CHAPTER 3

GENERALIZED REGULAR WEAKLY CLOSED SETS IN
TOPOLOGICAL SPACES

3.1 INTRODUCTION


Ahmad AL-Omari (2009) introduced and studied weaker forms of \( \omega \)-open sets and decompositions of continuity. Ahmad AL-Omari (2009) discussed the nature of weaker forms of open and closed functions via \( b-\theta \)-open sets. Ahmad AL-Omari (2009) investigated the weakly \( b \)-open functions in topological spaces.
3.2 GENERALIZED REGULAR WEAKLY CLOSED SETS IN TOPOLOGICAL SPACES

In this section a new class of sets called GRW-closed sets is introduced as generalization of rw-closed sets, some of their properties are brought out and $T_{GRW}$-space is defined and studied.

**Definition 3.2.1:** A subset $A$ of a topological space $(X, \tau)$ is called generalized regular weakly closed (GRW-closed) if $\text{cl}^*(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semi open in $X$. The set of all GRW-closed sets in $X$ is denoted by $\text{GRWC}(X)$.

**Theorem 3.2.2:** The union of two GRW-closed subsets of $X$ is also a GRW-closed set in $X$.

**Proof:** Assume that $A$ and $B$ are two GRW-closed subsets of $X$. Let $U$ be a RSO set in $X$ such that $A \cup B \subseteq U$, then, $A \subseteq U$ and $B \subseteq U$. Since $A$ and $B$ are GRW-closed subsets of $X$, $\text{cl}^*(A) \subseteq U$ and $\text{cl}^*(B) \subseteq U$, but $\text{cl}^*(A \cup B) = \text{cl}^*(A) \cup \text{cl}^*(B) \subseteq U$. Hence $A \cup B$ is also a GRW-closed subset of $X$.

**Remark 3.2.3:** The intersection of two GRW-closed sets of $X$ need not be a GRW-closed set from the following example.

**Example 3.2.4:** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. The sets $\{a, b\}$ and $\{a, d\}$ are GRW-closed sub sets of $X$, but $\{a\}$ which is the intersection of those GRW-closed sets is not a GRW-closed set.

**Theorem 3.2.5** If a subset $A$ of $X$ is GRW-closed set in $X$, then $\text{cl}^*(A) - A$ does not contain any nonempty regular semi open set in $X$. 
**Proof:** Suppose that $A$ is a GRW-closed set in $X$ such that $\text{cl}^*(A) - A$ contains some nonempty regular semi open set in $X$. Let $U$ be a regular semi open set such that $U \subseteq \text{cl}^*(A) - A$ and $U \neq \phi$. Now $U \subseteq \text{cl}^*(A) - A$ implies $U \subseteq X - A$ which implies $A \subseteq X - U$. Since $U$ is regular semi open, $X - U$ is also regular semi open in $X$ containing $A$. As $A$ is a GRW set in $X$, $\text{cl}^*A) \subseteq X - U$, that is $U \subseteq X - \text{cl}^*(A)$ and $U \subseteq \text{cl}^*(A)$. Therefore $U \subseteq \text{cl}^*(A) \cap (X - \text{cl}^*(A)) = \phi$. Thus $U = \phi$, which is a contradiction to the assumption that $U \neq \phi$. Hence $\text{cl}^*(A) - A$ does not contain any non-empty regular semi-open set in $X$.

**Remark 3.2.6:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.2.7:** Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. $A = \{a\}$ is GRW-closed set but, $\text{cl}^*(A) - A = \{c\}$ does not have any non-empty regular semi open set.

**Theorem 3.2.8:** For an element $x \in (X, \tau)$, the set $X - \{x\}$ is GRW-closed or regular semi-open.

**Proof:** Assume that $X - \{x\}$ is not a regular semi open set. Then $X$ is the only regular semi open set containing $X - \{x\}$. This implies $\text{cl}^*(X - \{x\}) \subseteq X$. Hence $X - \{x\}$ is a GRW-closed set in $X$.

**Theorem 3.2.9:** If $A$ is a regular open and GRW-closed set in $(X, \tau)$, then $A$ is $\tau^*$-closed.

**Proof:** Assume that $A$ is regular open and GRW-closed set. As every regular open set is regular semi open then, $\text{cl}^*(A) \subseteq A$ since $A$ is GRW-closed in $X$. Therefore $\text{cl}^*(A) = A$, hence $A$ is $\tau^*$-closed.
Theorem 3.2.10: If $A$ is GRW-closed in $(X, \tau)$, then $A$ is $\tau^*$-closed if and only if $\text{cl}^*(A) = A$ is regular semi open.

**Proof:** Suppose $A$ is $\tau^*$-closed, then $\text{cl}^*(A) = A$, so $\text{cl}^*(A) \cap A = \phi$, which is regular semi open in $X$. Conversely, assume that $\text{cl}^*(A) \cap A$ is regular semi open in $X$. Since $A$ is GRW-closed, by the theorem 3.2.5, $\text{cl}^*(A) \cap A$ does not contain any nonempty regular semi open set in $X$. Then $\text{cl}^*(A) \cap A = \phi$, which implies $\text{cl}^*(A) = A$, hence $A$ is $\tau^*$-closed in $X$.

Theorem 3.2.11: If $A$ is a GRW-closed subset of $X$ such that $A \subseteq B \subseteq \text{cl}^*(A)$, then $B$ is a GRW-closed set in $X$.

**Proof:** Let $A$ be a GRW-closed sub set of $X$ such that $A \subseteq B \subseteq \text{cl}^*(A)$. Let $U$ be a regular semi open sub set of $X$ such that $B \subseteq U$, then $A \subseteq U$. Since $A$ is GRW-closed, $\text{cl}^*(A) \subseteq U$. Now $\text{cl}^*(B) \subseteq \text{cl}^*(\text{cl}^*(A)) = \text{cl}^*(A) \subseteq U$. Therefore $B$ is a GRW-closed set in $X$.

Remark 3.2.12: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.13: Consider the topological space $(X, \tau)$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{c\}$ and $B = \{a, c\}$. Then $A$ and $B$ are GRW-closed sets in $(X, \tau)$, but $A \subseteq B \nsubseteq \text{cl}^*(A) = \{b, c\}$.

Theorem 3.2.14: Every rw-closed set in $X$ is GRW-closed in $X$.

**Proof:** Let $A$ be a rw-closed set in $X$ such that $A \subseteq U$, $U \in \text{RSO}(X)$. Then, $\text{cl}^*(A) \subseteq \text{cl}(A) \subseteq U$, as $A$ is rw-closed, hence $A$ is GRW-closed.

Remark 3.2.15: Following example shows that the converse of the above theorem need not be true.
Example 3.2.16: Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). In this topological space, the set \( \{a, d\} \) is GRW-closed set and it is not a rw-closed set.

Theorem 3.2.17: Every w-closed set in \( X \) is a GRW-closed set in \( X \).

Proof: Let \( A \) be a w-closed set in a \( X \) such that \( A \subseteq U \) where \( U \) is regular semi open set in \( X \), then \( \text{cl}(A) \subseteq U \) since \( A \) is w-closed and every regular semi open set in \( (X, \tau) \) is semi open and \( \text{cl}^r(A) \subseteq \text{cl}(A) \), hence \( \text{cl}^r(A) \subseteq U \) so \( A \) is GRW-closed.

Remark 3.2.18: Converse of the above theorem is not true from the following example.

Example 3.2.19: Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \). In this topological space \( \{a, b\} \) is GRW-closed set, but it is not w-closed.

Theorem 3.2.20: If a sub set \( A \) of a topological space \( (X, \tau) \) is g- closed set or closed then it is GRW-closed set in the topological space \( (X, \tau) \).

Proof: If \( A \) is g- closed or closed set then \( \text{cl}^*(A) = A \), whenever \( A \subseteq U \) and \( U \) is regular semi open in \( X \), \( \text{cl}^*(A) = A \subseteq U \), hence \( A \) is GRW-closed set.

