 1 \) i.e. \( q(P_n \circ C_n) = 2n^2 + n - 1 \)

**Proof**

\[
q(P_n \circ C_n) = \text{Number of edges in largest subgraph} + \text{Number of edges not in any of the largest subgraph} \\
= n \times (2n) + n - 1 \\
= 2n^2 + n - 1
\]

Therefore \( q(P_n \circ C_n) = 2n^2 + n - 1 \)

8.9 b-Chromatic number of Corona Product of Path with Complete Graph

8.9.1 Theorem

Let \( P_n \) and \( K_2 \) be Paths and Complete graphs with \( n \) vertices respectively. Then

\[
\varphi(P_n \circ K_2) = \begin{cases} 
  n + 1 & \text{for } n = 2 \\
  n & \text{for } n = 3 \text{ and } 4 \\
  n - 1 & \text{for } n = 5 \\
  5 & \text{for every } n > 6
\end{cases}
\]

**Example**

![Figure 17(a): \( \varphi(P_3 \circ K_2) = 3 \)](image)

![Figure 17(b): \( \varphi(P_4 \circ K_2) = 4 \)](image)
8.9.2 Structural Properties of $P_n \circ K_2$

The number of vertices in $P_n \circ K_2$ i.e. $p(P_n \circ K_2) = 3n$, number of edges in the $P_n \circ K_2$ i.e. $q(P_n \circ K_2) = 3n + (n-1)$. The maximum and minimum degree of $P_n \circ K_2 \space (n>3)$ is denoted as $\Delta = 4$ and $\delta = 3$ respectively. The number of vertices having maximum degree in $P_n \circ K_2$ is denoted by $n(p_\Delta) = n-2$ and the number of vertices having minimum degree in $P_n \circ K_2$ is denoted by $n(p_\delta) = 2$ respectively.

8.9.3 Theorem

For any integer $n$, $\varphi(P_n \circ K_n) = n+1$

Proof

Let $G_1 = P_n$ be a Path graph of length $n-1$ with $n$ vertices and $G_2 = K_n$ be a complete graph of $n$ vertices.

Consider the Corona product of $G_1$ and $G_2$ i.e. $G = P_n \circ K_n$ is obtained by taking unique copy of $P_n$ with $n$ vertices and $n$ copies of $K_n$ and joining the $i^{th}$ vertex of $P_n$ to every vertex in $i^{th}$ copy of $K_n$. The vertex set of $P_n \circ K_n$ is defined as follows:

$$V(G) = V(P_n) \cup V(K_{1,n}) \cup V(K_{2,n}) \cup \ldots \cup V(K_{n,n})$$

where $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $V(K_{i,n}) = \{u_i^j: 1 \leq i \leq n, 1 \leq j \leq n\}$.

By observation, we see that there are $n$ copies of disjoint subgraph which induces a clique of order $n+1$ (say $K_{n+1}$). Therefore we assign more than or equal to $n+1$ colours to every corona product of path graph with complete graph. Consider the colour class $C = \{c_1, c_2, c_3, c_4, \ldots, c_n, c_{n+1}\}$. Now assign a proper colouring to these vertices as follows. Suppose if we assign more than $n+1$ colours, it contradicts the definition of b-chromatic coloring.

Thus to make the above colouring as b-chromatic one, we cannot assign more than $n+1$ colours i.e. $\varphi(P_n \circ K_n) \leq n+1$. Therefore $\varphi(P_n \circ K_n) = n+1$. Thus by the colouring procedure the above said coloring is maximum and b-chromatic colouring.
8.9.4 Structural Properties of $P_n \circ K_n$

The number of vertices in $P_n \circ K_n$ i.e. $p(P_n \circ K_n) = n(n + 1)$, number of edges in the $P_n \circ K_n$ i.e. $q(P_n \circ K_n) = \left\lceil \frac{n^3 + n^2 + 2n - 2}{2} \right\rceil$. The maximum and minimum degree of $P_n \circ K_n$ is denoted as $\Delta = n + 2$ and $\delta = n$ respectively. The number of vertices having maximum and minimum degree in $P_n \circ K_n$ is denoted by $n(p_{\Delta}) = n - 2$ and $n(p_{\delta}) = n^2$ respectively.

