Chapter 3

The CPP as the Quark-antiquark Interaction Potential

3.1 Introduction

In recent years, the spectroscopy of heavy flavor hadrons has attracted much attention due to availability of number of experimental facilities such as the BES at the Beijing Electron Positron Collider (BEPC), E835 at Fermilab, and CLEO at the Cornell Electron Storage Ring (CESR) etc. They have been able to produce huge data samples in the heavy flavor sector, whereas B-meson factories, BaBar at PEP-II and Belle at KEKB, are functioning on the observation of new and possibly exotic hadronic states. These experiments are capable of finding new hadrons, new production mechanisms, new decays and transitions, and in general will be giving high-precision data samples with better statistics and a higher confidence level. Recently, the LHC (Large Hadron Collider) experiments at the CERN, Panda at the GSI etc, will be collecting large data sets which will provide greater scopes and problems especially in the field of heavy flavor physics.

Theoretically, the study of heavy quarkonia (QQ) gives the information related to QCD (quantum chromodynamics) at the hadronic scale and checks the authentication of perturbative QCD, potential models and lattice QCD computations [1]. The study of the properties of mesons made up of heavy quark and antiquark (cc, bc, bb) provides a deep insight into heavy quark dynamics which leads to the better understanding of the constituent quark masses. The theoretical study of the heavy quarkonia (cc, bb) and bc mesons have rich spectroscopy with numerous narrow states of charmonium lying under the threshold of open charm production [2] and of botomo-
nium lying under the threshold of B-B production. Many of these states have not
yet been confirmed or observed experimentally [3].
The nonperturbative effects linked with a complex structure of QCD vacuum nec-
essarily play a major role at the hadronic scale. But our limited knowledge about
the nonperturbative QCD provides a theoretical uncertainty in the $Q\bar{Q}$ potential at
large and intermediate distances [4]. Therefore, some better theoretical structure is
desirable which can give important information about the quark-antiquark in-
teractions and the behavior of QCD at the hadronic scale.
The most commonly employed potential for studying the hadron properties is the
Coulomb plus linear power potential, $V(r) = -\frac{4}{3} \frac{\alpha_s}{r} + \sigma r$, with the string tension $\sigma$.
This type of confining potential is known as Cornell potential and is supported by
the lattice QCD calculation [5]. In this potential, the Coulomb term is to be liable
for the interaction at small distances and linear term leads to the confinement. Since,
for the higher excited mesonic states it is contended that the string tension $\sigma$ must
depend on the $Q\bar{Q}$ separation [6]. This corresponds to widening of the conﬁnement
potential at larger $r(r \geq 1 fm)$. Furthermore, the study based on Regge trajectories
for meson states proposes the conﬁnement part of the potential to have the power $\frac{2}{3}$
in lieu of 1 [7]. This proposal inspired the many researchers to choose various power
form for the confining part of the interquark potential and discuss the properties of
heavy flavor $Q\bar{Q}$ systems by changing the power around 1.0. The solution to the
non-relativistic radial SE with spherical symmetric potentials plays an important
role in atomic and hadronic spectroscopy. For this, in the next section we discuss
the non-relativistic behavior of the heavy hadron.

3.1.1 Non-relativistic treatment for heavy flavor mesons
In general, the properties of heavy flavor mesons have been investigated are based on
potential models in the framework of relativistic as well as non-relativistic quantum
mechanics. Since in the center of mass frame, the momenta of the heavy quark and
antiquark are small in comparison with their rest mass, i.e. $m_{Q\bar{Q}} \gg \Lambda_{QCD} \sim |\vec{p}|$,
which forms the basis of the non-relativistic behavior. The non-relativistic quark
model [8] is commonly employed for knowing the behavior of heavy hadron. The non-
relativistic approximation is good for obtaining the mass spectra of heavy mesons
consisting of heavy quark and antiquark ($\Upsilon(b\bar{b}), \Psi(c\bar{c})$). This approximation pro-
vides a good description of static properties of heavy mesons such as mass spectra,
radius etc. while for dynamical properties such as decay, the relativistic corrections
are considered. Furthermore, the observations of the properties of quark-antiquark
system using non-relativistic methods are at par with the relativistic model predic-
tions [9, 10]. Generally, for dealing with a particular physical system, a potential
model is used. The potential model can provide a good amount of information about the problem. The spherically symmetrical potential model also presents a good description of heavy quarkonium mass spectra such as charmonium and bottomonium. The interaction potential $V(r)$, for the study of heavy quarkonia bound state systems such as $\Upsilon(b\bar{b})$, $\Psi(c\bar{c})$ non-relativistically, consists of a central term $V_c(r)$ and a spin dependent term $V_{SD}(r)$. The spin dependent part of the potential is mainly employed for computing the mass difference between different degenerate mesonic states. The central part $V_c(r)$ can be written as a combination of a vector (Coulomb) plus a scalar (confining) part as