Remark 3.2.21: Converse of the above theorem is false from the following example.

Example 3.2.22: Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\} \). In this topological space \( \{a, b\} \) is GRW-closed set but it is neither g-closed nor closed subset of \( X \).

Remark 3.2.23: GRW-closed sets are independent of gpr-closed sets in a topological space \( (X, \tau) \) from the following examples.
Example 3.2.24: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}$. In this topological space $\{c\}$ is gpr-closed but not GRW-closed.

Example 3.2.25: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. In this topological space the set $\{a\}$ is not gpr-closed but $\{a\}$ is GRW-closed.

Theorem 3.2.26: If $A$ is a pre-closed and GRW-closed set in a topological space $(X, \tau)$ then $A$ is rwg-closed in $X$.

Proof: Let $A$ be a pre-closed and GRW-closed subset of $X$ such that $A \subseteq U$, where $U$ is regular open. Regular open sets are regular semi open so, $\text{cl}(\text{int}(A)) \subseteq A \subseteq \text{cl}^*(A) \subseteq U$ which proves that $A$ is rwg-closed.

Theorem 3.2.27: If $A$ is regular open set in a topological space $(X, \tau)$ such that $\text{cl}^*(A) \subseteq U$ whenever $A \subseteq U$ where $U$ is regular open set then, $A$ is GRW-closed in $X$.

Proof: Let $U$ be any regular semi open set in $X$ such that $A \subseteq U$. $\text{cl}^*(A) \subseteq A \subseteq \text{cl}^*(A)$, since $A$ itself is a regular open set, hence $\text{cl}^*(A) = A \subseteq U$ and so $A$ is a GRW-closed set in $X$.

Theorem 3.2.28: If $A$ is regular closed set in a topological space $(X, \tau)$ then $A$ is GRW-closed in $X$.

Proof: If $A$ is a regular closed set in $X$, then $\text{cl}(\text{int}(A)) = A$ which implies $A$ is closed, hence $A$ is GRW-closed set in $X$.

Remark 3.2.29: Converse of the above theorem need not be true from the following example.
Example 3.2.30: Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{\emptyset, \{a\}, \{b\}$, $\{a, b\}, \{a, b, c\}, X\}$. In this topological space $\{a\}$ is GRW-closed but not regular closed in $(X, \tau)$.

Theorem 3.2.31: If a subset $A$ of a topological space $(X, \tau)$ is both regular semi open and GRW-closed, then it is $\tau^*$-closed.

Proof: Suppose a subset $A$ of a topological space $X$ is both regular semi open and GRW-closed, then $cl^*(A) \subseteq A$, therefore $cl^*(A) = A$. Hence $A$ is $\tau^*$-closed.

Corollary 3.2.32: Let $A$ be regular semi open and GRW-closed in $(X, \tau)$. Suppose $F$ is $\tau^*$-closed in $X$, then $A \cap F$ is a GRW-closed set in $X$.

Proof: Let $A$ be regular semi open and GRW-closed in $X$ and $F$ be $\tau^*$-closed. From the above theorem, $A$ is $\tau^*$-closed. Hence $A \cap F$ is a $\tau^*$-closed set in $X$, since finite intersection of $\tau^*$-closed sets is $\tau^*$-closed set in $X$, therefore $A \cap F$ is GRW-closed in $X$.

Theorem 3.2.33: Suppose $B \subseteq A \subseteq X$, $B$ is GRW-closed set relative to $A$ and $A$ is both regular open and GRW-closed subset of $(X, \tau)$, then $B$ is GRW-closed set relative to $X$.

Proof: Let $B \subseteq G$ and $G$ be a regular semi open set in $X$. It is given that $B \subseteq A \subseteq X$, therefore $B \subseteq A \cap G$. As $A \cap G$ is regular semi open in $A$. Since $B$ is GRW-closed relative to $A$, $cl^*_A(B) \subseteq A \cap G$. But $cl^*_A(B) = A \cap cl^*(B)$. Hence it follows that $A \cap cl^*(B) \subseteq A \cap G$. Consequently $A \cap cl^*(B) \subseteq G$. Since $A$ is regular open and GRW-closed, $cl^*(A) = A$, so $cl^*(B) \subseteq A$. Further $A \cap cl^*(B) = cl^*(B)$. Thus $cl^*(B) \subseteq G$ and hence $B$ is GRW-closed relative to $X$. 
**Theorem 3.2.34:** Let \( A \subseteq B \subseteq X \) and suppose that \( A \) is GRW-closed in \((X, \tau)\) then, \( A \) is GRW-closed in \( B \) provided \( Y \) is regular open in \( X \).

**Proof:** Let \( A \) be GRW-closed in \( X \) and \( B \) be a regular open subspace of \((X, \tau)\). Let \( U \) be any regular semi open set related to \( Y \) such that \( A \subseteq U \). But \( A \subseteq B \subseteq X \). \( U \) is regular semi open in \( X \). Since \( A \) is GRW-closed in \( X \), \( \text{cl}^{*}(A) \subseteq U \). That is \( \text{cl}^{*}_{B}(A) = B \cap \text{cl}^{*}(A) \subseteq B \cap U \). Thus \( \text{cl}^{*}_{B}(A) \subseteq U \). Hence \( A \) is GRW-closed in \( Y \).

**Theorem 3.2.35:** If a sub set \( A \) is g-closed in a topological space \((X, \tau)\), it is then, GRW-closed in \( X \).

**Proof:** Let \( A \) be a g-closed set in \( X \), then \( \text{cl}^{*}(A) = A \). Let \( A \subseteq U \) and \( U \) is a regular semi open in \( X \), then, \( \text{cl}^{*}(A) \subseteq U \). Thus \( A \) is GRW-closed in \( X \).

**Remark 3.2.36:** Converse of the above theorem need not be true from the following example.

**Example 3.2.37:** \( X = \{a, b, c, d\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\} \). In this topological space \( \{a, b\} \) is GRW-closed set but it is neither g-closed nor closed sub set of \( X \).

**Theorem 3.2.38:** If \( A \) is \( g^{*} \)-closed in a topological space \((X, \tau)\), then it is GRW-closed in \((X, \tau)\).

**Proof:** In a topological space \( g^{*} \)-closed sets are g-closed, therefore \( \text{cl}^{*}(A) = A \). If \( A \subseteq U \), where \( U \) is regular semi open in \( X \), then \( A = \text{cl}^{*}(A) \subseteq U \). Thus \( A \) is GRW-closed in \( X \).

**Remark 3.2.39:** Converse of the above theorem need not be true from the following example.
Example 3.2.40: \( X= \{ a, b, c, d \}, \tau = \{ \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ b, c \}, \{ a, c \}, \{ a, b, c \} \} \). In this topological space the set \{ a, b \} is GRW-closed set but it is not a \( g^* \)-closed subset of \( X \).

**Theorem 3.2.41:** If \( A \) is \( \tau^* \)-closed in a topological space \((X, \tau)\), then it is GRW-closed in \((X, \tau)\).

**Proof:** If \( A \) is a \( \tau^* \)-closed set in \( X \) then, \( \text{cl}^*(A) = A \). If \( A \subseteq U \), where \( U \) is regular semi open in \( X \), then \( A = \text{cl}^*(A) \subseteq U \). Thus \( A \) is GRW-closed in \( X \).

**Remark 3.2.42:** Converse of the above theorem need not be true from the following example.

Example 3.2.43: \( X= \{ a, b, c, d \}, \tau = \{ \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ b, c \}, \{ a, c \}, \{ a, b, c \} \} \). In this topological space \{ a, b, c \} is GRW-closed set but it is not a \( \tau^* \)-closed sub set of \( X \).