8.9.5 Theorem

For any Path $P_n$ and Complete graph graph $K_n$ the number of edges in Corona product of $P_n$ with $K_n$ is $\left\lceil \frac{n^3 + n^2 + 2n - 2}{2} \right\rceil$ i.e. $q(P_n \circ K_n) = \left\lceil \frac{n^3 + n^2 + 2n - 2}{2} \right\rceil$.

Proof

$q(P_n \circ K_n) = \text{Number of edges in all } K_{n+1}^+ + \text{Number of edges not in any of the } K_{n+1}$

$= n \times q(K_{n+1})^+ + \text{Number of edges not in any of the } K_{n+1}$

$= n \times \left( \frac{(n + 1)^2}{2} \right) + n - 1$

$= n \left\lceil \frac{n(n + 1)}{2} \right\rceil + n - 1$

$= n \left\lceil \frac{n^2 + n}{2} \right\rceil + n - 1$

$= \left\lceil \frac{n^3 + n^2 + 2n - 2}{2} \right\rceil + n - 1$

$= \left\lceil \frac{n^3 + n^2 + 2n - 2}{2} \right\rceil$

Therefore $q(P_n \circ K_n) = \left\lceil \frac{n^3 + n^2 + 2n - 2}{2} \right\rceil$.
8.10 b-Chromatic Number of Corona Product of Fan Graph with K₂

8.10.1 Theorem
\[ \phi(F_{1,n} \circ K_2) = \begin{cases} n + 1 & \text{for } 2 \leq n \leq 4 \\ 5 & \text{for every } n > 5 \end{cases} \]

8.10.2 Theorem
\[ \phi(F_{1,n} \circ K_n) = n + 1 \text{ for every } n > 2. \]

Proof
The proof of the theorem is similar to Theorem 8.6.1.

8.11 b-Chromatic Number of Corona Product of K₁,n with Kₙ

8.11.1 Theorem
If K₁,n and K₂ are star graph and Complete graphs with n vertices respectively, then
\[ \phi(K_{1,n} \circ K_2) = \begin{cases} n + 1 & \text{for } n \leq 3 \\ 4 & \text{for } n \geq 4 \end{cases} \]

8.11.2 Theorem
\[ \phi(K_{1,n} \circ K_n) = n + 1 \text{ for every } n > 2. \]

Proof
The proof of the theorem is similar to theorem (8.6.1)

8.12 b-Chromatic Number of Corona Product of Path Graph with K₁

8.12.1 Theorem
For any \( n \geq 3 \), \( \phi(P_n \circ (n-1) K_1) = n \)

Proof
Let \( P_n \) be a Path graph of length \( n-1 \) i.e. \( V(P_n) = \{v_i, v_{i+1}, \ldots, v_n\} \) and \( E(P_n) = \{e_i, e_{i+1}, \ldots, e_{n-1}\} \).

By the definition of Corona product, attach \( (n-1) \) copies of \( K_1 \) to each vertex of \( P_n \).

\[ V[P_n \circ (n-1) K_1] = \{v_i/1 \leq i \leq n\} \cup \{v_{ij}/1 \leq i \leq n, 1 \leq j \leq n-1\}. \]

