CHAPTER 7

b-Colouring of Middle Graph and Middle Graph of Central Graph

In this Chapter, the structural properties of Cycle, Path, Star graph, Fan graph, Sunlet graph, Double Star graph, Bistar, Complete Bipartite graph, are obtained along with its b-Chromatic number which are denoted as $M(C_n)$, $M(P_n)$, $M(K_{1,n})$, $M(F_{1,n})$, $M(S_n)$, $M(K_{1,n,n})$, $M(B_{n,n})$ and $M(K_{m,n})$ respectively. Also the b-Chromatic number of Middle graph of Central graph of Complete graph and Star graph are obtained along with its structural properties.

7.1 Introduction [78, 83]

The Middle graph of $G$, denoted by $M(G)$ is defined as the vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices $x, y$ in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds.

- $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$.
- $x$ is in $V(G)$, $y$ is in $E(G)$, $x, y$ are adjacent in $G$.

Example

![Figure 1(a): C_4](image1)

![Figure 1(b): Middle graph of C_4](image2)
7.2 b-Chromatic Number of Middle Graph of Cycle

7.2.1 Theorem

For any \( n \geq 5 \), \( \varphi[M(C_n)] = 5 \)

Proof

Let \( C_n \) be a Cycle with \( n \) vertices \( v_1, v_2, v_3, \ldots, v_n \). Let \( e_i = v_iv_{i+1} \) for \( 1 \leq i \leq n-1 \) and \( e_n = v_nv_1 \) be the edges of the Cycle \( C_n \). If \( M(C_n) \) be the Middle graph of Cycle \( C_n \) then \( v_1', v_2', v_3', \ldots, v_n' \) be the newly added vertices corresponding to the edges \( e_1, e_2, e_3, \ldots e_{n-1} \) in order to obtain \( M(C_n) \). The vertex set of Middle graph of cycle is defined as follows:

\[
V[M(C_n)] = \{v_i/1 \leq i \leq n\} \cup \{v_i'/1 \leq i \leq n\}
\]

Here the vertices \( v_1', v_2', v_3', \ldots, v_n' \) forms a four regular graph. Hence by colouring procedure we assign five colours to every \( M(C_n) \) to produce a b-chromatic colouring. Suppose if we assign more than five colours, it contradicts the definition of b-chromatic because the vertices in \( M(C_n) \) is incident with atmost four vertices. Note that rearrangement of colours also does not accommodate the new colour class. Thus by the colouring procedure the above said colouring is maximal and b-chromatic.

Example

![Diagram](image)

*Figure 2: \( \varphi[M(C_6)] = 5 \)
7.2.2 Corollary

For any $n<5$, $\varphi[M(C_n)] = n$

7.2.3 Corollary

The number of cycles in $M(C_n)$ has $n$ times 3 cycle and two times $n$ cycles.

Proof

In $M(C_n)$, every vertex $v_i$ is adjacent with $v_{i+1}$ and $v_{i-1}$ for every $i=2,3,..., n-1$, $v_1$ is adjacent with $v_n$ and $v_1$. Thus we see that every vertex $v_i$ ($i=1,2,3..n$) along with the edges incident with $v_i$ forms a 3 cycle and vertex $v_i$ for $i=1,2,3..n$ forms an $n$ cycle and $v_i$ for $i=1,2,3..n$ forms another $n$ cycle. Thus, there are $n$ times 3 cycles and two times $n$ cycles.

7.2.4 Structural Properties of Middle Graph of Cycle

The number of vertices in $M(C_n)$, for $n \geq 2$, i.e. $p[M(C_n)] = 2n$, the number of edges in $M(C_n)$ i.e. $q[M(C_n)] = 3n$ and the maximum and minimum degree of $M(C_n)$ are denoted as $\Delta = 4$ and $\delta = 2$ respectively. The number of vertices having maximum and minimum degree in $M(C_n)$ is denoted by $n(p_\Delta) = n$ and $n(p_\delta) = n$ respectively.

7.2.5 Remark

The number of vertices in $M(C_n)$ is two times the number of vertices in Cycle $C_n$.

7.2.6 Remark

The number of edges in $M(C_n)$ is three times the number of edges in Cycle $C_n$.

7.3 b-Chromatic Number of Middle graph of Path and its Structural Properties

7.3.1 Theorem

$$\varphi[M(P_n)] = \begin{cases} 
3 & \text{if } n = 3,4 \\
4 & \text{if } n = 5,6,7 \\
5 & \text{if } n \geq 8 
\end{cases}$$
Proof

Let $v_1, v_2, v_3... v_n$ and $e_1, e_2, e_3... e_{n-1}$ are respectively be the vertices and edges of the path graph $P_n$. If $M(P_n)$ be the Middle graph of Path $P_n$ then $v_1, v_2, v_3... v_n$ be the vertices of path and $v_1', v_2', v_3', ....v_{n-1}'$ be the newly added vertices corresponding to the edges $e_1, e_2, e_3... e_{n-1}$ in order to obtain $M(P_n)$. The vertex set of $M(P_n)$ is defined as follows:

$$i.e. \ V[M(P_n)] = \{v_i/ 1 \leq i \leq n\} \cup \{v_i'/ 1 \leq i \leq n-1\}$$

The result is easy to verify for $n=2,3,4... 7$. Now we consider for $n \geq 8$. Here the vertices $v_2', v_3', ....v_{n-1}'$ are incident with at most four edges. So there is a possibility of assigning five colours to every $M(P_n)$ to produce a b-chromatic colouring. Suppose if we assign more than five colours, it contradicts the definition of b-colouring. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.