**Theorem 3.2.44:** If an open subset \( A \) of a topological space \((X, \tau)\) such that \( \text{cl}^*(\text{int} A) \subseteq U \) where \( U \) is open, it is then GRW-closed in \((X, \tau)\).

**Proof:** Let \( A \) be an open sub set of \( X \) such that \( \text{cl}^*(\text{int} A) \subseteq U \) where \( U \) is open. Let \( A \subseteq U \) with \( U \) regular semi open in \( X \). By the hypothesis, \( \text{cl}^*(\text{int}(A)) \subseteq A \), as \( A \) is open and \( \text{cl}^*(A) \subseteq A \subseteq U \), Thus \( A \) is a GRW-closed set in \( X \).

**Theorem 3.2.45:** If a subset \( A \) of a topological space \((X, \tau)\) is both open and \( \text{wg} \)-closed, then it is GRW-closed.

**Proof:** Let a subset \( A \) of \( X \) be both open and \( \text{wg} \)-closed. If \( A \subseteq U \), where \( U \) is regular semi open in \( X \) then, as \( A \) is \( \text{wg} \)-closed, \( \text{cl}(\text{int}(A)) \subseteq A \), since \( A \) is
open. That is \( \text{cl}(A) \subseteq A \subseteq U \), \( \text{cl}^*(A) \subseteq \text{cl}(A) \subseteq A \subseteq U \), thus \( A \) is a GRW-closed set in \( X \).

**Theorem 3.2.46:** In a topological space \( (X, \tau) \), if \( \text{RSO}(X) = \{ \phi, X \} \), then every subset of \( X \) is a GRW-closed set in \( X \).

**Proof:** Let \( X \) be a topological space and \( \text{RSO}(X) = \{ \phi, X \} \). Let \( A \) be any subset of \( X \). Suppose \( A = \phi \), then \( \phi \) is a GRW-closed set in \( X \). If \( A \neq \phi \) then \( X \) is the only regular semi open set containing \( A \), so \( \text{cl}^*(A) \subseteq X \). Hence \( A \) is a GRW-closed in \( X \).

**Remark 3.2.47:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.2.48:** Let \( X = \{ a, b, c, d \}, \tau = \{ X, \phi, \{a, b\}, \{c, d\}\} \). Then every subset of \( X \) is a GRW-closed set in \( X \), but \( \text{RSO}(X) = \{ X, \phi, \{a, b\}, \{c, d\}\} \).

**Theorem 3.2.49:** In a topological space \( (X, \tau) \), \( \text{RSO}(X) \subseteq \{F \subseteq X: \text{cl}^*(F) = F\} \), if and only if every subset of \( (X, \tau) \) is GRW-closed.

**Proof:** (Necessary) Suppose that \( \text{RSO}(X, \tau) \subseteq \{F \subseteq X : \text{cl}^*(F) = F \} \). Let \( A \) be any subset of \( X \) such that \( A \subseteq U \), where \( U \) is regular semi open; \( U \in \text{RSO}(X, \tau) \subseteq \{F \subseteq X : \text{cl}^*(F) = F \} \) (ie) \( F \) is \( \tau^* \)-closed. That is \( U \in \{F \subseteq X : F \text{ is } \tau^* \text{-closed} \} \). Thus \( U \) is \( \tau^* \)-closed. Then \( \text{cl}^*(U) = U \). Also \( \text{cl}^*(A) \subseteq \text{cl}^*(U) = U \). Hence \( A \) is a GRW-closed set in \( X \).

(Sufficient) Suppose that every subset of \( (X, \tau) \) is GRW-closed. Let \( U \in \text{RSO}(X, \tau) \), \( U \) is GRW-closed so \( \text{cl}^*(U) \subseteq U \). Thus \( U^c \) is \( \tau^* \)-open, so \( U \in \{F \subseteq X : F^c \text{ is } \tau^* \text{-open in } X \} \). Therefore \( \text{RSO}(X, \tau) \subseteq \{F \subseteq X : \text{cl}^*(F) = F \} \).
Theorem 3.2.50: Let \( X \) be a regular space in which every regular semi open subset is open. If \( A \) is a compact subset of \( X \), then \( A \) is GRW-closed.

Proof: Suppose \( A \subseteq U \) and \( U \) is regular semi open. By the hypothesis, \( U \) is open. But \( A \) is a compact subset in the regular space \( X \). Hence there exists an open set \( V \) such that \( A \subseteq V \subseteq \text{cl}(V) \subseteq U \). Now \( A \subseteq \text{cl}(V) \) implies \( \text{cl}(A) \subseteq \text{cl}(\text{cl}(V)) = \text{cl}(V) \subseteq U \). That is \( \text{cl}(A) \subseteq U \). Hence \( \text{cl}^*(A) \subseteq U \), so \( A \) is GRW-closed in \( X \).

Theorem 3.2.51: A subset \( A \) of \((X, \tau)\) is GRW-closed if and only if \( \text{cl}^*(A) \subseteq \text{rsker}(A) \).

Proof: Suppose that \( A \) is GRW-closed. If \( x \in \text{cl}^*(A) \) and \( x \not\in \text{rsker}(A) \), then there is a regular semi open set \( U \) containing \( A \) such that \( x \) is not in \( U \). Since \( A \) is GRW-closed, \( \text{cl}^*(A) \subseteq U \), but \( x \) not in \( \text{cl}^*(A) \), which is a contradiction to \( x \subseteq \text{cl}^*(A) \). Hence \( x \in \text{rsker}(A) \) and so \( \text{cl}^*(A) \subseteq \text{rsker}(A) \). Conversely, let \( \text{cl}^*(A) \subseteq \text{rsker}(A) \). If \( U \) is any regular semi open set containing \( A \), then \( \text{rsker}(A) \subseteq U \). That is \( \text{cl}^*(A) \subseteq \text{rsker}(A) \subseteq U \). Therefore \( A \) is GRW-closed in \( X \).

Theorem 3.2.52: For any subset \( A \) of \((X, \tau)\), if \( \cap X_1 \cap \text{cl}^*(A) \subseteq A \), then \( A \) is GRW-closed in \( X \).

Proof: Let \( A \) be a subset of \((X, \tau)\). Suppose that \( X_1 \cap \text{cl}^*(A) \subseteq A \). It will be proved that \( A \) is GRW-Closed in \( X \). Then \( \text{cl}^*(A) \subseteq \text{rsker}(A) \), since \( A \subseteq \text{rsker}(A) \), \( \text{cl}^*(A) = X_1 \cap \text{cl}^*(A) = (X_1 \cup X_2) \cap \text{cl}^*(A) \). That is \( \text{cl}^*(A) = (X_1 \cap \text{cl}^*(A)) \cup (X_2 \cap \text{cl}^*(A)) \subseteq \text{rsker}(A) \), since \( X_1 \cap \text{cl}^*(A) \subseteq \text{rsker}(A) \) and \( X_2 \cap \text{cl}^*(A) \subseteq \text{rsker}(A) \). That is \( \text{cl}^*(A) \subseteq \text{rsker}(A) \). By the Theorem 3.2.51, \( A \) is GRW-closed in \( X \).

Remark 3.2.53: The converse of the above theorem need not be true in general as seen from the following example.
**Example 3.2.54:** Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Here $X_1 = \{c, d\}$ and $X_2 = \{a, b\}$. For the set $A = \{a, c\}$, $A$ is GRW-closed in $X$. But $X_1 \cap \text{cl}^*(A) = \{c, d\} \cap \{a, c, d\} = \{c, d\}$ is not a subset of $A$.

**Remark 3.2.55:** GRW-closed sets need not be a $\tau^*$-g closed and $\tau^*$-g closed sets need not be a GRW-closed set in $X$ from the following example.

**Example 3.2.56:** Consider the set $X= \{a, b, c, d\}$, $\tau= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. In this topological space $\{d\}$ is $\tau^*$-g closed set, but not GRW-closed. The set $\{a, b, c\}$ is GRW-closed but not $\tau^*$-g closed.

