Chapter 4

Applications of the CPP to Quantum Dots

4.1 Introduction

The properties of confined two dimensional (2D) electronic systems in semiconductor materials, that named as 2D quantum dots (QDs), are attracting a lot of attention of researchers recently [1, 2, 3]. Such structures are expected to give the platform for future generations of device technologies such as threshold-less lasers and ultra-dense memories. 2D semiconductor QDs generally contain only a few electrons in contrast to standard bulk semiconductor devices, so they give the ultimate limit of the semiconductor device scaling. The physics of QDs exhibits many parallels with the behavior of naturally occurring quantum systems in atomic and nuclear physics. Like a real atom, the energy levels in a QD emerge as quantized and discrete due to confinement of electrons. Unlike real atoms, however, QDs can be easily joined to electrodes and are therefore wonderful tools for obtaining atomic-like properties. Furthermore, the electrons in a QD act lethargically, reluctant to move themselves to provide way for another electron, even on a timescale of milliseconds. Most strangely, the large Coulomb repulsion can make it seem that electrons attract one another. Thus, various theoretical [4, 5, 6, 7, 8] and experimental [9, 10, 11] techniques were developed to obtain the spectroscopic structure of interacting electrons confined in a 2D parabolic QD under the effect of an applied magnetic field. The 2D disc-like gallium arsenide (GaAs) QDs with two electrons are of particular current interest [12, 13, 14, 15]. Theoretically, a system of two interacting electrons confined in a 2D parabolic potential under a uniform magnetic
field is a quasi-exactly solvable quantum mechanical problem. This type of problem is studied by applying the various numerical and approximation methods. Xie [16] obtained the ground state transition, found the effect of distance between a double layer dot on the Coulomb interaction and presented that when the inter-dot distance increases, a higher magnetic field is required to produce ground state transition and for sufficient large distance no transition occurs. Dineykhan et al. [17] obtained the solution for the two interacting electrons in the presence of harmonic term under the influence of magnetic field. Taut obtained the analytical solutions of the 2D SE for two interacting electrons in a uniform magnetic field [18]. Ghosh and Samanta have investigated the solutions for the two electrons employing the ansatz method by considering the confining potential as the sum of harmonic, linear and Coulomb terms in the absence of spin interaction and magnetic field [19]. Zue et al. investigated the quantum size effects on the energy spectra of two electrons by using the power series method [20]. The other techniques such as the WKB approximation [21], shifted 1/N method [22] and the AIM method have also been employed by the researcher for studying the two electrons QDs [23]. The study of 2D QD had also been carried out with magnetic field using different techniques [24, 25, 26, 27, 28].

4.2 Solutions for the QD confine potential

Here in the present chapter, we investigate the solutions for the CPP by employing the methodology given in [29] and the AIM, details of which is given in chap.-1. Then, as a special case we obtain the energy spectra in lower dimensions and compare these with other studies [30, 31]. Since the CPP is the combination of Coulomb, linear and harmonic terms. If we make the linear term zero, then the CPP acts as the confining potential for QDs which is written as

\[ U(r) = ar^2 - \frac{b}{r}, \quad a > 0. \]  \hspace{1cm} (4.1)

This confining potential is now used for investigating two electron GaAs and InGaAs quantum dots in a magnetic field in 2D. The SE in N-dimensional space for the CPP is written as

\[ \left[ \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{l(l+N-2)}{r^2} + \frac{2\mu}{\hbar^2}(E - ar^2 - br + \frac{c}{r}) \right] \psi(r) = 0, \]  \hspace{1cm} (4.2)

where \(m, N\) and \(\mu\) denote the angular momentum quantum number, dimensionality and reduced mass of the particles. \(E\) are the energy eigenvalues corresponding to the radial eigenfunctions \(\psi(r)\) in an arbitrary dimensional space.
On substituting $\psi(r) = r^{-\frac{N-1}{2}} R(r)$ in above equation, we obtain

$$\left[ \frac{d^2}{d r^2} - \frac{4l(l + N - 2) + (N - 1)(N - 3)}{4r^2} + \frac{2\mu}{\hbar^2} (E - ar^2 - br + \frac{c}{r}) \right] R(r) = 0.$$ (4.3)