Example

![Figure 3: $\phi[M(P_8)] = 5$](image)

7.3.2 Corollary

Every $M(P_n)$ has $(n-2)$ edge disjoint cycles.

Proof

In $M(P_n)$, the vertices $v_i'$ is adjacent with the vertices $v_i, v_{i+1}$ and $v_i, v_{i+1}'$ and $v_i+2'$ for $i=2,3,4...n-2$ and the vertices $v_i$ is adjacent with $v_i, v_{i+1}$, $v_i$ and $v_{n-1}$ is adjacent with $v_{n-1}, v_n$ and $v_{n-2}$. Therefore for every $i=1,2,3...n-2$ the vertices $v_i'$ and $v_i+1$ forms a cycle. Thus there are $(n-2)$ edge disjoint cycles in every $M(P_n)$. 

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7.3.3 Structural Properties of Middle Graph of Path

The number of vertices in $M(P_n), (n>4)$ i.e. $p[M(P_n)]= 2n-1$, The number of edges in $M(P_n)$ i.e. $q[M(P_n)]=3n-4$ and the maximum and minimum degree of $M(P_n)$ are denoted as $\Delta = 4$ and $\delta =1$ respectively. The number of vertices having maximum and minimum degree in $M(P_n)$ is denoted by $n(p_{\Delta})=n-3$ and $n(p_{\delta})= 2$ respectively.

7.3.4 Theorem

For every $n>2$, the number of edges in Middle graph of Path graph is $3n-4$.

Proof

$q[M(P_n)] = (n-2) \times n + (n-2)$

$= (n-2)q(K_3) + \text{Remaining edges not in } K_3$

$= (n-2)\times 3 + \text{Remaining edges not in } K_3$

$= 3n-6+2$

$= 3n-4$

Therefore $q[M(P_n)] = 3n-4$

7.4 b-Chromatic Number of Middle Graph of Star Graph

7.4.1 Theorem

For $n \geq 3$, $\varphi[M(K_{1,n})] = n + 1$

Proof

Consider the Star graph $K_{1,n}$ with $V(K_{1,n}) = v_1, v_2, v_3, ..., v_n, v$, where $v$ is the root vertex. By the definition of Middle graph, each edge $vv_i$ for $1 \leq i \leq n$ of $K_{1,n}$ is subdivided by a vertex $v_i'$ for $1 \leq i \leq n$ in $M(K_{1,n})$.

Now assign a proper colouring to these vertices as follows. Consider the colour class $C = \{c_1, c_2, c_3, ..., c_n, c_{n+1}\}$. Assign the colour $c_i$ to the vertex $v_i'$ for $i = 1, 2, ..., n$ and assign the colour $c_{n+1}$ to the root vertex $v$, since the vertex $v_i'$ along with the vertex $v$ forms a Complete
graph. Therefore any colouring of \(<v_i,v_i'>\) will be a b-chromatic colouring. Now we cannot colour any \(v_i\) with a new colour, since all the vertices \(v_i's\) are pendant vertices. Thus by the colouring procedure it is the maximum and b-colouring.

Therefore \(\varphi[M(k_{1,n})] = n + 1, n \geq 3\).

Example

![Image](image.png)

*Figure 4: \(\varphi[M(k_{1,4})] = 5\)*

### 7.4.2. Structural Properties of Middle Graph of Star Graph

The number of vertices in \(M(K_{1,n}), (n>2)\) i.e. \(p[M(K_{1,n})] = 2n + 1\). The number of edges in \(M(K_{1,n})\) i.e. \(q[M(K_{1,n})] = \frac{n(n+3)}{2}\) and the maximum and minimum degree of \(M(K_{1,n})\) are denoted as \(\Delta = n+1\) and \(\delta = 1\) respectively. Here \(n\) vertices of degree \(n+1\), \(n\) vertices of degree \(1\) and one vertex of degree \(n\).