\[ E[C_n \circ (n-3) K_1] = \{e_i/1 \leq i \leq n-1\} \cup \{e_{ij}/1 \leq i \leq n, 1 \leq j \leq n-1\}. \]
Consider the colour class \( C = \{c_1, c_2, ..., c_n\} \). Assign a proper colouring to the vertices as follows. Give the colour \( c_i \) to vertex \( v_i \) for \( i = 1, 2, 3, ..., n \) and assign the colour \( c_{n+j} \) to \( v_{ij} \)'s for \( i = 1, 2, ..., n \) and \( j = 1, 2, 3, ..., n-1 \). We see that each vertex \( v_i \) is adjacent with \( v_{i-1} \) and \( v_{i+1} \) for \( i = 2, 3, ..., n-1 \), \( v_1 \) is adjacent with \( v_2 \) and \( v_n \) is adjacent with \( v_{n-1} \). Due to this non-adjacency condition of \( v_i \) for \( i = 1, 2, 3, ..., n \) does not realize its own colour, which does not produce a \( b \)-chromatic colouring. To make the colouring as \( b \)-chromatic one, assign the colouring to \( v_i \), \( v_{ij} \)'s as follows. For \( 1 \leq i \leq n \), assign the colour \( c_i \) to \( v_i \). For \( i = 1, 2, 3, ..., n \), \( j = 1, 2, 3, ..., n-1 \), assign the colour \( c_{i+j} \) to \( v_{ij} \) when \( i+j \leq n \) and assign the colour \( c_{i+j-n} \) when \( i+j > n \). Now the vertices \( v_i \) for \( i = 1, 2, 3, ..., n \) realizes its own colour, which produces a \( b \)-chromatic colouring. Thus by the colouring procedure the above said colouring is maximum and \( b \)-chromatic.

**Example**

![Figure 19](image)

![Figure 20](image)

**8.12.2 Structural Properties of \( P_n \circ (n-1) K_1 \)**

The number of vertices in \( P_n \circ (n-1) K_1 \) i.e. \( p[P_n \circ (n-1) K_1] = n^2 \), number of edges in the \( P_n \circ (n-1) K_1 \) i.e. \( q[P_n \circ (n-1) K_1] = n^2 - 1 \). The maximum and minimum degree of \( P_n \circ K_n \) is denoted as \( \Delta = n+1 \) and \( \delta = 1 \) respectively. The number of vertices with maximum and minimum degree is \( n(p_\Delta) = n+1 \), \( n(p_\delta) = n(n-1) \). Here \( n \) vertices with degree \( n+1 \) and \( n(n-1) \) vertices with degree \( 1 \).
8.13 b-Chromatic Number of Corona Product of Cycle with $K_1$

8.13.1 Theorem

For any $n \geq 3$, $\varphi [C_n \circ (n-3) K_1] = n$

Proof

Let $C_n$ be a Cycle of length $n$ i.e. $V(C_n) = \{v_1, v_2, v_3, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, e_3, ..., e_n\}$. By the definition of Corona product, attach $(n-3)$ copies of $K_1$ to each vertex of $C_n$.

i.e. $V[C_n \circ (n-3) K_1] = \{v_i/1 \leq i \leq n\} \cup \{v_{ij}/1 \leq i \leq n, 1 \leq j \leq n - 3\}$

$E[C_n \circ (n-3) K_1] = \{e_i/1 \leq i \leq n\} \cup \{e_{ij}/1 \leq i \leq n, 1 \leq j \leq n - 3\}$

Here $|V(G)| = n(n-2)$ and $|E(G)| = n(n-2)$

Consider the colour class $C=\{c_1, c_2, c_3, ..., c_n\}$. Assign a proper colouring to these vertices as follows. Assign the colour $c_i$ to vertex $v_i$ for $i=1,2,3...n$ and assign the colour $c_{n+1}$ to the vertex $v_{ij}$ for $i=1,2,3...n$ and $j=1,2,3..n-3$. Here the vertex $v_i$ for $i=1,2,3..n$ does not realize its own colour because each $v_i$ is adjacent with $v_{i-1}$ and $v_{i+1}$ for $i=2,3,..n-1$, $v_1$ is adjacent with $v_2,v_n$ and $v_n$ is adjacent with $v_{n-1}$ and $v_1$. To make the colouring as b-chromatic, assign the colouring to $v_i$, $v_{ij}$’s as follows. For $1 \leq i \leq n$, assign the colour $c_i$ to $v_i$. For $1 \leq i \leq n$, $1 \leq j \leq n-3$ assign the colour $c_{i+j+1}$ to $v_{ij}$’s when $i+j < n$ and assign the colour $c_{i+j+1-n}$ to remaining $v_{ij}$’s when $i+j \geq n$.

Now, here all the vertices $v_i$ for $i=1,2,3..n$ realizes its own colour, which produces a b-colouring. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.