**Remark 3.2.57:** From the following example GRW-closed sets are independent of mg-closed sets.

**Example 3.2.58:** Consider the topological space $(X, \tau)$, where $X=\{a, b, c, d\}$, $\tau= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. In this topological space $\{a, c\}$ is GRW-closed but not mg-closed. The set $\{a\}$ is mg-closed but not GRW-closed.

**Remark 3.2.59:** Following example shows that GRW-closed sets are independent of rwg-closed sets.

**Example 3.2.60:** Consider the topological space $(X, \tau)$, where $X=\{a, b, c, d\}$, $\tau= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. In this topological space $\{a\}$ is not GRW-closed but it is rwg-closed.

**Example 3.2.61:** Consider the topological space $(X, \tau)$, where $X=\{a, b, c\}$ with the topology $\tau= \{X, \emptyset, \{d\}, \{b, d\}, \{b, c, d\}, \{a, c, d\}\}$, the set $\{a\}$ is GRW-closed but it is not rwg- closed.
Remark 3.2.62: Following example proves that GRW-closed set need not be a $\delta$-generalized closed set, $\theta$-generalized closed set in $(X, \tau)$.

Example 3.2.63: Let $X = \{a, b, c, d\}$ be with the topology

$$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}. $$

Then

(i) $\{a, b\}$ is GRW-closed but not $\theta$-generalized closed in $(X, \tau)$.

(ii) $\{a, b\}$ is GRW-closed but not $\delta$-generalized closed in $(X, \tau)$.

Remark 3.2.64: The following example shows that GRW-closed sets are independent of $\text{wg}$-closed sets, semi closed, $\alpha$-closed sets, $g\alpha$-closed sets, $\alpha$ $g$-closed sets, $sg$-closed, $gs$-closed sets, $gsp$-closed sets, $\beta$-closed sets, pre-closed sets, gp-closed sets, swg-closed sets and $\pi g$-closed sets.

Example 3.2.65: Let $X = \{a, b, c, d\}$ be with the topology, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then

(i) $\{c\}$ is $\text{wg}$-closed set but not GRW-closed in $(X, \tau)$ and $\{a, b\}$ is GRW-closed but not $\text{wg}$-closed $(X, \tau)$.

(ii) $\{c\}$ is semi-closed set but not GRW-closed in $(X, \tau)$ and $\{a, b\}$ is GRW-closed but not semi-closed $(X, \tau)$.

(iii) $\{c\}$ is $\alpha$-closed set but not GRW-closed in $(X, \tau)$ and $\{a, b\}$ is GRW-closed but not $\alpha$-closed $(X, \tau)$.

(iv) $\{c\}$ is $g\alpha$-closed set but not GRW-closed in $(X, \tau)$ and $\{a, b\}$ is GRW-closed but not $g\alpha$-closed $(X, \tau)$.

(v) $\{c\}$ is $\alpha g$-closed set but not GRW-closed in $(X, \tau)$ and $\{a, b\}$ is GRW-closed but not $\alpha g$-closed $(X, \tau)$. 
(vi) \( \{c\} \) is sg-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not sg-closed \((X, \tau)\).

(vii) \( \{c\} \) is gs-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not gs-closed \((X, \tau)\).

(viii) \( \{c\} \) is gsp-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not gsp-closed \((X, \tau)\).

(ix) \( \{c\} \) is \(\beta\)-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not \(\beta\)-closed \((X, \tau)\).

(x) \( \{c\} \) is pre-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not pre-closed \((X, \tau)\).

(xi) \( \{c\} \) is gp-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not gp-closed \((X, \tau)\).

(xii) \( \{c\} \) is swg-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not swg-closed \((X, \tau)\).

(xiii) \( \{c\} \) is \(\pi_g\)-closed set but not GRW-closed in \((X, \tau)\) and \(\{a, b\}\) is GRW-closed but not \(\pi_g\)-closed \((X, \tau)\).

**Remark 3.2.66:** From the above discussion the following Implications can be seen with some known sets (Figure 3.1).
Remark 3.2.67: From the above discussion and known results the following independent relationships with some of the existing sets is brought out (Figure 3.2).

Figure 3.1 Implication relationships of GRW-closed sets

Figure 3.2 Independent relationships of GRW-closed set
3.3 GRW-OPEN SETS IN TOPOLOGICAL SPACES

In this section, GRW-open sets in topological spaces and GRW-neighborhoods in topological spaces by using the notion of GRW-open sets are defined and some of their properties are obtained.

**Definition 3.3.1** A subset $A$ in $(X, \tau)$ is called GRW-open (briefly GRW-open) in $X$ if $A^c$ is GRW-closed in $(X, \tau)$. The family of all GRW-open sets in $X$ is denoted by GRWO($X$).

**Theorem 3.3.2:** Every singleton set in a topological space $(X, \tau)$ is either GRW-open or regular semi open.

**Proof:** Let $X$ be a topological space and $x \in X$. By the theorem 3.2.8, the set $X - \{x\}$ is either GRW-closed or regular semi open.

**Theorem 3.3.3:** If a subset $A$ of a topological space $(X, \tau)$ is w-open then it is GRW-open, but not conversely.

**Proof:** Let $A$ be a w-open set in a topological space $(X, \tau)$. Then $A^c$ is w-closed set. By the theorem 3.2.16, $A^c$ is GRW-closed. Therefore $A$ is GRW-open set in $X$.

**Theorem 3.3.4:** Every open set in a topological space $(X, \tau)$ is GRW-open set but not conversely.

**Proof:** Let $A$ be an open set in a topological space $(X, \tau)$. Then $A^c$ is a closed set. By the theorem 3.2.20, $A^c$ is GRW-closed. Therefore $A$ is GRW-open set in $X$.

**Remark 3.3.5:** The converse of the above theorems need not be true, as seen from the following example.
Example 3.3.6: Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} \). In this topological space the set \( \{a\} \) is GRW-open but, not w-open and not open in \( X \).

Theorem 3.5.7: Every g-open set in topological space (\( X, \tau \)) is GRW-open set but not conversely.

Proof: Let \( A \) be a g-open set in \( X \), then \( A^c \) is g-closed, since every g-closed set is GRW-closed by the theorem 3.2.20. Hence \( A^c \) is GRW-closed and hence \( A \) is GRW-open.

3.4 GRW-NEIGHBORHOODS IN TOPOLOGICAL SPACES

In this section GRW-neighborhoods in a topological space are defined and some of their properties are studied.

Definition 3.4.1: Let (\( X, \tau \)) be a topological space and let \( x \in X \). A subset \( N \) of \( X \) is said to be a GRW-neighborhood of \( x \) if and only if there exists a GRW-open set \( G \) such that \( x \in G \subseteq N \). GRW-neighborhood can be denoted as GRW-nbd.

Definition 3.4.2: A subset \( N \) of space \( X \), is called as a GRW-nbd of \( A \subseteq X \) if and only if there exists a GRW-open set \( G \) such that \( A \subseteq G \subseteq N \).

Remark 3.4.3: The GRW-nbd \( N \) of \( x \in X \) need not be a GRW-open in \( X \).

Example 3.4.4: Let \( X = \{a, b, c, d\} \), \( \tau = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\} \). The set \( \{c, d\} \) is GRW-nbd, however, it is not a GRW-open set in \( X \).

Theorem 3.4.5: Every neighborhood \( N \) of a point \( x \in X \) of a topological space (\( X, \tau \)) is a GRW-nbd of \( X \).
Proof: Let $N$ be a nbd of a point $x \in X$. By the definition of GRW-nbd, there exists an open set $G$ such that $x \in G \subseteq N$. As every open set is GRW-open set and $x \in G \subseteq N$, $N$ is GRW- nbd of $x$.

Remark 3.4.6: In general, a GRW- nbd $N$ of a point $x \in X$ need not be a nbd of $x$ in $X$, as seen from the following example.