### 7.5 b-Chromatic Number of Middle Graph of Fan Graph

#### 7.5.1 Theorem

\[\varphi[M(F_{m,n})] = n+1 \text{ for } m=1 \text{ and } n \geq 2\]

**Proof**

Let \((X,Y)\) be the bipartition of \(F_{m,n}\) with \(|X|=m\) and \(|Y|=n\). Let \(v\) be the vertex of \(X\) and \(y=\{v_1,v_2,v_3..v_n\}\). By the definition of Middle graph each vertex \(vv_i\) for \(1 \leq i \leq n\) is subdivided by the newly introduced vertex \(v_i'\) for \(1 \leq i \leq n\) in \(M(F_{m,n})\). Also \(u_i' (1 \leq i \leq n-1)\) is the another newly introduced vertex between \(v_iv_{i+1}\) for \(i=1,2,3..n-1\) in \(M(F_{m,n})\).
\[ V[M(F_{m,n})] = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u_i' : 1 \leq i \leq n-1\} \]

Here in \( M(F_{m,n}) \), the vertices \( v'_1, v'_2, v'_3, ..., v'_n, v \) induces a clique of order \( n+1 \). Therefore we say that the b-chromatic number of \( M(F_{m,n}) \geq n+1 \).

Consider the colour class \( C = \{c_1, c_2, c_3, ..., c_n, c_{n+1}\} \). Now assign a proper colouring to the vertices as follows. Assign the colour \( c_{i+1} \) to \( v'_i \) for \( i = 1, 2, 3, ..., n \) and \( c_1 \) to the root vertex \( v \). Here the vertices \( v, v_i \) for \( i = 1, 2, 3, ..., n \) realizes its own colour, which produces b-chromatic colouring. Next assign the colour \( c_{n+2} \) to the vertices \( v_i \) for \( i = 1, 2, 3, ..., n \) or to the vertices \( u_i' \) for \( i = 1, 2, 3, ..., n-1 \), here the vertices \( v_1, v_2, v_3, ..., v_n \) and \( u_i' (1 \leq i \leq n-1) \) does not realizes the new colour.

Hence there is a possibility of assigning only the existing colours to the vertices \( v'_1, v'_2, v'_3, ..., v'_n \) and \( u_i' (1 \leq i \leq n) \). So, we say that the b-chromatic number of \( M(F_{m,n}) \leq n+1 \). Therefore \( \varphi[M(F_{m,n})] = n+1 \). Note that rearrangement of colours also does not accommodate the new colour class. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.

**Example**

\[ \text{Figure 5(a)}: F_{(1,3)} \quad \text{Figure 5(b)}: M[F_{(1,3)}] = 4 \]

### 7.5.2 Structural Properties of Middle graph of Fan graph

The number of vertices in \( M(F_{l,n}) \), for \( n \geq 2 \), i.e. \( p[M(F_{l,n})] = 3n \). The maximum and minimum degree of \( M(F_{l,n}) \) are denoted as \( \Delta = n+3 \) and \( \delta = 2 \) respectively. The number of edges in \( M(F_{l,n}) \) i.e. \( q[M(F_{l,n})] = \frac{n^2+13n-12}{2} \).
7.5.3 Theorem

For every $n \geq 2$, $q[M(F_{1,n})] = \left\lfloor \frac{n^2 + 13n - 12}{2} \right\rfloor$

Proof

$q[M(F_{1,n})] = \text{Number of edges in } K_{n+1} + (n-2) \text{ Number of edges in } K_4 + \text{ Remaining edges}

= \left\lfloor \frac{n(n+1)}{2} \right\rfloor + (n-2) q(K_4) + 2q(K_3)

= \left\lfloor \frac{n(n+1)}{2} \right\rfloor + (n-2) 6 + 2(3)

= \left\lfloor \frac{n^2 + n + 12n - 24 + 12}{2} \right\rfloor

= \left\lfloor \frac{n^2 + 13n - 12}{2} \right\rfloor$

Therefore $q[M(F_{1,n})] = \left\lfloor \frac{n^2 + 13n - 12}{2} \right\rfloor$

7.6 b-Chromatic Number of Middle Graph of Sunlet Graph

7.6.1 Theorem

The b-chromatic number of Middle graph of Sunlet graph is 5.

Proof

The Sunlet graph is the graph on $2n$ vertices obtained by attaching $n$ pendant edges to a Cycle $C_n$. Consider the Middle graph of Sunlet graph. Here, the Middle graph of every Sunlet graph has $n$-edge disjoint subgraph which induces $K_4$. Also we see that every $n$-edge disjoint $K_4$ is attached to the pendant vertex. Thus, by proper colouring we can assign a maximum of five colours for producing a b-chromatic colouring. Suppose if we assign more than five colours it contradicts the definition of b-colouring. Thus, by the colouring procedure the b-chromatic number of Middle graph of Sunlet graph is five.
Example

Figure 6: $\varphi[M(S_5)] = 5$

7.6.2 Structural Properties of Middle Graph of Sunlet Graph

In the Middle graph of Sunlet graph ($n \geq 3$) the number of vertices in $M(S_n)$ i.e. $p[M(S_n)] = 4n$, number of edges in $M(S_n)$ i.e. $q[M(S_n)] = 7n$, the maximum and minimum degree of $M(S_n)$ is denoted by $\Delta = 6$ and $\delta = 1$ respectively. Here there are $n$-copies of edge disjoint complete graph of order 4. The number of vertices having maximum and minimum degree in $M(S_n)$ is denoted by $n(p_\Delta) = n$ and $n(p_\delta) = n$ and remaining $n$ vertices are with degree $n+1$.