This equation can further be written into a simplified form as

$$\left[ \frac{d^2}{d r^2} - \frac{\eta^2 - \frac{1}{4}}{r^2} + (\xi - a_1 r^2 - b_1 r + \frac{c_1}{r}) \right] R(r) = 0,$$ (4.4)

with $\eta = l + \frac{N-2}{2}$, $\xi = \frac{2\mu}{\hbar^2} E$, $a_1 = \frac{2\mu}{\hbar^2} a$, $b_1 = \frac{2\mu}{\hbar^2} b$ and $c_1 = \frac{2\mu}{\hbar^2} c$.

An approximate solution of Eq.(4.4) can be obtained by taking the appropriate choice of the wavefunction as

$$R(r) = \exp(-\alpha r^2 - \beta r) g(r),$$ (4.5)

where $\alpha$ and $\beta$ are positive parameters whose values are to be determined in terms of potential parameters $a$, $b$ and $c$. After substituting Eq.(4.5) into Eq.(4.4), we get

$$r g''(r) + \left[ -4\alpha r^2 - 2\beta r + (2\eta + 1) \right] g'(r) + \left[ (4\alpha^2 - a_1) r^3 \right. \left. + (4\alpha \beta - b_1) r^2 + (\beta^2 - 4(\eta + 1)\alpha + \xi) r + c_1 - (2\eta + 1)\beta \right] g(r) = 0.$$ (4.6)

The above equation can be solved by considering $g(r) = \sum_{k=0}^{n} a_k r^k$. To this effect, we follow the theorem 5 of [29] which defines the polynomial solutions of a class of differential equations of the form

$$(a_{3,0} x^3 + a_{3,1} x^2 + a_{3,2} x + a_{3,3}) y'' + (a_{2,0} x^2 + a_{2,1} x + a_{2,2}) y' - (\tau_{1,0} x + \tau_{1,1}) y = 0,$$ (4.7)

where $a_{3,i}, i = 0, 1, 2, 3, a_{2,j}, j = 0, 1, 2$ and $\tau_{1,k}, k = 0, 1$ are arbitrary constant parameters.

The second order linear differential Eq.(4.7) has a polynomial solution of degree $n$ if

$$\tau_{1,0} = n(n-1)a_{3,0} + na_{2,0}, \quad n = 0, 1, 2, \ldots.$$ (4.8)
along with the vanishing of \((n + 1) \times (n + 1)\)-determinant \(\Delta_{n+1}\) given as

\[
\begin{vmatrix}
\beta_0 & \alpha_1 & \eta_1 \\
\gamma_1 & \beta_1 & \alpha_2 & \eta_2 \\
\gamma_2 & \beta_2 & \alpha_3 & \eta_3 \\
\vdots & \vdots & \vdots & \vdots \\
\gamma_{n-2} & \beta_{n-2} & \alpha_{n-1} & \eta_{n-1} \\
\gamma_{n-1} & \beta_{n-1} & \alpha_n \\
\gamma_n & \beta_n
\end{vmatrix}
= 0,
\]

where the entries of the above determinant are expressed in terms of the parameters of Eq.(4.7) by

\[
\beta_n = \tau_{1,1} - n((n - 1)a_{3,1} + a_{2,1}),
\]

\[
\alpha_n = -n((n - 1)a_{3,2} + a_{2,2}),
\]

\[
\gamma_n = \tau_{1,0} - (n - 1)((n - 2)a_{3,0} + a_{2,0}),
\]

\[
\eta_n = -n(n + 1)a_{3,3}.
\]

Here, \(\tau_{1,0}\) is established by Eq.(4.8) for a given value of \(n\) i.e. the degree of the polynomial solution.

Now in order to cast Eq.(4.6) in the form of Eq.(4.7), we take \(\alpha = \sqrt{\frac{a_1}{2}}\) and \(\beta = \frac{b_1}{2\sqrt{a_1}}\).