$$V_c(r) = -\frac{4\alpha_s}{3} r + Ar^v. \quad (3.1)$$

This potential acts as static quark-antiquark interaction potential. This potential is the special case of the general potential, $-Cr^{-a} + Dr^b + V_0$ [11, 12, 13] with $V_0 = 0, a = 1, b = v$. There exist various types of potentials of this general form, such as Cornell potential [11] corresponds to $a = b = 1$, Martin potential conforms to $a = 0, b = 0.1$ [14], Lichtenberg potential [15] corresponds to $a = b = 0.75$, and the logarithmic potential [16] of Quigg and Rosner when $a = b \to 0$. Most of these potentials have already been used for the description of hadron properties [17]. Potentials with values of $a$ and $b$ in the ranges $0 \leq a \leq 1.2$ and $0 \leq b \leq 1.1$ are also being investigated [18]. Recently some authors have also been studied the quark-antiquark interaction using Coulomb plus power potential, where power index values varies in the range $0.5 < v < 1.5$ [19]. Inspite of significant development to various potential models for studying the various characteristic of heavy quarkonia, two important questions are still important in potential model framework i.e. what is the adequate form of the $Q\bar{Q}$ potential ? How can one obtain a proper quantitative non-relativistic treatment of the $Q\bar{Q}$ bound systems ?

### 3.2 Solution to the SE for a $Q\bar{Q}$ Potential

The various potential models mentioned above give more or less accurate results to the spectroscopy of heavy flavor mesons. With this motivation, here in the present chapter, we investigate one more potential model for the quark-antiquark system and try to explain the static property such as mass spectra of heavy flavor mesons (charmonium and bottomonium). We consider the CPP which consists of harmonic, linear and Coulomb potential terms and acts as the interaction potential for the quark-antiquark systems. This type of potential has recently been studied using different techniques [20, 21]. The quark-antiquark interaction potential, also
known as quarkonium potential, can be written as

\[ U(r) = ar^2 + br - \frac{c}{r}, \quad a > 0. \]  \hspace{1cm} (3.2)

Although the problem of obtaining the interquark potential is still unsolved but its solution is necessary for finding mass spectra for coupled states and for describing electromagnetic characteristics of mesons.

In view of importance of generalized results, we extend the dimensionality limit i.e. we solve the SE in N-dimensional space for the system (3.2) by using the series expansion method [22] and also obtain the asymptotic solutions using the AIM methodology [23]. These powerful and easy methods available in literature can handle a variety of physical problems.

In N-dimensional Hilbert space the SE for two particles interacting via a spherically symmetric potential (3.2) is written as

\[ \left[ \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{l(l+N-2)}{r^2} + 2\mu(E - ar^2 - br + \frac{c}{r}) \right] R(r) = 0, \]  \hspace{1cm} (3.3)

where \( l \) is the angular momentum quantum number, \( N \) denotes dimensionality and \( \mu = \frac{m_1 m_2}{m_1 + m_2} \) is the reduced mass of the particles of masses \( m_1 \) and \( m_2 \). The reduced mass, however, for charmonium system \( (m_c = m_1 = m_2) \) is \( \mu = \frac{m_c}{2} \) and for bottomonium system \( (m_b = m_1 = m_2) \) is \( \mu = \frac{m_b}{2} \). \( E \) are the energy eigenvalues corresponding to the radial eigenfunctions \( R(r) \) in an arbitrary dimensional space.