7.7 b-Chromatic Number of Middle Graph of Double Star Graph

7.7.1 Theorem

For any integer $n > 2$, the b-Chromatic number of Middle graph of any Double Star graph is $n+1$ i.e. $\varphi[M(K_{I,n,n})] = n+1$

Proof

Consider the Double star graph $K_{I,n,n}$ with $V(K_{I,n,n}) = \{v_l / 1 \leq l \leq n\} \cup \{v'_l / 1 \leq l \leq n\} \cup \{v\}$. In $M(K_{I,n,n})$, by the definition of Middle graph each edge $vv_l$ is subdivided by the new vertex $u_l$ and each edge $v_l v'_l$ is subdivided by the another new vertex $u'_l$. Here we see that the vertices $u'_l$s are mutually adjacent to each other and the vertices $u'_l$ are adjacent with $u_l, v_l$ and $v_l'$ for $i=1,2,3..n$. The vertex set of $M(K_{I,n,n})$ is defined as follows:

i.e. $V[M(K_{I,n,n})]=\{v_l / 1 \leq l \leq n\} \cup \{u_l / 1 \leq l \leq n\} \cup \{v\} \cup \{v'_l / 1 \leq l \leq n\} \cup \{u'_l / 1 \leq l \leq n\}$. 

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Here the vertices $u_1, u_2, u_3 \ldots u_n$ along with root vertex $v$ induces a clique of order $n+1$ in $M(K_{1,n,n})$. Therefore we say that the b-chromatic number of $M(K_{1,n,n}) \geq n+1$. Now we will prove the inequality $\varphi[M(K_{1,n,n})] \leq n+1$, for that assign a proper colouring to these vertices as follows. Consider the colour class $C = \{c_1, c_2, c_3 \ldots c_n, c_{n+1}\}$. By proper colouring we assign the colour $c_i$ to the vertex $u_i$ for $i=1,2,3 \ldots n$ and $c_{n+1}$ to the vertex $v$ (induces a clique of order $n+1$), which produces a b-chromatic colouring. Suppose if we assign any new colour to the remaining vertices, it will not produce a b-chromatic colouring due to the above mentioned non-adjacency condition. So we should assign only the presused colours such as $c_{n+1}$ to the vertices $v_i$ and $v_i'$ for $i=1,2,3 \ldots n$ and $c_{i+1}$ to $u_i'$ for $i=1,2,3 \ldots n-1$ and $c_1$ to $u_n'$. Now, we say that the b-chromatic number of $M(K_{1,n,n}) \leq n+1$. Therefore $\varphi[M(K_{1,n,n})] = n+1$. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.

**Example**

![Figure 7: $\varphi[M(K_{1,4,4})] = 5$](image)

### 7.7.2 Structural Properties of Middle Graph of Double Star Graph

In the Middle graph of Double star graph, number of vertices in $M(K_{1,n,n})$ i.e. $p[M(K_{1,n,n})] = 4n+1$, number of edges in $M(K_{1,n,n})$ i.e. $q[M(K_{1,n,n})] = \left\lfloor \frac{n^2+9n}{2} \right\rfloor$, the maximum and minimum degree of Middle graph of Double star graph is denoted by $\Delta = n+2$ and $\delta = 1$ respectively. The number of vertices having $\Delta$ in $M(K_{1,n,n})$ is denoted by $n(p_{\Delta}) = n$, similarly $n(p_\delta) = n$. 

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7.7.3 Theorem

For every $n \geq 2$, $q[ M(K_{1,n,n}) ] = \left[ \frac{n^2 + 9n}{2} \right]$

Proof

$q[ M(K_{1,n,n}) ] = \text{Number of edges in } K_{n+1} + \text{Number of edges in } K_3 + \text{Remaining edges}$

$= q( K_{n+1} ) + n( \text{Number of edges in } K_3 ) + n$

$= \left[ \frac{n(n+1)}{2} \right] + 3n + n$

$= \left[ \frac{n^2 + n + 8n}{2} \right]$

$= \left[ \frac{n^2 + 9n}{2} \right]$

Therefore $q[ M(K_{1,n,n}) ] = \left[ \frac{n^2 + 9n}{2} \right]$

7.8 b-Chromatic number of Middle Graph of Bistar Graph

7.8.1 Theorem

The b-Chromatic number of Middle graph of Bistar has $n+2$ colours for every $n \geq 2$
i.e. $\varphi [ M(B_{n,n}) ] = n+2.$