Thus, using the values of \(\alpha\) and \(\beta\) in Eq.(4.6), we get

\[
rg''(r) + \left[ -2\sqrt{a_1} r^2 - \frac{b_1}{\sqrt{a_1}} r + (2\eta + 1) \right]g'(r) + \left[ \frac{b_1^2}{4a_1} - 2(\eta + 1)\sqrt{a_1 + \xi} r + c_1 - (2\eta + 1)\frac{b_1}{2\sqrt{a_1}} \right]g(r) = 0,
\]
So the polynomial solution of Eq.(4.10), using Eq.(4.8), is given as

\[ \xi = 2\sqrt{a_1}(n + \eta + 1) - \frac{b_1^2}{4a_1}. \]  \hspace{1cm} (4.11)

The constraint conditions on the potential parameters are decided by the vanishing of the tridiagonal determinant with entries

\[ \beta_n = -c_1 + (2n + 2\eta + 1)\frac{b_1}{2\sqrt{a_1}}, \]  \hspace{1cm} (4.12)
\[ \alpha_n = -n(n + 2\eta), \]
\[ \gamma_n = 2(n - k - 1)\sqrt{a_1}, \]
\[ \eta_n = 0, \]

i.e., the vanishing of the \((n + 1) \times (n + 1)\)-determinant \(\triangle_{n+1}\) given by

\[
\begin{vmatrix}
-c_1 + (2\eta + 1)\frac{b_1}{2\sqrt{a_1}} & -(1 + 2\eta) \\
-2k\sqrt{a_1} & -c_1 + (2\eta + 3)\frac{b_1}{2\sqrt{a_1}} & -2(2 + 2\eta) \\
2(1 - k)\sqrt{a_1} & -c_1 + (2\eta + 5)\frac{b_1}{2\sqrt{a_1}} & -3(3 + 2\eta) \\
& \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & \ddots 
\end{vmatrix} = 0
\]

where \(k = n\) is fixed by the size of the determinant \(\triangle_{n+1} = 0\) and represents the degree of polynomial solution of Eq.(4.10). One can easily obtain the remaining entries of the above determinant using Eq.(4.12).

In particular for \(n = 0\), we have \(\triangle_1 = 0\), the ground state energy from Eq.(4.11) becomes

\[ \xi = 2\sqrt{a_1}(\eta + 1) - \frac{b_1^2}{4a_1}. \]  \hspace{1cm} (4.13)

and condition on the potential parameters will be

\[ c_1 = (2\eta + 1)\frac{b_1}{2\sqrt{a_1}}. \]  \hspace{1cm} (4.14)
When \( n = 1 \) i.e. \( \triangle_2 = \)

\[
\begin{vmatrix}
-c_1 + (2\eta + 1)\frac{b_1}{2\sqrt{a_1}} & -(1 + 2\eta) \\
-2\sqrt{a_1} & -c_1 + (2\eta + 3)\frac{b_1}{2\sqrt{a_1}}
\end{vmatrix} = 0.
\]

Then the first excited state energy becomes

\[
\xi = 2\sqrt{a_1}(\eta + 2) - \frac{b_1^2}{4a_1},
\]

and constraint condition is written as

\[
c_1^2 - 2(\eta + 1)\frac{b_1}{\sqrt{a_1}}c_1 + (4\eta^2 + 8\eta + 3)\frac{b_1^2}{4a_1} - 2(1 + 2\eta)\sqrt{a_1} = 0.
\] (4.16)

In this way, one can obtain the complete energy spectra as well as accompanying constraint relations on the present system.