Example 3.4.7: Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. In this topological space $\{c, d\}$ is a GRW- nbd of $c$ but it is not a neighborhood of $c$.

Theorem 3.4.8: If a subset $N$ of a topological space $(X, \tau)$ is GRW-open, then $N$ is a GRW- nbd of each of its points.

Proof: Suppose $N$ is GRW-open. Let $x \in N$. Since $N$ is a GRW-open set such that $x \in N \subseteq N$. Since $x$ is an arbitrary point of $N$, it follows that $N$ is a GRW- nbd of each of its points.

Remark 3.4.9: The converse of the above theorem is not true as seen from the following example.

Example 3.4.10: Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. The set $\{c, d\}$ is GRW-nbd of the points $c$ and $d$, since the GRW-open sets $\{c\}$ and $\{d\}$ are such that $c \in \{c\} \subseteq \{c, d\}$ and $d \in \{d\} \subseteq \{c, d\}$. However, the set $\{c, d\}$ is not a GRW-open set in $X$.

Theorem 3.4.11: In a topological space $(X, \tau)$, if $F$ is a GRW-closed subset of $X$, and $x \in F^c$ then there exists a GRW-nbd $N$ of $x$ such that $N \cap F = \emptyset$.

Proof: Let $F$ be GRW-closed subset of $X$ and $x \in F^c$. Then $F^c$ is GRW-open set of $X$. So, by the theorem 3.4.8, $F^c$ is a GRW- nbd of each of its points.
Hence there exists a GRW-nbd $N$ of $x$ such that $x \in N \subseteq F^c$. That is $N \cap F = \emptyset$.

**Definition 3.4.12:** Let $x$ be a point in a topological space $(X, \tau)$. The set of all GRW-nbd of $x$ is called the GRW-nbd system at $x$, and is denoted by $\text{GRW-N}(x)$.

**Theorem 3.4.13:** Let $(X, \tau)$ be a topological space and for each $x \in X$, Let $\text{GRW-N}(x)$ be the collection of all GRW-nbd of $x$. Then,

(i) $\text{GRW-N}(x) \neq \emptyset$, $\forall x \in X$,

(ii) $N \in \text{GRW-N}(x) \Rightarrow x \in N$.

(iii) $N \in \text{GRW-N}(x)$, $M \supseteq N \Rightarrow M \in \text{GRW-N}(x)$.

(iv) $N \in \text{GRW-N}(x)$, $M \in \text{GRW-N}(x) \Rightarrow N \cap M \in \text{GRW-N}(x)$.

(v) $N \in \text{GRW-N}(x) \Rightarrow$ there exists $M \in \text{GRW-N}(x)$ such that $M \subseteq N$ and $M \in \text{GRW-N}(y)$ for every $y \in M$.

**Proof:**

(i) Since $X$ is a GRW-open set, it is a GRW-nbd of every $x \in X$. Hence there exists at least one GRW-nbd (namely - $X$) for each $x \in X$. Hence $\text{GRW-N}(x) \neq \emptyset$ for every $x \in X$.

(ii) If $N \in \text{GRW-N}(x)$, then $N$ is a GRW-nbd of $x$, so by the definition of GRW-nbd, $x \in N$.

(iii) Let $N \in \text{GRW-N}(x)$ and $M \supseteq N$. Then there is a GRW-open set $G$ such that $x \in G \subseteq N$. Since $N \subseteq M$, $x \in G \subseteq M$, so $M$ is GRW-nbd of $x$. Hence $M \in \text{GRW-N}(x)$. 

(iv) Let \( N \in \text{GRW-N}(x) \) and \( M \in \text{GRW-N}(x) \). By the definition of GRW-nbd, there exists GRW-open sets \( G_1 \) and \( G_2 \) such that \( x \in G_1 \subseteq N \) and \( x \in G_2 \subseteq M \). Hence \( x \in G_1 \cap G_2 \subseteq N \cap M \). Since \( G_1 \cap G_2 \) is a GRW-open set, it follows that \( N \cap M \) is a GRW-nbd of \( x \). Hence \( N \cap M \in \text{GRW-N}(x) \).

(v) If \( N \in \text{GRW-N}(x) \), then there exists a GRW-open set \( M \) such that \( x \in M \subseteq N \). Since \( M \) is a GRW-open set, it is GRW-nbd of each of its points. Therefore \( M \in \text{GRW-N}(y) \) for every \( y \in M \).

**Theorem 3.4.14:** Let \( X \) be a non-empty set, and for each \( x \in X \), let \( \text{GRW-N}(x) \) be a non-empty collection of subsets of \( X \) satisfying the following conditions.

(i) \( N \in \text{GRW-N}(x) \) implies \( x \in N \)

(ii) \( N \in \text{GRW-N}(x) \), \( M \in \text{GRW-N}(x) \) implies \( N \cap M \in \text{GRW-N}(x) \). Let \( \tau \) consists of the empty set and all those subsets of \( X \) of \( X \), having the property that \( x \in G \) implies that there exists a \( N \in \text{GRW-N}(x) \) such that \( x \in N \subseteq G \), then \( \tau \) is a topology for \( X \).

**Proof:** (1) \( \emptyset \in \tau \) by definition. (2) Let \( x \) be an arbitrary element of \( X \). Since \( \text{GRW-N}(x) \) is nonempty, there is an \( N \in \text{GRW-N}(x) \) and so \( x \in N \) by (i). Since \( N \) is a subset of \( X \), \( x \in N \subseteq X \). Hence \( X \in \tau \). (3) Let \( G_1 \in \tau \) and \( G_2 \in \tau \). If \( x \in G_1 \cap G_2 \) then, \( x \in G_1 \) and \( x \in G_2 \). Since \( G_1 \in \tau \) and \( G_2 \in \tau \), there exists \( N \in \text{GRW-N}(x) \) and \( M \in \text{GRW-N}(x) \), such that \( x \in N \subseteq G_1 \) and \( x \in M \subseteq G_2 \). Then \( x \in N \cap M \subseteq G_1 \cap G_2 \). But \( N \cap M \in \text{GRW-N}(x) \) by (2). Hence \( G_1 \cap G_2 \in \tau \). (4) Let \( G_\lambda \in \tau \) for every \( \lambda \in \Lambda \). If \( x \in \bigcup \{ G_\lambda : \lambda \in \Lambda \} \), then \( x \in G_\lambda \) for some \( \lambda \in \Lambda \).
Since $G_x \in \tau$, there exists an $N \in \text{GRW-N}(x)$ such that $x \in N \subset G_x$ and consequently $x \in N \subset \bigcup\{G_x : \lambda \in \Lambda\}$. Hence $\bigcup\{G_x : \lambda \in \Lambda\} \in \tau$. It follows that $\tau$ is a topology for $X$.

### 3.5 GRW-INTERIOR OPERATOR OF A SET IN A TOPOLOGICAL SPACE

In this section GRW-interior operator of a subset of the topological space $(X, \tau)$ is defined and some of its properties are analyzed.

**Definition 3.5.1:** Let $A$ be a subset of the topological space $(X, \tau)$. A point $x \in A$ is said to be GRW-interior point of $A$, if $A$ is a GRW-nbd of point $x$. The set of all GRW-interior points of $A$ is denoted by $\text{GRW-int}(A)$.

**Theorem 3.5.2:** If $A$ is a subset of a topological space $(X, \tau)$, then, $\text{GRW-int}(A) = \bigcup\{G : G \text{ is GRW-open, } G \subseteq A\}$.