Example

![Diagram](image)

Figure 21: $\varphi [C_6 \circ 3K_1] = 6$
8.13.2 Structural Properties of $C_n \circ (n-3) K_1$

The number of vertices and edges in $C_n \circ (n-3) K_1 = n(n-2)$. The maximum and minimum degree of $C_n \circ (n-3) K_1$ are denoted as $\Delta = n-1$ and $\delta=1$ respectively. The number of vertices having maximum degree in $C_n \circ (n-3) K_1$ is denoted by $n(p_\Delta)=n$ and the number of vertices having minimum degree in $C_n \circ (n-3) K_1$ is denoted by $n(p_\delta)=n(n-3)$ respectively.

8.14 b-Chromatic Number of Corona Product of Star Graph with Complete Graph

8.14.1 Theorem

For any $n \geq 2$, $\varphi [K_{1,n} \circ (n-1) K_1] = n+1$ where $n \geq 2$.

Proof

Let $v_1,v_2,v_3,\ldots,v_n$ be the Pendant vertices of the graph $K_{1,n}$ with $v$ as the root vertex. i.e. $V[K_{1,n}] = \{v\} \cup \{v_i / 1 \leq i \leq n\}$. Here $v$ is adjacent with $v_i$ for $i=1,2,3\ldots n$.

Now consider the vertex set of $K_{1,n} \circ (n-1) K_1$ is defined as follows:

i.e. $V[K_{1,n} \circ (n-1) K_1] = \{v\} \cup \{v_i / 1 \leq i \leq n\} \cup \{v_{ij} / 1 \leq i \leq n, 1 \leq j \leq n-1\}$

$\cup \{v'_i / 1 \leq i \leq n-1\}$.

Consider the colour class $C=\{c_1,c_2,c_3,\ldots,c_n,c_{n+1}\}$. Assign a proper colouring to these vertices as follows. Give the colour $c_i$ to vertex $v_i$ for $i=1,2,3\ldots n$ and assign the colour $c_{n+1}$ to the root vertex $v$. Here the root vertex realizes its own colour but the vertices $v_i$ for $i=1,2,3\ldots n$ does not realize its own colour, which does not produce a b-chromatic colouring.

Thus to make the colouring as b-chromatic one, assign the colouring as follows:

For $1 \leq i \leq n$, assign the colour $c_i$ to $v_i$. Assign the colour $c_{n+1}$ to root vertex $v$. Next assign proper colouring to the $v_{ij}$'s as follows. For $i=1,2,3\ldots n, j=1,2,3\ldots n-1$, assign the colour $c_{i+j}$ to $v_{ij}$ when $i+j \leq n$ and assign the colour $c_{i+j-n}$ when $i+j>n$. For the remaining vertex $v'_i$ assign any colour $c_i$ for $i=1,2,\ldots n-1$.

Now, all the vertices $v_i$ for $i=1,2,3\ldots n$ and the root vertex $v$ realizes its own colour, which produces a b-chromatic colouring. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.
Example

Figure 22: $\phi [K_{1,4} \circ (n-1) K_1] = 5$

8.14.2 Structural Properties of $K_{1,n} \circ (n-1)K_1$

- Number of vertices in $K_{1,n} \circ (n-1)K_1$ i.e. $p[K_{1,n} \circ (n-1)K_1] = n(n+1)$.
- Number of edges in $K_{1,n} \circ (n-1)K_1$ i.e. $q[K_{1,n} \circ (n-1)K_1] = n^2 + n - 1$.
- Maximum degree of $K_{1,n} \circ (n-1)K_1$ is $\Delta = 2n - 1$.
- Minimum degree of $K_{1,n} \circ (n-1)K_1$ is $\delta = 1$.
- The number of vertices having maximum degree $\Delta$ in $K_{1,n} \circ (n-1)K_1$ is $n(\rho_\Delta) = l$.
- The number of vertices having minimum degree $\delta$ in $K_{1,n} \circ (n-1)K_1$ is $n(\rho_\delta) = n^2 - 1$.
- $n$ vertices with degree $n$. 