### 4.2.1 Special case

With a view to realize the strength and weakness of the present approach, here we solve the problem for \( n = 0 \) case with \( \bar{h} = \mu = 1 \). Under this condition, the analytical solution is obtained from Eq. (4.13), after substituting \( \eta = l + \frac{N-2}{2} \) and values of other variables, as

\[
E_{nmN} = \sqrt{\frac{\pi}{2}}(2l + N) - \frac{b_1^2}{4a_1}.
\] (4.17)

For this case the problem is also solved numerically. A comparison of the analytical and numerical results along with the literature results is presented in the following table.
TABLE 4.1: lowest energy eigenvalues for $l = 0, 1, 2$ in two, three and four dimensions

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$m$</th>
<th>$N$</th>
<th>From Eq.(4.17)</th>
<th>Numerical Results</th>
<th>Results of [30]</th>
</tr>
</thead>
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<tr>
<td>$\frac{1}{32}$</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-7.750</td>
<td>-7.750</td>
<td>–</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>-7.500</td>
<td>-7.500</td>
<td>–</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>1</td>
<td>10</td>
<td>2</td>
<td>2</td>
<td>-7.250</td>
<td>-7.250</td>
<td>–</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>-7.625</td>
<td>-7.625</td>
<td>-7.625</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
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<td>8</td>
<td>1</td>
<td>3</td>
<td>-7.375</td>
<td>-7.375</td>
<td>-7.375</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
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<td>12</td>
<td>2</td>
<td>3</td>
<td>-7.125</td>
<td>-7.125</td>
<td>-7.125</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>-7.500</td>
<td>-7.500</td>
<td>-7.500</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>4</td>
<td>-7.250</td>
<td>-7.250</td>
<td>-7.250</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>1</td>
<td>14</td>
<td>2</td>
<td>4</td>
<td>-7.000</td>
<td>-7.000</td>
<td>-7.000</td>
</tr>
</tbody>
</table>

The values of potential parameters $a$, $b$ and $c$ conform the constraint condition (4.14). It is apparent that the results of present study are in good agreement with the other studies [30, 31, 32].

4.2.2 Application to the GaAs QD

As a physical application of the potential (4.1), we derive the ground state energy spectra of a two electron quantum dot in a magnetic field. The corresponding Hamiltonian for the two electron quantum dot in a magnetic field is written as

$$H = \frac{1}{2m_e^*} \sum_{i=1}^{2} (p_i - \frac{e}{c} A)^2 + V(r_1, r_2) + H_{\text{spin}},$$

(4.18)

with

$$V(r_1, r_2) = \sum_{i=1}^{2} \frac{1}{2} m_e^* \omega_0^2 r_i^2 + \frac{e^2}{\epsilon |r_1 - r_2|},$$

(4.19)

and

$$H_{\text{spin}} = g^* \mu_B (S_1 + S_2) B.$$  

(4.20)

Here $H_{\text{spin}}$, $\mu_B$, $m_e$, $g^*$ and $\epsilon$ denote the Zeeman energy, the Bohr magneton, the effective mass, the effective Lande factor and the relative dielectric constant of a
We consider the QD as a 2D one because the confinement in the z-direction is assumed to be much stronger than the one in the xy-plane. Let us consider the vector potential with a gauge of the form \(A = \frac{1}{2}B \times r = \frac{1}{2}B(-y, x, 0)\) and \(\omega_0\) constant. The Hamiltonian now becomes
\[
H = \sum_{i=1}^{2} \left( \frac{p_i^2}{2m_e} + \frac{1}{2}m_e^*\omega^2 r_i^2 \right) + \frac{e^2}{\epsilon|r_1 - r_2|} + \frac{1}{2}\omega_z L_z + g^*\mu B(S_1 + S_2)B, \tag{4.21}
\]
with \(\omega = \sqrt{\omega_0^2 + \omega_c^2}\) and \(\omega_c = \frac{eB}{\mu m_e^*}\) is the cyclotron frequency and \(L_z\) is the total orbital angular momentum along z-direction. On introducing the relative and center of mass coordinates as \(r = r_1 - r_2\), \(R = \frac{r_1 + r_2}{2}\), Eq.(4.21) can be reduced in the following form
\[
H = H_{cm} + H_r + H_{spin}, \tag{4.22}
\]
with
\[
H_{cm} = \frac{P_{cm}^2}{2M^*} + \frac{1}{2}M^*\omega^2 R^2 + \frac{1}{2}\omega_c L_{zcm}, \tag{4.23}
\]
and
\[
H_r = \frac{p_r^2}{2\mu^*} + \frac{1}{2}\mu^*\omega^2 r^2 + \frac{e^2}{\epsilon r} + \frac{1}{2}\omega_z L_z, \tag{4.24}
\]
with \(M^* = 2m_e^*\), \(\mu^* = \frac{m_e^*}{2}\). \(P_{cm}\) and \(p_r\) are the momenta corresponding to the center of mass and relative coordinates respectively.