**Proof:** Let $A$ be a subset of $X, x \in \text{GRW-int}(A)$.

\[\iff x \text{ is a GRW-interior point of } A.\]

\[\iff A \text{ is a GRW-nbd of point } x.\]

\[\iff \text{there exists GRW-open set } G \text{ such that } x \in G \subseteq A.\]

\[\iff x \in \bigcup\{G : G \text{ is GRW-open, } G \subseteq A\}. \text{Hence } \text{GRW-int}(A) = \bigcup\{G : G \text{ is GRW-open, } G \subseteq A\}.\]

**Theorem 3.5.3:** Let $A$ and $B$ be subsets of the topological space $(X, \tau)$. Then

(i) $\text{GRW-int}(X) = X$ and $\text{GRW-int}(\emptyset) = \emptyset$. 


(ii) \( \text{GRW-int}(A) \subseteq A \).

(iii) If \( B \) is any GRW-open set contained in \( A \), then \( B \subseteq \text{GRW-int}(A) \).

(iv) If \( A \subseteq B \), then \( \text{GRW-int}(A) \subseteq \text{GRW-int}(B) \).

(v) \( \text{GRW-int} (\text{GRW-int}(A)) = \text{GRW-int}(A) \).

Proof:

(i) Since \( X \) and \( \phi \) are GRW-open sets, by the theorem 3.5.2.

\[
\text{GRW-int}(X) = \bigcup \{G : G \text{ is GRW-open, } G \subseteq X\} = X \cup \{\text{all GRW-open sets}\} = X.
\]

That is \( \text{GRW-int}(X) = X \). Since \( \phi \) is the only GRW-open set contained in \( \phi \), \( \text{GRW-int}(\phi) = \phi \).

(ii) Let \( x \in \text{GRW-int}(A) \) implies that \( x \) is a GRW-interior point of \( A \), that is \( A \) is a GRW-nbd of \( x \), hence \( x \in A \). Thus \( x \in \text{GRW-int}(A) \) implies \( x \in A \). Hence \( \text{GRW-int}(A) \subseteq A \).

(iii) Let \( B \) be any GRW-open sets such that \( B \subseteq A \). Let \( x \in B \), since \( B \) is a GRW-open set contained in \( A \), \( x \) is a GRW-interior point of \( A \). That is \( x \in \text{GRW-int}(A) \). Hence \( B \subseteq \text{GRW-int}(A) \).

(iv) Let \( A \) and \( B \) be subsets of \( X \) such that \( A \subseteq B \). Let \( x \in \text{GRW-int}(A) \). Then \( x \) is a GRW-interior point of \( A \) and so \( A \) is a GRW-nbd of \( x \). Since \( B \supseteq A \), \( B \) is also a GRW-nbd of \( x \). This implies that \( x \in \text{GRW-int}(B) \). Thus \( x \in \text{GRW-int}(A) \) implies \( x \in \text{GRW-int}(B) \), (ie) \( \text{GRW-int}(A) \subseteq \text{GRW-int}(B) \).
(v) From (ii) and (iv) $\text{GRW-int}(\text{GRW-int}(A)) \subseteq \text{GRW-int}(A)$. Let $x \in \text{GRW-int}(A)$, this implies $A$ is a neighborhood of $x$, so there exists a GRW-open set $G$ such that $x \in G \subseteq A$, every element of $G$ is a GRW-interior of $A$, hence $x \in G \subseteq \text{GRW-int}(A)$, which means that $x \in \text{GRW-int}(A)$ that is $\text{GRW-int}(\text{GRW-int}(A)) \subseteq \text{GRW-int}(\text{GRW-int}(A))$. That is $\text{GRW-int}(\text{GRW-int}(A)) = \text{GRW-int}(A)$.

**Theorem 3.5.4:** If a subset $A$ of the topological space $(X, \tau)$ is GRW-open then, $\text{GRW-int}(A) = A$.

**Proof:** Let $A$ be GRW-open subset of $X$. $\text{GRW-int}(A) \subseteq A$. Also, $A$ is GRW-open set contained in $A$. From the theorem 3.5.2. (iii) $A \subseteq \text{GRW-int}(A)$. Hence $\text{GRW-int}(A) = A$.

**Remark 3.5.5:** The converse of the above theorem need not be true, as seen from the following example.

**Example 3.5.6:** Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. $\text{GRW-int}(\{a, d\}) = \{a\} \cup \{d\} = \{a, d\}$, but $\{a, d\}$ is not a GRW-open set in $X$.

**Theorem 3.5.7:** If $A$ and $B$ are subsets of the topological space $(X, \tau)$, then $\text{GRW-int}(A) \cup \text{GRW-int}(B) \subseteq \text{GRW-int}(A \cup B)$.

**Proof:** For any two sub sets $A$ and $B$ of $X$, $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By the theorem 3.5.3.(iv), $\text{GRW-int}(A) \subseteq \text{GRW-int}(A \cup B)$ and $\text{GRW-int}(B) \subseteq \text{GRW-int}(A \cup B)$. Hence $\text{GRW-int}(A) \cup \text{GRW-int}(B) \subseteq \text{GRW-int}(A \cup B)$.

**Theorem 3.5.8:** If $A$ and $B$ are subsets of the topological space $(X, \tau)$, then $\text{GRW-int}(A \cap B) = \text{GRW-int}(A) \cap \text{GRW-int}(B)$. 
**Proof:** For any sub sets A and B of X, A ∩ B ⊆ A and A ∩ B ⊆ B. By the theorem 3.5.3 (iv), GRW-int (A ∩ B) ⊆ GRW-int (A) and GRW-int (A ∩ B) ⊆ GRW-int (B). This implies that GRW-int (A ∩ B) ⊆ GRW-int (A) ∩ GRW-int (B). Let x ∈ GRW-int (A) ∩ GRW-int (B) then, x ∈ GRW-int (A) and x ∈ GRW-int (B). Hence x is a GRW-interior point of A as well as B. It follows that whenever A and B are GRW-nbds of x, A ∩ B is also a GRW-nbd of x. Hence x ∈ GRW-int (A ∩ B). Thus x ∈ GRW-int (A) ∩ GRW-int (B) implies that x ∈ GRW-int (A ∩ B). Therefore, GRW-int (A) ∩ GRW-int (B) ⊆ GRW-int (A ∩ B) and hence, GRW-int (A ∩ B) = GRW-int (A) ∩ GRW-int (B).

**Theorem 3.5.9:** If A is a subset of the topological space (X, τ), then int(A) ⊆ GRW-int (A).

**Proof.** Let A be a subset of a space X and x ∈ int(A), this implies that x ∈ ∪{G : G is open, G ⊆ A}, that is, there exists an open set G such that x ∈ G ⊆ A, that is there exist a GRW-open set G such that x ∈ G ⊆ A, as every open set is a GRW-open set in X. That is, x ∈ ∪ {G: G is GRW-open, G ⊆ A}, this implies that x ∈ GRW-int (A). Thus x ∈ int (A), implies x∈GRW-int(A). Hence int (A) ⊆ GRW-int (A).

**Remark 3.5.10:** The converse of the above theorem need not be true from the following example.

**Example 3.5.11:** Let X = {a, b, c, d} with the topology τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X} and int{c} = ϕ but, GRW-int{c} = {c} in X.

**Theorem 3.5.12:** If A is a subset of the topological space (X, τ), then w-int(A) ⊆ GRW-int(A), where w-int(A) is given by w-int(A) = ∪{G : G is a w-open, G ⊆ A}. 
Proof: Let \( A \) be a subset of a topological space \( X \). Let \( x \in w\text{-}int(A) \) this implies that \( x \in \bigcup \{ G \subseteq X : G \text{ is a w-open, } G \subseteq A \} \), (ie)there exists a w-open set \( G \) such that \( x \in G \subseteq A \) (ie)there exists a GRW- open set \( G \) such that, \( x \in G \subseteq A \), as every w-open set is a GRW-open set in \( X \). Hence \( x \in GRW\text{-}int(A) \). Thus \( x \in w\text{-}int(A) \) implies \( x \in GRW\text{-}int (A) \). Hence \( w\text{-}int (A) \subseteq GRW\text{-}int (A) \).