Proof

Consider the Bistar $B_{n,n}$. Let $V( B_{n,n} ) = \{ u, v, u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n \}$ and $E( B_{n,n} ) = \{ uu_i, vv_j : 1 \leq i \leq n, 1 \leq j \leq n \}$. By the definition of Middle graph, introduce a new vertex $u_i'$ in the edge connecting $uu_i$ and let $v_j'$ be the new vertex in edge connecting $vv_j$ and $s$ be the vertex in between $u$ and $v$ in $M( B_{n,n} )$. The vertex set of $M( B_{n,n} )$ is defined as follows:

i.e. $V[ M(B_{n,n}) ] = \{ u, v, u_i, v_j, u_i', v_j', s / 1 \leq i \leq n, 1 \leq j \leq n \}$

Here $v_j'$ for $i=1,2,3..n$ along with the vertex $s$ forms a clique of order $n+2$, also $u_i'$ for $i=1,2,3..n$ with $s$ forms another clique of $n+2$. Thus we see there are two edge disjoint subgraph of order $K_{n+2}$. Therefore we say $\varphi [ M(B_{n,n}) ] \geq n+2.$
Consider the colour class $C = \{c_1, c_2, c_3 \ldots c_n, c_{n+1}, c_{n+2}\}$. Hence by proper colouring we assign $n+2$ colours to every $M(B_{n,n})$. Suppose if we assign more than $n+2$ colours it contradicts the definition of b-chromatic colouring. Note that rearrangement of colours also does not accommodate the new colour class. Thus by the colouring procedure the above said colouring is maximal and b-chromatic.

**Example**

![Middle graph of Bistar](image)

*Figure 8: $\varphi[M(B_{3,3})] = 5$*

### 7.8.2 Corollary

The Middle graph of any Bistar ($n \geq 2$) is a Separable graph.

**Proof**

From the above theorem and example, clearly we say that the vertex connectivity of Middle graph of BiStar is one. By definition, a graph is said to be separable if its vertex connectivity is one. Therefore the Middle graph of any BiStar is a Separable graph.

#### 7.8.3.1 Structural Properties of Middle Graph of Bistar

In the Middle graph of Bistar graph, number of vertices in $M(B_{n,n})$ for every $n \geq 2$, i.e. $p[M(B_{n,n})] = 4n + 3$, number of edges in $M(B_{n,n})$ for every $n \geq 2$, i.e. $q[M(B_{n,n})] = n^2 + 5n + 2$, the maximum and minimum degree of Middle graph of Bistar graph is denoted by $\Delta = 2(n+1)$ and $\delta = 1$. The number of vertices having $\Delta$ in $M(B_{n,n})$ is denoted by $n(p_\Delta) = 1$, similarly $n(p_\delta) = 2n$ respectively.
7.8.4 Theorem

For every \( n \geq 2 \), \( q[M(B_{n,n})] = n^2 + 5n + 2 \)

Proof

\[
q[M(B_{n,n})] = 2 \left( \text{Number of edges in } K_{n+2} \right) + \text{Remaining edges not in } K_{n+2}
\]

\[
= 2q[K_{n+2}] + \text{Remaining edges not in } K_{n+2}
\]

\[
= 2 \left( \frac{(n+2)(n+1)}{2} \right) + 2n
\]

\[
= (n+2)(n+1) + 2n
\]

\[
= n^2 + n + 2n + 2 + 2n
\]

\[
= n^2 + 5n + 2
\]

Therefore \( q[M(B_{n,n})] = n^2 + 5n + 2 \)

7.9 b-Chromatic Number of Middle Graph of Complete Bipartite Graph

7.9.1 Theorem

Let \( K_{m,n} \) be a Complete bipartite graph on \( m \) and \( n \) vertices, then \( \varphi[M(K_{m,n})] = n + 1 \)

for every \( m \leq n \).

Proof

Consider the Complete bipartite graph \( K_{m,n} \) with bipartition \((X,Y)\) where \( X = \{ u_1, u_2, u_3, \ldots, u_m \} \) and \( Y = \{ v_1, v_2, v_3, \ldots, v_n \} \). By the definition of Middle graph, let \( u_{ij} \) be the newly introduced vertex in the edge connecting \( u_i \) and \( v_j \)

i.e. \( M(K_{m,n}) = \{ u_i \mid 1 \leq i \leq m \} \cup \{ v_j \mid 1 \leq j \leq n \} \cup \{ u_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n \} \).

Here every vertex set \( \{ u_i \mid 1 \leq i \leq m \} \) are not adjacent with vertex set \( \{ v_j \mid 1 \leq i \leq n \} \). Clearly \( u_1, u_2, u_3, \ldots, u_m \) and \( v_1, v_2, v_3, \ldots, v_n \) are independent sets with \( |u_i| = m \) and \( |v_i| = n \). Here \( u_i \) along with \( u_{ij} 's \) for \( i = 1, 2, 3 \ldots, m \) and \( j = 1, 2, 3 \ldots, n \) forms a complete graph of clique of order \( m \) in \( M(K_{m,n}) \). Also we see that there are \( m \) copies of edge disjoint \( K_{m+1} \) in \( M(K_{m,n}) \).
Consider the colour class $C=\{c_1, c_2, c_3 \ldots c_m, c_{m+1}\}$. Now assign a proper colouring to $M(K_{m,n})$ as follows. Assign the colour $c_i$ to the vertex $u_i$ for $i=1,2,3 \ldots m$ and assign the colour $c_{j+1}$ to $u_{1j}$, $c_{j+2}$ to $u_{2j}$ and so on for $j=1,2,3 \ldots n$. Here the vertex $u_i$ for $i=1,2,3 \ldots m$ do not realize the colour $c_{i+j}$, which does not produce a b-chromatic colouring.