The center of mass Hamiltonian \((H_{cm})\) in 2D has well known eigenvalues i.e.
\[
E_{cm}(n_1, |m_1|) = (2n_1 + |m_1| + 1)\hbar\omega + \frac{1}{2}\omega_c m_1 \hbar, \tag{4.25}
\]
The SE for Hamiltonian \((H_r)\) in 2D can be written as
\[
\left[ \frac{d^2}{dr^2} - \frac{m_2^2 - \frac{1}{r^2}}{r^2} + (\zeta - a_2 r^2 + \frac{b_2}{r}) \right] R(r) = 0, \tag{4.26}
\]
where \(\zeta = \frac{2\mu^*}{\hbar^2}(E_r - 1/2\omega_c m_\hbar)\), \(a_2 = \frac{\mu^* \omega^2}{\hbar^2}\) and \(b_2 = -\frac{2\mu^* \omega^2}{\hbar^2}\).

For \(N = 2\) and \(b_1 = 0\), Eq.(4.4) has same form as Eq.(4.26). Based on the previous calculations, the energy eigenvalues for the above equation can be written as
\[
\zeta = 2\sqrt{a_2}(n_2 + |m_2| + 1). \tag{4.27}
\]
Then the final expression for \(E_r\) becomes
\[
E_r = (n_2 + |m_2| + 1)\hbar\omega + \frac{1}{2}\omega_c m_2 \hbar. \tag{4.28}
\]
The additional Zeeman energy corresponding to spin of two electrons can be written directly [17] as

$$E_s = g^* \mu_B S_z = \frac{\hbar \omega_0}{4} (1 - (-1)^{m_2}) g^* m_e^* \omega_c. \quad (4.29)$$

Finally, the expression for energy eigenvalues of two electrons QD is written as i.e.

$$E = E_{cm} + E_r + E_s$$

$$E_{n_1,m_1,n_2,m_2} = (2n_1 + |m_1| + |m_2| + 2) \hbar \omega_0 \sqrt{1 + \frac{1}{4} (\frac{\omega_c}{\omega_0})^2} + \frac{1}{2} \frac{\omega_c}{\omega_0} m_1 \hbar \omega_0 + \frac{1}{4} (1 - (-1)^{m_2}) g^* m_e^* \omega_c. \quad (4.30)$$

For \(n_2 = 0\), the ground state energy becomes

$$E_{n_1,m_1,0,m_2} = (2n_1 + |m_1| + |m_2| + 2) \hbar \omega_0 \sqrt{1 + \frac{1}{4} (\frac{\omega_c}{\omega_0})^2} + \frac{1}{2} \frac{\omega_c}{\omega_0} m_1 \hbar \omega_0 + \frac{1}{4} (1 - (-1)^{m_2}) g^* m_e^* \omega_c. \quad (4.31)$$

and the constraint is \(b_2 = 0\). This implies in the framework of present approach the effect of Coulomb term on this state reduces to zero and only harmonic confinement for the electrons in QD is effective.

For \(n_2 = 1\), the 1st excited state energy is

$$E_{n_1,m_1,1,m_2} = (2n_1 + |m_1| + |m_2| + 3) \hbar \omega_0 \sqrt{1 + \frac{1}{4} (\frac{\omega_c}{\omega_0})^2} + \frac{1}{2} \frac{\omega_c}{\omega_0} m_1 \hbar \omega_0 + \frac{1}{4} (1 - (-1)^{m_2}) g^* m_e^* \omega_c. \quad (4.32)$$

and the constraint is \(b_2^2 - 2 \sqrt{a_2}(2|m_2| + 1) = 0\).