Remark 3.5.13: The containment relation in the above theorem 3.5.12 may be proper as seen from the following example.

Example 3.5.14: Let \( X = \{a, b, c\} \) with topology \( \tau = \{\emptyset, \{a\}, X\} \). Then \( GRW\text{-}int(\{a, b\}) = \{a, b\} \) and \( w\text{-}int (\{a, b\}) = \{a\} \). It follows that \( w\text{-}int (A) \subseteq GRW\text{-}int (A) \) and \( w\text{-}int(A) \neq GRW\text{-}int (A) \).

3.6 GRW-CLOSURE OPERATOR IN A TOPOLOGICAL SPACE

In this section GRW-closure operator of a subset of a topological space \((X, \tau)\) is defined and some of its properties are studied.

Definition 3.6.1: Let \( A \) be a subset of the topological space \((X, \tau)\). The GRW-closure of \( A \) is defined as the intersection of all GRW-closed sets containing \( A \). GRW-cl(\( A \)) = \( \bigcap \{ F : A \subseteq F \in GRWC(X) \} \). GRW-cl (\( A \)) is denoted as \( cl^{GRW}(A) \).

Definition 3.6.2: A topological space \((X, \tau)\) is called GRWC-space if every GRW-closed set in \( X \) is closed in \( X \). It is denoted by \( T_{GRW}\text{-}space \).

Theorem 3.6.3: If \( A \) and \( B \) are subsets of a topological space \((X, \tau)\). Then
(i) $\text{cl}^{\text{GRW}}(X) = X$ and $\text{cl}^{\text{GRW}}(\emptyset) = \emptyset$.

(ii) $A \subseteq \text{cl}^{\text{GRW}}(A)$.

(iii) If $B$ is any GRW-closed set containing $A$, then $\text{cl}^{\text{GRW}}(A) \subseteq B$.

(iv) If $A \subseteq B$ then $\text{cl}^{\text{GRW}}(A) \subseteq \text{cl}^{\text{GRW}}(B)$.

(v) $\text{cl}^{\text{GRW}}(A) = \text{cl}^{\text{GRW}}(\text{cl}^{\text{GRW}}(A))$.

**Proof.**

(i) By the definition of $\text{cl}^{\text{GRW}}(A)$, $X$ is the only GRW-closed set containing $X$. Therefore $\text{cl}^{\text{GRW}}(X)$ is the intersection of all the GRW-closed sets containing $X$. That is, $\text{cl}^{\text{GRW}}(X) = \cap \{ X \} = X$. (ie) $\text{cl}^{\text{GRW}}(X) = X$. $\text{cl}^{\text{GRW}}(\emptyset)$ is the intersection of all the GRW-closed sets containing $\emptyset$, (ie) $\text{cl}^{\text{GRW}}(\emptyset) = \emptyset$.

(ii) By the definition of $\text{cl}^{\text{GRW}}(A)$, $A \subseteq \text{cl}^{\text{GRW}}(A)$.

(iii) Let $B$ be any GRW-closed set containing $A$. Since $\text{cl}^{\text{GRW}}(A)$ is the intersection of all GRW-closed sets containing $A$, $\text{cl}^{\text{GRW}}(A)$ is contained in every GRW-closed set containing $A$. Hence in particular $\text{cl}^{\text{GRW}}(A) \subseteq B$.

(iv) Let $A$ and $B$ be subsets of $X$ such that $A \subseteq B$. By the definition of $\text{cl}^{\text{GRW}}$, $\text{cl}^{\text{GRW}}(B) = \cap \{ F : B \subseteq F \in \text{GRWC}(X) \}$. If $B \subseteq F \in \text{GRWC}(X)$, then $\text{cl}^{\text{GRW}}(B) \subseteq F$. Since $A \subseteq B$, $A \subseteq F \in \text{GRWC}(X)$, $\text{cl}^{\text{GRW}}(A) \subseteq F$. Therefore $\text{GRW-cl}(A) \subseteq \cap \{ F : B \subseteq F \in \text{GRWC}(X) \} = \text{cl}^{\text{GRW}}(B)$. That is $\text{cl}^{\text{GRW}}(A) \subseteq \text{cl}^{\text{GRW}}(B)$. 

(v) Let $A$ be any subset of a topological space $X$, then,
$$\text{cl}^{GRW}(A) = \bigcap \{ F : A \subseteq F \in \text{GRWC}(X) \}. $$ If $A \subseteq F \in \text{GRWC}(X)$ then, $\text{cl}^{GRW}(A) \subseteq F$. Since $F$ is GRW-closed set containing $\text{cl}^{GRW}(A)$, hence $\text{GRW-cl}(\text{GRW-cl}(A)) \subseteq \bigcap \{ F : A \subseteq F \in \text{GRWC}(X) \} = \text{cl}^{GRW}(A).$ (ie) $\text{cl}^{GRW}(\text{cl}^{GRW}(A)) = \text{cl}^{GRW}(A).

**Theorem 3.6.4:** Let $(X, \tau)$ be a topological space. If $A \subseteq X$ is a GRW-closed then $\text{cl}^{GRW}(A) = A$.

**Proof.** Let $A$ be a GRW-closed subset of $X$. By the definition of $\text{cl}^{GRW}(A), A \subseteq \text{cl}^{GRW}(A)$. Also $A \subseteq A$ and $A$ is GRW-closed. By the theorem 3.6.3 (iii) $\text{cl}^{GRW}(A) \subseteq A$. Hence $\text{cl}^{GRW}(A) = A$.

**Remark 3.6.5:** The converse of the theorem need not be true as seen from the following example.

**Example 3.6.6:** Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. $\text{cl}^{GRW}(A) = \text{cl}^{GRW}(\{a, c\}) = \{a, b, c\} \cap \{a, c, d\} = \{a, c\}$, but $\{a, c\}$ is not a GRW-closed set.

**Theorem 3.6.7:** If $A$ and $B$ are subsets of a topological space $(X, \tau)$, then $\text{cl}^{GRW}(A \cap B) \subseteq \text{cl}^{GRW}(A) \cap \text{cl}^{GRW}(B)$.

**Proof:** Let $A$ and $B$ be subsets of $X$. Clearly $A \cap B \subseteq A$ and $A \cap B \subseteq B$. By the theorem 3.6.3 (iv), $\text{cl}^{GRW}(A \cap B) \subseteq \text{cl}^{GRW}(A)$ and $\text{cl}^{GRW}(A \cap B) \subseteq \text{cl}^{GRW}(B)$. Hence $\text{cl}^{GRW}(A \cap B) \subseteq \text{cl}^{GRW}(A) \cap \text{cl}^{GRW}(B)$.

**Theorem 3.6.8:** If $A$ and $B$ are subsets of a topological space $(X, \tau)$, then $\text{cl}^{GRW}(A \cup B) = \text{cl}^{GRW}(A) \cup \text{cl}^{GRW}(B)$.
Proof: Let $A$ and $B$ be any two subsets of $X$. Clearly $A \subset A \cup B$ and $B \subset A \cup B$. Hence $\text{cl}^{\text{GRW}}(A) \cup \text{cl}^{\text{GRW}}(B) \subset \text{cl}^{\text{GRW}}(A \cup B)$. Now to prove that $\text{cl}^{\text{GRW}}(A \cup B) \subset \text{cl}^{\text{GRW}}(A) \cup \text{cl}^{\text{GRW}}(B)$. Let $x \in \text{cl}^{\text{GRW}}(A \cup B)$ and suppose $x \notin \text{cl}^{\text{GRW}}(A) \cup \text{cl}^{\text{GRW}}(B)$ then there exists GRW-closed sets $A_1$ and $B_1$ with $A \subset A_1$, $B \subset B_1$ and $x \notin A_1 \cup B_1$. $A \cup B \subset A_1 \cup B_1$ and $A_1 \cup B_1$ is GRW-closed set by the theorem 3.2.2 such that $x \notin A_1 \cup B_1$. Thus $x \notin \text{cl}^{\text{GRW}}(A \cup B)$ which is a contradiction to $x \in \text{cl}^{\text{GRW}}(A \cup B)$. Hence $\text{cl}^{\text{GRW}}(A \cup B) \subset \text{cl}^{\text{GRW}}(A) \cup \text{cl}^{\text{GRW}}(B)$. Therefore, $\text{cl}^{\text{GRW}}(A \cup B) = \text{cl}^{\text{GRW}}(A) \cup \text{cl}^{\text{GRW}}(B)$.