To make the colouring as b-chromatic, we assign the colour $c_1$ to $u_i$ for $i=1,2,3 \ldots m$ and assign the colour $c_{i+j}$ to $u_{ij}$’s when $i+j \leq n+1$ and for remaining $u_{ij}$’s assign $c_{i+j-n}$ when $i+j \geq n+1$ for $i=2,3,4 \ldots m$ and $j=1,2,3 \ldots n$, which produces a b-chromatic colouring. Suppose if we assign any new colour to $v_j$ for $j=1,2,3 \ldots n$ it does not produce a b-chromatic colouring because each $u_i$ for $1 \leq i \leq m$ is not adjacent with $v_j$ for $1 \leq j \leq n$. So we have to assign only the preused colours to $v_j$ i.e. assign the colour $c_1$ to $v_j$ for $j=1,2,3 \ldots n$. Thus by the colouring procedure, we assign a maximum of $n+1$ colours to every $M(K_{m,n})$ to produce a b-chromatic colouring.

Therefore $\phi[M(K_{m,n})] = n+1$ for $m \leq n$.

**Example**

![Diagram](image)

*Figure 9: $\phi[M(K_{3,3})] = 4$*
7.9.2 Structural Properties of the Middle Graph of Complete Bipartite Graph

The number of vertices in $M(K_{m,n})$, for $n \geq 2$, i.e. $p[M(K_{m,n})] = m + n + mn$. The maximum and minimum degree of $M(K_{m,n})$ are denoted as $\Delta = m + n$ and $\delta = \min(m,n)$ respectively. Here we find there are $m$ copies of vertex disjoint subgraph $K_m$. The number of edges in $M(K_{m,n})$ is $q[M(K_{m,n})] = \frac{3mn + mn^2}{2} + \sum_{i=1}^{m} n(m - i)$.

7.9.3 Theorem

$$q[M(K_{m,n})] = \frac{3mn + mn^2}{2} + \sum_{i=1}^{m} n(m - i)$$

Proof

$$q[M(K_{m,n})] = \text{Number of edges in all } K_{n+1} + mn + \text{Remaining edges}$$

$$= mq(K_{n+1}) + nm + \text{Remaining edges}$$

$$= m\left[ \frac{n(n+1)}{2} \right] + nm + \sum_{i=1}^{m} n(m - i)$$

$$= \frac{mn + mn^2 + 2mn}{2} + \sum_{i=1}^{m} n(m - i)$$
\[ q[M(K_{m,n})] = \frac{3mn+mn^2}{2} + \sum_{i=1}^{m} n(m - i) \]

Therefore \( q[M(K_{m,n})] = \frac{3mn+mn^2}{2} + \sum_{i=1}^{m} n(m - i) \)

### 7.10 b-Chromatic Number of Middle Graph of Central Graph of Complete Graph

#### 7.10.1 Theorem

For any \( n \geq 3 \), \( \phi(M[C(K_n)]) = n+2 \)

**Proof**

Let \( K_n \) be a Complete graph with \( n \) vertices and \( \left( \begin{array}{c} n \\ 2 \end{array} \right) \) edges. By the definition of Central graph, let \( v_{ij} \) be the newly introduced vertex in the edge connecting \( v_i, v_j \) \((1 \leq i \leq n-1, i+1 \leq j \leq n)\) in every \( K_n \). Let us consider the edges as \( e_{ij} = e_{ij} \) and \( e_{ij}, v_j = e_{ij}' \). Here we consider the undirected graph of \( K_n \), so we can have \( e_{ij} = e_{ij} \) and \( e_{ij}' = e_{ij}' \). By the definition of Middle graph, the vertices and edges in \( C(K_n) \) corresponds to the vertices of \( M[C(K_n)] \).

\[ i.e. \ V(M[C(K_n)]) = \{v_i, v_{ij}, e_{ij}, e_{ij}' / 1 \leq i \leq n-1, i+1 \leq j \leq n\}. \]

Here each \( v_{ij} \) is adjacent with \( e_{ij} \) and \( e_{ij}' \), so that \( v_{ij}, e_{ij}, e_{ij}' \) forms a \( K_3 \) in every \( M[C(K_n)] \). By the definition of the Middle graph, the edges incident with vertex \( v_i \) together with the vertex \( v_i \) forms a complete graph of order \( n \). Hence it contains \( n \) copies of vertex disjoint \( K_n \) as subgraphs. Let us name the Complete subgraphs in anticlockwise direction namely \( K_n^1, K_n^2, ..., K_n^n \), and label the vertices of each \( K_n^i \) in anticlockwise direction as \( v_i^j \) for \( i=1,2,3..n, j=1,2,3..n \).