Using the constraint condition the above equation can be written in a more simplified form as

$$E_{n_1,m_1,1,m_2} = (2n_1 + |m_1| + |m_2| + 2) \hbar \omega_0 \sqrt{1 + \frac{1}{4} (\gamma)^2} + \frac{1}{2} \gamma m_1 \hbar \omega_0 + \frac{1}{4} (1 - (-1)^{m_2}) g^* m_e^* \omega_c \gamma. \quad (4.33)$$

where \(\lambda = \frac{e^2 \alpha}{\hbar \omega_0}\) is the dimensionless Coulomb correlation parameter, \(\alpha = \sqrt{\frac{m_e^* \omega_c}{\omega_0}}\) is the harmonic oscillator inverse length and \(\gamma = \frac{\omega_c}{\omega_0}\) is the dimensionless magnetic field parameter.

In a similar way one can obtain the energy spectra for higher states along with
constraints relation of a 2D QD.

In table 4.2, the ground state energy spectra of 2D QD of GaAs semiconductor is calculated using Eq.(4.31) and results are compared with other similar studies.

TABLE 4.2: Ground state energy spectra of two electrons QD of GaAs:
\[ m_e^* = 0.067m_e, \; g^* = -0.44 \; \text{and} \; \epsilon = 12.5 \; \text{in} \; 2D \; \text{in} \; \hbar\omega_0 \; \text{units} \]

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( \gamma )</th>
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<td>2</td>
<td>23.79899</td>
<td>–</td>
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</table>

From the above table, we find that our analytical results are in good correspondence with the results of other studies.

Fig.(4.1) shows the variations of ground state energy spectrum with magnetic field for angular momentum \( m_2 \leq 0 \) when the both the electrons are free i.e. there is no Coulomb interaction between them.

### 4.3 Asymptotic Solutions to the Quantum Dot Confine Potential

The asymptotic study to the N-dimensional radial SE for QD interaction potential Eq.(4.1) using the AIM via an ansatz to the wavefunction is obtained. This methodology employed in a very simple manner such as the problem is studied into three parts i.e. the energy spectra is obtained for ground state, for large ‘\( r \)’ and for
small ‘r’. This type of study is also carried out for the quark-antiquark interaction potential in chapter 3. As a special case of this study, the energy spectra of InGaAs QD is obtained in 2D from the above generalized study.

In N-dimensional Hilbert space the SE for two particles interacting via a spherically symmetric potential (4.1) is written as

$$\left[ \frac{d^2}{dr^2} + \frac{N - 1}{r} \frac{d}{dr} - \frac{l(l + N - 2)}{r^2} + 2\mu(E - ar^2 + b) \right] \Psi(r) = 0.$$  \(4.34\)

Now inserting \(\Psi(r) = r^{-\frac{N-1}{2}} R(r)\) in equation (4.34) we obtain

$$\left[ \frac{d^2}{dr^2} - \frac{\eta^2 - \frac{1}{4}}{r^2} + 2\mu(E - ar^2 + \frac{b}{r}) \right] R(r) = 0,$$  \(4.35\)

with \(\eta = l + \frac{N-2}{2}\). For developing an approximate solution of Eq.(4.35), we start with the following choice of the wavefunction

$$R(r) = r^{\eta + \frac{1}{2}} \exp(-ar^2) f(r),$$  \(4.36\)
with $\alpha$ as positive parameter whose value is to be determined in terms of potential parameters $a$ and $b$. The function $f(r)$ in above equation assigns the number of nodes to the wavefunction. Thus, on substituting Eq.(4.36) into Eq.(4.35), we get
\[
rf''(r) + \left[-4\alpha r^2 + 2(\eta + \frac{1}{2})\right]f'(r) + \left[(\eta + \frac{1}{2})(-4\alpha r) + 4\alpha^2 r^3 - 2\alpha r + 2\mu(Er - ar^3 + b)\right]f(r) = 0.
\] (4.37)