Now we find an approximate solution of Eq.(3.3) by making the following choice of the wave function

\[ R(r) = exp(-\alpha r^2 - \beta r) F(r), \]  \hspace{1cm} (3.4)

where \( \alpha \) and \( \beta \) are positive parameters whose values are to be determined in terms of potential parameters \( a, b \) and \( c \). On substituting the Eq.(3.4) into Eq.(3.3), we obtain

\[ \left[ \frac{d^2}{dr^2} + \left( \frac{N-1}{r} - 4\alpha r - 2\beta \right) \frac{d}{dr} + (4\alpha^2 - 2\mu a)r^2 + (4\alpha \beta - 2\mu b)r + (2\mu c - \beta(N-1)) \frac{1}{r} - l(l + N - 2) \right] F(r) = 0. \]  \hspace{1cm} (3.5)

Now let us consider a series solution of above equation as

\[ F(r) = \sum_{l=0}^{\infty} a_l r^{\frac{3n}{2} + l}, \]  \hspace{1cm} (3.6)

where \( a_l \) are expansion coefficients to be determine later. The reason to select such type of power series solutions is to remove the degeneracy in the lower states of the
present system.

On putting Eq. (3.6) into Eq. (3.5) and collecting like powers of ‘r’, we get

\[ \sum_{l=0}^{\infty} a_l \left[ ((3n + 2l)(3n + 2l + 2N - 4) - 4l(l + N - 2)) \right. \]
\[ r^{3n+l-2} + 4(2\mu c - (N - 1)\beta - \beta(3n + 2l)) \]
\[ r^{3n+l-1} + 4(\beta^2 - 2\alpha N + 2\mu E - 2\alpha(3n + 2l)r^{3n+l} \]
\[ + 8(2\alpha \beta - \mu b)r^{3n+l+1} + 8(2\alpha^2 - \mu a)r^{3n+l+2} \right] = 0. \tag{3.7} \]

This equation, after equating the each coefficient of ‘r’ to zero, gives the following relations

\[ E = \frac{2\alpha(N + 3n + 2l) - \beta^2}{2\mu}, \tag{3.8} \]
\[ a = \frac{2\alpha^2}{\mu}, \tag{3.9} \]
\[ b = \frac{2\alpha \beta}{\mu}, \tag{3.10} \]
\[ c = \frac{\beta(3n + 2l + N - 1)}{2\mu}, \tag{3.11} \]
\[ (3n + 2l)(3n + 2l + 2N - 4) - 4l(l + N - 2) = 0. \tag{3.12} \]

The ansatz parameters \( \alpha \) and \( \beta \) may be obtained from Eqs. (3.9) and (3.10) respectively as

\[ \alpha = \sqrt{\frac{a\mu}{2}}, \tag{3.13} \]
\[ \beta = b\sqrt{\mu \frac{2}{2a}}, \quad a > 0. \tag{3.14} \]

Thus the energy eigenvalue Eq. (3.8) becomes

\[ E = \sqrt{\frac{a}{2\mu}}(N + 3n + 2l) - \frac{b^2}{4a}. \tag{3.15} \]

Eqs. (3.11) and (3.12) act as constraints on the given system with \( l \geq 0 \) and \( n \geq 0 \).

It is to be noted that the constraint relation (3.12) will not appear if one replaces \( l \) with \( l + \frac{3n}{2} \) in the third term of Eq. (3.3).
3.2.1 Recurrence relation and convergence of series

For obtaining the recurrence relation, which can connect various expansion coefficients \(a_n\), we make identical powers of \(r\) in Eq. (3.7), i.e. equating the coefficients of \(r^{3n+l-1}\) equal to zero. So the relation becomes

\[
\begin{align*}
\left[(3n+2l+2)(3n+2l+2N-2) - 4(l+1)(l+N-1)\right]a_{l+1} + 4\left[2\mu c - \beta(3n+2l+N-1)\right]a_l + 4\left[\beta^2 + 2\mu E - 2\alpha(3n+3l+2l-2)\right]a_{l-1} + 8\left[2\alpha \beta - \mu b\right]a_{l-2} + 8\left[2\alpha^2 - \mu a\right]a_{l-3} &= 0. \\
\end{align*}
\]

This recursion relation can further be written in a more plausible way by using Eqs. (3.8)-(3.11) as

\[
a_{l+1} = \frac{16\alpha}{(3n+2l+2)(3n+2l+2N-2) - 4(l+1)(l+N-1)} a_{l-1},
\]

where \(l = 1, 2, ..., \) with \(a_0 \neq 0\) and \(a_1 \neq 0\). It is worth to mention that the above relation provides various coefficients of power series in terms of \(a_0\) and \(a_1\).