Here in \( M[C(K_n)] \) we see that in each \( K_n^{(i)} \) except the vertex \( v_i^j \) for \( i=j \), all the remaining vertices are adjacent with the two vertices other than \( v_i^j \) in all \( K_n^{(i)} \). By observation we see there are \( \frac{n(n-1)}{2} \) edge disjoint triangles in every \( M[C(K_n)] \).

Consider the colour class \( C=\{c_1,c_2,c_3...c_n,c_{n+1},c_{n+2}\} \). Now assign a proper colouring to these vertices as follows. Assign the colour \( c_j \) to the vertex \( v_i^j \) of \( K_n^j \) for \( j=1,2,3..n \) and assign \( c_{n+1} \) and \( c_{n+2} \) to the remaining two vertices of the triangle which is attached to the vertices of \( K_n^j \).

Now in \( K_n^j \) all the vertices \( v_i^j \) for \( j=2,3..n \) realizes its own colour except the vertex \( v_i^j \), which does not produce a b-chromatic colouring.
Thus to make the colouring as b-chromatic, assign a proper colouring i.e. assign only the preused colours to $K_n^i$ for $i=2,3,...n$ such that every vertices realizes its own colour class. Suppose if we assign any new colour to any $K_n^i$ ($2 \leq i \leq n$), it contradicts the definition of b-chromatic colouring. Thus by colouring procedure, the above said colouring is maximal and b-chromatic.

**Example**

![Figure 11: $\varphi\{M[C(K_4)]\} = 6$](image)

### 7.10.2 Structural Properties of Middle Graph of Central Graph of Complete Graph

In $M[C(K_n)]$ for $n \geq 2$, number of vertices in $M[C(K_n)] = \left[\frac{n(3n-1)}{2}\right]$, number of edges in $M[C(K_n)] = \left[\frac{n(n^2+2n-3)}{2}\right]$, maximum and minimum degree of $M[C(K_n)]$ are denoted as $\Delta = n+1$ and $\delta = 2$ respectively. In $M[C(K_n)]$ there are $\left[\frac{n^2-n}{2}\right]$ vertices of degree 2, $n$ vertices of degree $(n-1)$, $n(n-1)$ vertices of degree $n+1$. 

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7.11 b-Chromatic number of Middle Graph of Central Graph of Star Graph

7.11.1 Theorem

For any integer $n \geq 2$, $\varphi\{M[C(K_{1,n})]\} = n+3$

Proof

Let $K_{1,n}$ be a Star graph with vertices $v, v_1, v_2, v_3, \ldots v_n$ i.e. $V[K_{1,n}] = \{v\} \cup \{v_i / I \leq i \leq n\}$ and $E[K_{1,n}] = \{e_i / I \leq i \leq n\}$ where $deg(v) = n$. By the definition of Central graph, let $v_i'$ ($I \leq i \leq n$) be the newly introduced vertices in the edge connecting $v v_i$ in $K_{1,n}$. Let $e_i$ be the edge between $v v_i'$ and $e_i'$ be the edge $u_i v_i$ for $I \leq i \leq n$ i.e. $V[C(K_{1,n})] = \{v\} \cup \{v_i / I \leq i \leq n\} \cup \{v_i'/I \leq i \leq n\}$ and $E[C(K_{1,n})] = \{e_i / I \leq i \leq n\} \cup \{e_i'/I \leq i \leq n\} \cup \{e_{ij} : I \leq i \leq n-1, 2 \leq j \leq n\}$. By the definition of Middle graph, the vertex set and the edge set in $C(K_{1,n})$ corresponds to the vertex set of $M[C(K_{1,n})]$. i.e. $V[M[C(K_{1,n})]] = \{v\} \cup \{v_i / I \leq i \leq n\} \cup \{v_i'/I \leq i \leq n\} \cup \{e_i / I \leq i \leq n\} \cup \{e_i'/I \leq i \leq n\} \cup \{u_i / I \leq i \leq n\}$

where $u_i = \{e_{ij} / j=1,2,3..i-1,i+1,\ldots n, I \leq i \leq n\}$ and let $u'' = u_1 + u_2 + u_3 + \ldots u_n$.

Clearly $|u''| = \frac{n(n-1)}{2}$. Here we see that the vertices $v, e_1, e_2, \ldots e_n$ induces a clique of order $n+1$. Also the vertices $v_i$ and $e_i'$ together with $u_i$ induces another clique of order $n+1$ but there is only one vertex common to both cliques of order $n+1$ i.e. each vertex $e_i$ is adjacent only with $e_i'$ for $i=1,2,3,\ldots n$.