For the ground state, however, $f(r)$ is constant ($f(r)=1$), then Eq.(4.37) becomes
\[
(\eta + \frac{1}{2})(-4\alpha r) + 4\alpha^2 r^3 - 2\alpha r + 2\mu(E_0r - ar^3 + b) = 0.
\] (4.38)

For obtaining the ground state energy and wavefunction parameter $\alpha$, equate the coefficients of same powers of $r$ equal to zero. This provides a set of equations which give
\[
E_0 = \frac{4\alpha(\eta + 1)}{2\mu},
\] (4.39)

\[
a = \frac{2\alpha^2}{\mu}.
\] (4.40)

The ansatz parameter $\alpha$ may be calculated from Eq.(4.40) as
\[
\alpha = \sqrt{\frac{a\mu}{2}}, \quad a > 0.
\] (4.41)

Thus the ground state energy eigenvalue Eq.(4.39) turns out to be
\[
E_0 = \sqrt{\frac{2a}{\mu}(\eta + 1)}.
\] (4.42)

Essentially to obtain the complete spectra for a present system, in what follows, we establish the asymptotic solutions i.e. $r \to \infty$ and $r \to 0$ to Eq.(4.37). For this, recast Eq.(4.37) in the following form, after using Eq.(4.38)
\[
f''(r) + \left(\frac{2(\eta + \frac{1}{2})}{r} - 4\alpha r\right)f'(r) + 2\mu(\delta E_n + \frac{b}{r})f(r) = 0,
\] (4.43)

where $\delta E_n = E_n - E_0$.

Firstly, to find the solution of Eq.(4.43) for large $r$, set $r \to \infty$, which lead Eq.(4.43) to
\[
f''(r) - (4\alpha r)f'(r) + 2\mu(\delta E_n)f(r) = 0,
\] (4.44)
On comparison of Eq. (4.44) with Eq. (1.13), and using recursion Eq. (1.16), we get

\[
\begin{align*}
\lambda_0 &= 4\alpha r, \quad s_0 = -2\mu \delta E_n, \\
\lambda_1 &= 4\alpha - 2\mu \delta E_n + (4\alpha r)^2, \quad s_1 = -2\mu \delta E_n(4\alpha r), \\
\lambda_2 &= (4\alpha r)((8\alpha - 2\mu \delta E_n) + 4\alpha - 2\mu \delta E_n + (4\alpha r)^2), \\
\lambda_3 &= (4\alpha r)\left((8\alpha - 2\mu \delta E_n) + 4\alpha - 2\mu \delta E_n + (4\alpha r)^2\right) + (4\alpha r)^2, \\
\lambda_4 &= (4\alpha r)\left((8\alpha - 2\mu \delta E_n) + 4\alpha - 2\mu \delta E_n + (4\alpha r)^2\right) + (4\alpha r)^2, \\
&\vdots
\end{align*}
\]

and so on. On substituting above values in Eq. (1.17), we obtain

\[
\begin{align*}
\lambda_1 s_0 - \lambda_0 s_1 &= 0, \Rightarrow \delta E_1 = \frac{2\alpha}{\mu}, \\
\lambda_2 s_1 - \lambda_1 s_2 &= 0, \Rightarrow \delta E_2 = \frac{4\alpha}{\mu}, \\
\lambda_3 s_2 - \lambda_2 s_3 &= 0, \Rightarrow \delta E_3 = \frac{6\alpha}{\mu}, \\
\lambda_4 s_3 - \lambda_3 s_4 &= 0, \Rightarrow \delta E_4 = \frac{8\alpha}{\mu},
\end{align*}
\]

and so on. Finally, this trend leads us to

\[
\delta E_n = \frac{2n\alpha}{\mu}, \quad n = 1, 2, 3, \ldots
\]

Now the eigenfunction \(f(r)\) can be determined using Eqs. (4.44) and (1.18). On comparison of these two equations, we have \(D = 0, u = 2\alpha, v = 0\) and \(t = -1\). Therefore, we obtain \(\rho = \frac{1}{2} + \frac{2\alpha}{v}\) and \(\sigma = \frac{1}{2}\). Then wavefunction \(f(r)\) becomes

\[
f(r) = (-2)^n C_2 \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2}\right)} \, \, _1F_1(-n, \frac{1}{2}; 2\alpha r^2).
\]