Eq. (3.17) gives one power series solution in terms of even power of \(\cdot r\) and another in terms of odd power of \(\cdot r\).

Now we check the behavior of power series for large value of \(\cdot r\) i.e. convergence of series. For convergence, take the ratio of two successive terms i.e. \(\frac{a_{l+1}}{a_{l-1}}\) which for large \(\cdot r\) becomes

\[
\frac{a_{l+1}}{a_{l-1}} \approx \frac{4\alpha}{l^2}.
\]

It is apparent from the above relation that the power series converges to 0 when \(l \to \infty\).

The coefficients \(a_0\) and \(a_1\) are appropriate normalization constants for the even and odd power series respectively and can be determined using the normalization condition

\[
\int_0^\infty |R(r)|^2 r^{N-1} dr = 1.
\]

After solving Eq. (3.19), we get the normalized constants

\[
a_0 = \sqrt{\frac{(4\alpha)^{3n+N-\frac{1}{2}}}{(3n+N-2)\sqrt{\pi}2^\alpha}exp\left(-\frac{\beta}{2\alpha}\right)},
\]

\[
a_1 = \sqrt{\frac{(4\alpha)^{3n+N+\frac{1}{2}}}{(3n+N)\sqrt{\pi}2^\alpha}exp\left(-\frac{\beta}{2\alpha}\right)}.
\]
Finally the normalized wave function for quark-antiquark potential is written as

$$R(r) = \left[ \sum_{l=0}^{even} a_l r^{\frac{3n}{2}+l} + \sum_{l=1}^{odd} a_l r^{\frac{3n}{2}+l} \right] \exp\left(-\sqrt{\frac{a\mu}{2}} r^2 - b\sqrt{\frac{\mu}{2a}} r \right),$$  \hfill (3.22)

where the different expansion coefficients of the series $a_n$ are determined from the recursion relation (3.17) in terms of either $a_0$ or $a_1$ which are given by Eqs.(3.20) and (3.21).

### 3.2.2 Mass spectra of heavy $Q\bar{Q}$ systems

Here we determine the mass spectra of the heavy quarkonium systems consisting quark and antiquark of same flavor. For calculating the mass spectra, we use the following relation \cite{19, 20}

$$M = 2m + E_{nl}. \hfill (3.23)$$

Using Eq.(3.15) into Eq.(3.23), the analytic expression for mass spectra of charmonium becomes

$$M_c = 2m_c + \sqrt{\frac{a}{m_c}} (3 + 3n + 2l) - \frac{b^2}{4a}. \hfill (3.24)$$

Similarly the mass spectra for bottomonium is determined from Eq.(3.24) by replacing $m_c$ with $m_b$.

In order to check the accuracy of the obtained analytic results, we have solved the given problem numerically using the Finite Difference Method in MATLAB within the ambit of constraining relation (3.11). The results are given in the following two tables.
Table 3.1: Mass spectra of charmonium \((m_c = 1.48 GeV, a = 0.042 GeV^3, b = 0.255 GeV^2)\) in GeV

<table>
<thead>
<tr>
<th>States</th>
<th>c</th>
<th>From Eq.(3.24)</th>
<th>Numerical Results</th>
<th>Experimental [24]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1S</td>
<td>1.023</td>
<td>3.078</td>
<td>3.078</td>
<td>3.068</td>
</tr>
<tr>
<td>1P</td>
<td>2.046</td>
<td>3.415</td>
<td>3.415</td>
<td>3.525</td>
</tr>
<tr>
<td>2S</td>
<td>2.557</td>
<td>3.581</td>
<td>3.455</td>
<td>3.663</td>
</tr>
<tr>
<td>2P</td>
<td>3.580</td>
<td>3.917</td>
<td>3.768</td>
<td>–</td>
</tr>
<tr>
<td>3S</td>
<td>4.092</td>
<td>4.085</td>
<td>4.250</td>
<td>4.159</td>
</tr>
<tr>
<td>4S</td>
<td>5.626</td>
<td>4.589</td>
<td>4.661</td>
<td>4.421</td>
</tr>
</tbody>
</table>