Consider the colour class $C = \{c_1, c_2, c_3, \ldots c_n, c_{n+1}, c_{n+2}\}$. Now assign a proper colouring to $M[C(K_{1,n})]$ as follows. Assign the colour $c_1$ to the root vertex $v$ and assign the colour $c_{i+1}$ to the vertex $e_i$ for $i=1,2,3,\ldots n$. Next assign the colour $c_{n+2}$ to $v_i'$ for $i=1,2,3,\ldots n-1$ and for remaining $v_i'$ assign the colour $c_{n+3}$. Similarly, assign the colour $c_{n+3}$ to $e_i'$ for $i=1,2,3,\ldots n-1$ and for remaining $e_i'$ assign the colour $c_{n+2}$. For $i=2,3,\ldots n$ assign the colour $c_i$ to $v_i$ and assign the colour $c_{n+1}$ to $v_1$. Next assign the colour $c_i$ to the vertex $\frac{u_n(n-1)}{2}$ and assign only the preused colours to the vertices of $u_i$ other than the previously coloured vertex, such that the vertices in $M[C(K_{1,n})]$ realizes $n+3$ colours, which produces a b-chromatic colouring. Note that rearrangement of colours will also be not possible for more than $n+3$ colours. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.
Example

Figure 12: $C(K_{1,3})$

Figure 13: $\varphi\{M[C(K_{1,3})]\} = 6$

7.12 b-Chromatic Number of Middle Graph of Complete Graph

7.12.1 Theorem

If $K_n$ be a Complete graph on $n$ vertices, then $\varphi[M(K_n)] = n$ for every $n > 2$. 
Proof

Let \( K_n \) be a complete graph with vertex set \( V(K_n)=\{v_1,v_2,v_3...v_n\} \). By the definition of the Middle graph, for \( 1\leq i \leq n-1, i+1\leq j \leq n \), let \( v_{ij} \) represent the newly introduced vertex in the edge connecting \( v_i \) and \( v_j \). Here we have considered only the undirected graph so we can have \( v_{ij}=v_{ji} \). Middle graph of \( K_n \) has the vertex set \( \{v_i: 1\leq i \leq n\} \cup \{v_{ij}: 1\leq i \leq n-I, i+1\leq j \leq n\} \). Here there are \( n \) vertices of degree \( n-1 \) and \( \binom{n}{2} \) vertices of degree \( 2n-2 \). By the definition of Middle graph, the vertex \( v_i \) together with incident edges induces a clique of order \( n \) i.e. (say \( K_n \)) in \( M(K_n) \).

In \( M(K_n) \), we find \( n \) copies of edge disjoint \( K_n \). Let \( K_n^i \) be the cliques in \( M(K_n) \) for \( i=1,2,3..n \). Now assign a proper colouring to the above vertices as follows. Let us consider the colours \( c_1, c_2, c_3,...,c_n \). First consider the complete graph \( K_n^i \) for \( i=1 \), assign the colour \( c_1 \) to the vertices of \( K_n^i \), which produces a b-chromatic colouring.

Next suppose if we assign any new colour to the remaining vertices of \( K_n^i \) for \( i=2,3..n \), it contradicts the definition of b-chromatic colouring because here all \( K_n^i \) are edge disjoint complete subgraphs. Hence we should assign only the same set of colours to the remaining subgraphs. Note that rearrangement of colours also fails to accommodate the new colour class. Thus by the colouring procedure the above said colour is maximum and b-chromatic.

Example

![Diagram](attachment:image.png)

*Figure 14: \( \varphi[M(K_3)]=3 \)
7.12.2 Structural Properties of the Middle Graph of Complete Graph

The number of vertices in $M(K_n)$, for $n \geq 2$, i.e. $p[M(K_n)] = \left\lceil \frac{n(n+1)}{2} \right\rceil$, the maximum and minimum degree of $M(K_n)$ are denoted as $\Delta = 2n-2$ and $\delta = n-1$ respectively. We find $n$ copies of edge disjoint subgraph $K_n$ and there are $n$ vertices of degree $n-1$, $\left\lceil \frac{n(n-1)}{2} \right\rceil$ vertices of degree $2n-2$. The number of edges in $M(K_n)$, i.e. $q[M(K_n)] = n\left( \begin{array}{c} n \\ 2 \end{array} \right)$

7.12.3 Theorem

For any integer $n > 2$, the number of edges in Middle graph of Complete graph is $q[M(K_n)] = n\left( \begin{array}{c} n \\ 2 \end{array} \right)$

Proof

\[ q[M(K_n)] = n \times \text{edges in all } K_n = n \times q(K_n) = n \left( \left\lceil \frac{n(n-1)}{2} \right\rceil \right) = n\left( \begin{array}{c} n \\ 2 \end{array} \right) \]

Therefore $q[M(K_n)] = n\left( \begin{array}{c} n \\ 2 \end{array} \right)$

7.12.4 Theorem

For every $n \geq 3$, $\phi\{M[L(K_{1,n})]\} = n$

Proof

$L(K_{1,n}) \cong K_n$. By theorem 7.12.1 we have $\phi\{M[L(K_{1,n})]\} = n$. 

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