Secondly, for \(r \to 0\), Eq. (4.43) for small \(r\) can be written as

\[
f''(r) + \left(\frac{2(\eta + \frac{1}{2})}{r}\right)f'(r) + 2\mu(\delta E_n + \frac{b}{r})f(r) = 0.
\]

Again compare above Eq. (4.51) with Eq. (1.13) and applying the methodology of the previous case, we obtain

\[
\begin{align*}
\lambda_0 &= -\left(\frac{2(\eta + \frac{1}{2})}{r}\right), \quad s_0 = -2\mu(\delta E_n + \frac{b}{r}), \\
\lambda_1 &= \frac{2(\eta + \frac{1}{2})}{r^2} - 2\mu(\delta E_n + \frac{b}{r}) + (-\left(\frac{2(\eta + \frac{1}{2})}{r}\right))^2, \quad s_1 = 2\mu(\delta E_n + \frac{b}{r})\left(\frac{2(\eta + \frac{1}{2})}{r}\right), \\
\lambda_2 &= -\left(\frac{2(\eta + \frac{1}{2})}{r}\right)\left[4\mu(\delta E_n + \frac{b}{r}) - \left(\frac{2(\eta + \frac{1}{2})}{r}\right)^2\right], \\
s_2 &= -2\mu(\delta E_n + \frac{b}{r})\left[-2\mu(\delta E_n + \frac{b}{r}) - \left(\frac{2(\eta + \frac{1}{2})}{r}\right)^2\right].
\end{align*}
\]
On substituting above values in Eq.(1.17), we obtain

\[ \lambda_1 s_0 - \lambda_0 s_1 = 0, \Rightarrow \delta E_1 = 0, \]
\[ \lambda_2 s_1 - \lambda_1 s_2 = 0, \Rightarrow \delta E_2 = 0, \]  
(4.53)

and so on. Therefore, from above trend, we write

\[ \delta E_n = 0. \]  
(4.54)

Further, the complete energy spectrum for the present system is obtained by adding Eqs.(4.49) and (4.54) as

\[ \delta E_n = E_n - E_0 = \frac{2n\alpha}{\mu}. \]  
(4.55)

Finally, above equation is written in a more compact form by using Eqs.(4.41) and (4.42), and replacing \( \eta = l + \frac{N-2}{2} \) as

\[ E_{nlN} = \sqrt{\frac{a}{2\mu}}(2n + 2l + N). \]  
(4.56)

It is to be noted for dimensionality \( N = 3 \), the above analytical result reduces to the existing literature results [35] when \( n \to 0 \) and \( \mu = 1 \).

### 4.3.1 Application to the InGaAs QD

If we replace dimensionality \( N = 2 \), \( a = a_2 = \frac{\mu^2 \omega^2}{\hbar^2} \), \( b = b_2 = -\frac{2\mu^* e^2}{\hbar^2} \) and \( E = \zeta = \frac{2\mu^*}{\hbar^2}(E_r - \frac{1}{2} \omega_m \hbar) \), then Eq.(4.35) is similar to the Eq.(4.26). Thus the total energy eigenvalues for the 2-electron InGaAs QD in magnetic field can be obtained from Eq.(4.31).
TABLE 4.3: Energy spectra of two electrons InGaAs QD:

\[ m_e^* = 0.064m_e, \quad g^* = 0.44, \quad \epsilon = 12.5, \quad \hbar \omega_0 = 2.9meV \text{ and } \frac{\omega}{\omega_0} = 2 \text{ in 2D.} \]

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In the present chapter, we derived the ground state energy spectra of 2-electron GaAs and InGaAs QDs in a magnetic field using two different techniques. The variation of ground state energy with the magnetic field for different angular momentum values are studied. The presented theoretical results are in excellent matching with the other experimental and theoretical studies.
Bibliography