Table 3.2: Mass spectra of bottomonium \((m_b = 4.68 GeV, a = 0.143 GeV^3, b = 0.465 GeV^2)\) in GeV

<table>
<thead>
<tr>
<th>States</th>
<th>c</th>
<th>From Eq.(3.24)</th>
<th>Numerical Results</th>
<th>Experimental [24]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1S</td>
<td>0.57</td>
<td>9.510</td>
<td>9.506</td>
<td>9.460</td>
</tr>
<tr>
<td>1P</td>
<td>1.14</td>
<td>9.862</td>
<td>9.841</td>
<td>9.900</td>
</tr>
<tr>
<td>2S</td>
<td>1.42</td>
<td>10.038</td>
<td>10.365</td>
<td>10.023</td>
</tr>
<tr>
<td>1D</td>
<td>1.71</td>
<td>10.214</td>
<td>10.196</td>
<td>10.161</td>
</tr>
<tr>
<td>2P</td>
<td>1.99</td>
<td>10.390</td>
<td>10.205</td>
<td>10.260</td>
</tr>
<tr>
<td>3S</td>
<td>2.28</td>
<td>10.566</td>
<td>10.716</td>
<td>10.355</td>
</tr>
<tr>
<td>4S</td>
<td>3.13</td>
<td>11.094</td>
<td>11.161</td>
<td>10.580</td>
</tr>
</tbody>
</table>

The obtained mass spectra of the charmonium and bottomonium systems are approximately same with the experimental results as well as with the results of other works. The power series method, for the system considered in the present chapter, has some shortcoming which can be observed from Eqs.(3.11) and (3.17). Here the dependence of potential parameter \(c\), Eq.(3.11), on quantum numbers \(n, l\) and dimensionality \(N\) will give different values of \(c\) for different eigenstates. But nevertheless, the power series method is a powerful technique and capable of producing good results for the systems like (3.2).
3.3 Asymptotic solutions to the SE for the $Q\bar{Q}$ potential

In this section, an asymptotic study is carried out to the N-dimensional radial SE for the $Q\bar{Q}$ interaction potential (3.2) exploiting the AIM via an ansatz to the wavefunction. This methodology has an edge over the PSM in dealing $Q\bar{Q}$ system. The AIM is applied to obtain the energy eigenvalues of heavy $Q\bar{Q}$ system for ground state, for large ‘$r$’ and for small ‘$r$’. From this analysis, the mass spectra of heavy quarkonia is obtained in three dimensions. This study has resolved some issues related to the $Q\bar{Q}$ system such as degeneracy in the lower states has not occurred and there is no constraint condition on $n$ i.e. we can vary $n$ freely.

On inserting $Ψ(r) = r^{-\frac{N-1}{2}} R(r)$ in Eq.(3.3), we obtain

$$\left[ \frac{d^2}{dr^2} - \frac{\eta^2 - \frac{1}{4}}{r^2} + 2\mu(E - ar^2 - br + \frac{c}{r}) \right] R(r) = 0,$$

(3.25)

with $\eta = l + \frac{N-2}{2}$. For developing an approximate solution of Eq.(3.25), we start with the following choice of the wavefunction

$$R(r) = r^{\eta+\frac{1}{2}}\exp(-ar^2 - \beta r)g(r),$$

(3.26)

with $\alpha$ and $\beta$ as positive parameters whose values are to be determined in terms of potential parameters $a, b$ and $c$. The function $g(r)$ in above equation assigns the number of nodes to the wavefunction. Thus, on substituting Eq.(3.26) into Eq.(3.25), we get

$$rg''(r) + \left[ 2r(-2\alpha r - \beta) + 2(\eta + \frac{1}{2}) \right] g'(r) + \left[ 2(\eta + \frac{1}{2})(-2\alpha r - \beta) + r(-2\alpha r - \beta)^2 - 2\alpha r + 2\mu(Er - ar^3 - br^2 + c) \right] g(r) = 0.$$

(3.27)

For the ground state, however, $g(r)$ is constant ($g(r)=1$), then Eq.(3.27) emerges as

$$2(\eta + \frac{1}{2})(-2\alpha r - \beta) + r(-2\alpha r - \beta)^2 - 2\alpha r + 2\mu(E_0r - ar^3 - br^2 + c) = 0.$$

(3.28)

For obtaining the ground state energy and wavefunction parameters $\alpha$ and $\beta$, equate the coefficients of same powers of $r$ equal to zero. This provides following set of equations

$$E_0 = \frac{4\alpha(\eta + 1) - \beta^2}{2\mu},$$

(3.29)

$$a = \frac{2\alpha^2}{\mu},$$

(3.30)
\[ b = \frac{2\alpha \beta}{\mu}, \]  
\[ c = \frac{\beta (\eta + \frac{1}{2})}{\mu}. \]  

The ansatz parameters \( \alpha \) and \( \beta \) may be calculated from Eqs. (3.30) and (3.31) respectively as

\[ \alpha = \sqrt{\frac{a \mu}{2}}, \]  
\[ \beta = b \sqrt{\frac{\mu}{2a}}, \]  

for \( a > 0 \). Thus the ground state energy eigenvalue Eq. (3.29) for quarkonium system turns out to be

\[ E_0 = \sqrt{\frac{2a \mu}{\mu}} (\eta + 1) - \frac{b^2}{4a}. \]  

Note that Eq. (3.32) act as a constraint on the given system. There is no constraint condition on \( n \) (cf. Eq. (3.12)) as in the case of the PSM. Thus one can take any value of \( n \). This is one more advantage of the AIM.

Essentially to find the complete spectra for quarkonium system, in what follows, we establish the asymptotic solutions i.e. \( r \to \infty \) and \( r \to 0 \) to Eq. (3.27). To achieve this goal, recast Eq. (3.27) in the following form, after using Eq. (3.28)

\[ g''(r) + \left( \frac{2(\eta + \frac{1}{2})}{r} - 4\alpha r - 2\beta \right)g'(r) + 2\mu \delta E_n g(r) = 0, \]  

where \( \delta E_n = E_n - E_0 \).

Firstly, we determine the solution of Eq. (3.36) for large \( r \). So, when \( r \to \infty \), Eq. (3.36) becomes

\[ g''(r) - (4\alpha r + 2\beta)g'(r) + 2\mu \delta E_n g(r) = 0, \]  

On comparison of Eq. (3.37) with Eq. (1.13), and using recursion Eq. (1.16), we obtain

\[ \lambda_0 = 4\alpha r + 2\beta, \quad s_0 = -2\mu \delta E_n, \]  
\[ \lambda_1 = 4\alpha - 2\mu \delta E_n + (4\alpha r + 2\beta)^2, \quad s_1 = -2\mu \delta E_n (4\alpha r + 2\beta), \]  
\[ \lambda_2 = (4\alpha r + 2\beta)((8\alpha - 2\mu \delta E_n) + 4\alpha - 2\mu \delta E_n + (4\alpha r + 2\beta)^2), \]  
\[ s_2 = -2\mu \delta E_n (8\alpha - 2\mu \delta E_n + (4\alpha r + 2\beta)^2), \]  

and so on. On substituting above values in Eq. (1.17), we obtain

\[ \lambda_1 s_0 - \lambda_0 s_1 = 0 \Rightarrow \delta E_1 = \frac{2\alpha}{\mu}, \]  

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\[ \lambda_2 s_1 - \lambda_1 s_2 = 0, \Rightarrow \delta E_2 = \frac{4\alpha}{\mu}, \]  
(3.40)  
\[ \lambda_3 s_2 - \lambda_2 s_3 = 0, \Rightarrow \delta E_3 = \frac{6\alpha}{\mu}, \]  
(3.41)  
and so on. Finally, this trend leads us to  
\[ \delta E_n = \frac{2n\alpha}{\mu}, \quad n = 1, 2, 3, \ldots \]  
(3.42)  
Now the eigenfunction \( g(r) \) can be determined using Eqs. (3.37) and (1.18). On comparison of these two equations, we have \( D = 0, u = 2\alpha, v = 0 \) and \( t = -(\beta r + 1) \). Therefore, we obtain \( \rho = \frac{-2(2\beta r + 1)}{r} + \frac{2\alpha}{v} \) and \( \sigma = \frac{1-2\beta r}{2} \). Then wavefunction \( g(r) \) becomes  
\[ g(r) = (-2)^n C_2 \frac{\Gamma\left(\frac{1-2\beta r}{2} + n\right)}{\Gamma\left(\frac{1-2\beta r}{2}\right)} \, {}_1F_1\left(-n, \frac{1-2\beta r}{2}; 2\alpha r^2\right). \]  
(3.43)  
Secondly, for \( r \to 0 \), Eq. (3.36) for small ‘r’ can be written as  
\[ g''(r) + \left(\frac{2(\eta + \frac{1}{2})}{r} - 2\beta\right)g'(r) + 2\mu\delta E_n g(r) = 0. \]  
(3.44)  
Again compare above Eq. (3.44) with Eq. (1.13), and applying the methodology of the previous case, we obtain  
\[ \lambda_0 = 2\beta - \left(\frac{2(\eta + \frac{1}{2})}{r}\right), \quad s_0 = -2\mu\delta E_n, \]  
\[ \lambda_1 = \frac{2(\eta + \frac{1}{2})}{r^2} - 2\mu\delta E_n + (2\beta - \frac{2(\eta + \frac{1}{2})}{r})^2, \quad s_1 = -2\mu\delta E_n(2\beta - \frac{2(\eta + \frac{1}{2})}{r}), \]  
\[ \lambda_2 = (2\beta - \frac{2(\eta + \frac{1}{2})}{r})\left[\frac{2\beta}{r^2} - \frac{2\beta}{r^2} + 4\mu\delta E_n - (2\beta - \frac{2(\eta + \frac{1}{2})}{r})^2\right], \]  
\[ s_2 = -2\mu\delta E_n\left[\frac{2\beta}{r^2} - 2\mu\delta E_n - (2\beta - \frac{2(\eta + \frac{1}{2})}{r})^2\right]. \]  
(3.45)  
For the present case, Eq. (1.17) is not exactly solvable. Thus, we find \( r = \frac{(\eta + \frac{1}{2})}{\beta} \) as a root of equation \( \lambda_0(r) = 0 \), which is also the maximum value point of the asymptotic wavefunction. So for this particular value of ‘r’, Eq. (1.17) provides  
\[ \lambda_1 s_0 - \lambda_0 s_1 = 0, \Rightarrow \delta E_1 = \frac{\beta^2}{(\eta + \frac{1}{2})\mu}, \]  
\[ \lambda_2 s_1 - \lambda_1 s_2 = 0, \Rightarrow \delta E_2 = \frac{2\beta^2}{(\eta + \frac{1}{2})\mu}, \]  
(3.46)  
and so on. Therefore, from above trend, we write  
\[ \delta E_n = \frac{n\beta^2}{(\eta + \frac{1}{2})\mu}, \quad n = 1, 2, 3, \ldots \]  
(3.47)
Further, on comparison of Eqs. (3.44) and (1.18), the wavefunction $g(r)$ is written as

$$g(r) = (-2)^n C_2 \frac{\Gamma(2\eta + 1 + n)}{\Gamma(2\eta + 1)} \, {}_1F_1(-n, 2\eta + 1; 2\beta r).$$

(3.48)

The complete energy spectrum for the quark-antiquark system is obtained by adding Eqs. (3.42) and (3.47) as

$$\delta E_n = E_n - E_0 = \frac{2n\alpha}{\mu} + \frac{n\beta^2}{(\eta + \frac{1}{2})\mu}.$$ 

(3.49)

Finally, above equation is written in a more elucidated form by using Eqs. (3.33), (3.34) and (3.35), and replacing $\eta = l + \frac{N-2}{2}$ as

$$E_{n\ell N} = \sqrt{\frac{a}{2\mu}} (2n + 2l + N) + \frac{(4n - 2l - N + 1)b^2}{(2l + N - 1)4a}. 

(3.50)

It is to be noted that the above analytical result reduces to the PSM results (cf. Eq. (2.13)) and the other literature results [29, 30] when $n \to 0$ and $\mu = 1$.

### 3.3.1 Mass spectra of heavy $Q\bar{Q}$ system

In order to derive the mass spectra of the heavy $Q\bar{Q}$ systems such as charmonium and bottomonium which consist of quark and antiquark of same flavor. For determining the mass spectra in three dimensions, we use the relation (3.23). Thus, using Eq. (3.50) into Eq. (3.23), the mass spectra of charmonium becomes

$$M_c = 2m_c + \sqrt{\frac{a}{m_c}} (2n + 2l + 3) + \frac{(2n - l - 1)b^2}{4a(l + 1)}. 

(3.51)

Similarly, for obtaining the mass spectra of bottomonium, change $M_c$ and $m_c$ by $M_b$ and $m_b$ respectively in Eq. (3.51).

With a view to check the effectiveness of the analytic calculations made by the AIM, we also computed numerically the mass spectra of charmonium and bottomonium systems. A comparison of various results is listed in tables 3.3 and 3.4.
Table 3.3: Mass spectra of charmonium \((m_c = 1.48 GeV, a = 0.042 GeV^3, b = 0.255 GeV^2)\) in GeV

<table>
<thead>
<tr>
<th>States</th>
<th>c</th>
<th>From Eq.(3.51)</th>
<th>Numerical Results</th>
<th>Experimental [24]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1S</td>
<td>1.023</td>
<td>3.078</td>
<td>3.078</td>
<td>3.068</td>
</tr>
<tr>
<td>1P</td>
<td>2.046</td>
<td>3.415</td>
<td>3.415</td>
<td>3.525</td>
</tr>
<tr>
<td>2S</td>
<td>2.046</td>
<td>4.192</td>
<td>3.787</td>
<td>3.663</td>
</tr>
<tr>
<td>2P</td>
<td>3.069</td>
<td>4.143</td>
<td>4.030</td>
<td>–</td>
</tr>
<tr>
<td>3S</td>
<td>3.069</td>
<td>5.497</td>
<td>4.028</td>
<td>4.159</td>
</tr>
<tr>
<td>4S</td>
<td>4.092</td>
<td>6.222</td>
<td>4.280</td>
<td>4.421</td>
</tr>
</tbody>
</table>

Table 3.4: Mass spectra of bottomonium \((m_b = 4.68 GeV, a = 0.143 GeV^3, b = 0.465 GeV^2)\) in GeV

<table>
<thead>
<tr>
<th>States</th>
<th>c</th>
<th>From Eq.(3.51)</th>
<th>Numerical Results</th>
<th>Experimental [24]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1S</td>
<td>0.57</td>
<td>9.510</td>
<td>9.506</td>
<td>9.460</td>
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<tr>
<td>1P</td>
<td>1.14</td>
<td>9.862</td>
<td>9.841</td>
<td>9.900</td>
</tr>
<tr>
<td>2S</td>
<td>1.14</td>
<td>10.618</td>
<td>9.772</td>
<td>10.023</td>
</tr>
<tr>
<td>1D</td>
<td>1.71</td>
<td>10.214</td>
<td>10.196</td>
<td>10.161</td>
</tr>
<tr>
<td>2P</td>
<td>1.71</td>
<td>10.944</td>
<td>10.468</td>
<td>10.260</td>
</tr>
<tr>
<td>3S</td>
<td>1.71</td>
<td>11.915</td>
<td>10.488</td>
<td>10.355</td>
</tr>
<tr>
<td>4S</td>
<td>2.28</td>
<td>12.834</td>
<td>10.747</td>
<td>10.580</td>
</tr>
</tbody>
</table>

From the above tables, it is to be noted that our analytical and numerical results are in good agreement. These results are approximately match with the experimental results and the results of other studies [19, 20, 25, 31]. From the above studies, we conclude that the low lying states masses of charmonium and bottomonium are in good agreement with the experimental results but for higher states the discrepancy becomes more pronounced. This discrepancy is due to the fact that these excited states are near the threshold of open charm and bottom production, which results the comparison with the experimental data more complicated.
Bibliography