Theorem 3.6.9: Let $(X, \tau)$ be a topological space. For an element $x \in X$, $x \in \text{cl}^{\text{GRW}}(A)$ if and only if $V \cap A \neq \emptyset$, for every GRW-open set $V$ containing $x$.

Proof: Let $x \in V$ and $x \notin \text{cl}^{\text{GRW}}(A)$. To prove that $V \cap A \neq \emptyset$ for every GRW-open set $V$ containing $x$. Suppose there exists a GRW-open set $V$ containing $x$ such that $V \cap A = \emptyset$. Then $A \subset X - V$ and $X - V$ is GRW-closed. $\text{cl}^{\text{GRW}}(A) \subset X - V$. This shows that $x \notin \text{cl}^{\text{GRW}}(A)$, which is a contradiction. Hence $V \cap A \neq \emptyset$ for every GRW-open set $V$ containing $x$. Conversely, let $V \cap A \neq \emptyset$ for every GRW-open set $V$ containing $x$. To prove $x \notin \text{cl}^{\text{GRW}}(A)$, suppose $x \notin \text{cl}^{\text{GRW}}(A)$. Then there exists a GRW-closed subset $F$ containing $A$ such that $x \notin F$, which implies that $x \in X - F$ and $X - F$ is GRW-open. Also $(X - F) \cap A = \emptyset$, which is a contradiction to $V \cap A \neq \emptyset$ for every GRW-open set $V$ containing $x$. Hence $x \notin \text{cl}^{\text{GRW}}(A)$.

Theorem 3.6.10: If $A$ is a subset of the topological space $(X, \tau)$, then $\text{cl}^{\text{GRW}}(A) \subset \text{cl}(A)$ and $\text{cl}^{\text{GRW}}(A) \subset \text{cl}^\star(A)$.

Proof: Let $A$ be a subset of a space $X$. By the definition of closure, $\text{cl}(A) = \bigcap \{ F \subset X : A \subset F \ \text{and} \ C(X) \}$. If $A \subset F \ \text{and} \ C(X)$, then $A \subset F \ \text{and} \ \text{GRWC}(X)$, because every closed set is GRW-closed. That is $\text{cl}^{\text{GRW}}(A) \subset F$. Therefore $\text{cl}^{\text{GRW}}(A) \subset$
∩ {F ⊆ X : A ⊆ F ∈ C(X)} = cl(A). Hence cl^{GRW}(A) ⊆ cl(A).cl^*(A) = ∩ {F ⊆ X : A ⊆ F ∈ GC(X)}. If A ⊆ F ∈ GC(X) then, A ⊆ F. GRW C(X), because every g-closed set is GRW-closed. (ie) cl^{GRW}(A) ⊆ F. Therefore cl^{GRW}(A) ⊆ ∩ {F ⊆ X : A ⊆ F ∈ GC(X)} = cl^*(A). Hence cl^{GRW}(A) ⊆ cl^*(A).

Remark 3.6.11: The containment relation in the above theorem 3.6.10, may be proper as seen from following example.

Example 3.6.12: Let X = {a, b, c} with topology τ = {X, φ, {a}, {a, b}}. Then GRW-cl({a}) = {a} and cl^*({a}) = X. It follows that cl^{GRW}({a}) ⊆ cl^*({a}) and cl^{GRW}({a}) ≠ cl^*({a}).

Theorem 3.6.13: If A is subset of a topological space (X, τ ) then, cl^{GRW}(A) ⊆ w-cl(A), where w-cl(A) is given by w-cl(A) = ∩ {F ⊆ X : A ⊆ F and F is w-closed set in X}.

Proof: Let A be a subset of X. By definition, w-cl(A) = ∩ {F ⊆ X : A ⊆ F and F is w-closed set in X}. If A ⊆ F and F is w-closed subset of X then, A ⊆ F GRW C(X), because every w-closed is GRW-closed subset in X. That is cl^{GRW}(A) ⊆ F. Therefore cl^{GRW}(A) ⊆ ∩ {F ⊆ X : A ⊆ F and F is w-closed} = w-cl(A). Hence GRW-cl(A) ⊆ w-cl(A).

Remark 3.6.14: The containment relation in the above theorem 3.6.13 may be proper as seen from following example.

Example 3.6.15: Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Let A = {a}. Then GRW-cl(A) = {a} and w-cl(A) = {a, c}. That is cl^{GRW}(A) ⊆ w-cl(A) and cl^{GRW}(A) ≠ w-cl(A).
Definition 3.6.16: Let $\tau_{GRW}$ be the topology on $X$ generated by $cl^{GRW}$. That is $\tau_{GRW}= \{ U \subseteq X : cl^{GRW}(U^c) = U^c \}$.

Theorem 3.6.17: For any topology $\tau$ on $X$, $\tau \subseteq \tau_w \subseteq \tau_{GRW}$, where $\tau_w=\{ U \subseteq X : w-cl(U^c) = U^c \}$

Proof: $\tau \sqsubset \tau_w$, since $\tau$-closed sets are $\tau_w$-closed. To prove $\tau_w \subseteq \tau_{GRW}$.

Let $U \sqsubset \tau_w$ which implies $w-cl(U^c) = U^c$, it follows that $U^c$ is a $w$-closed set. Now $U^c$ is GRW-closed, as every $w$-closed set is GRW-closed and so $cl^{GRW}(U^c) = U^c$. That is $U \sqsubset \tau_{GRW}$ and so $\tau_w \subseteq \tau_{GRW}$. Hence $\tau \subseteq \tau_w \subseteq \tau_{GRW}$.

Theorem 3.6.18: Let $A$ be any subset of $X$. Then, (i) $(GRW-int(A))^c = cl^{GRW}(A^c)$ (ii) $GRW-int(A) = (cl^{GRW}(A^c))^c$ (iii) $cl^{GRW}(A) = (cl^{GRW}(int(A^c))^c$.

Proof: Let $x \not\in (GRW-int(A))^c$. Then $x \not\in GRW-int(A)$. That is every GRW-open set $U$ containing $x$ is such that $U \not\subset A$. That is every GRW-open set $U$ containing $x$ is such that $U \cap A^c \neq \emptyset$, so $x \not\in cl^{GRW}(A^c)$, therefore $(GRW-int(A))^c \subseteq GRW-cl(A^c)$. Conversely, let $x \not\in cl^{GRW}(A^c)$ then, by the theorem 3.6.9, every GRW-open set $U$ containing $x$ is such that $U \cap A^c \neq \emptyset$. That is every GRW-open set $U$ containing $x$ is such that $U \not\subset A$, this implies by the definition of $GRW-int(A)$, $x \not\in GRW-int(A)$. That is $x \not\in \cap (GRW-int(A))^c$ and hence $GRW-cl(A^c) \subseteq (GRW-int(A))^c$. (GRW-int(A))^c = cl^{GRW}(A^c).(ii) Follows by taking complements in (i).(iii) Replacing $A$ by $A^c$ in (i), $cl^{GRW}(A) = (cl^{GRW}(int(A^c))^c$.

3.7 SEPARATION AXIOMS - GRW - $T_1$ SPACES

In this section separation axioms on GRW-closed sets are discussed and some of their properties are studied.
**Definition 3.7.1**: Let \((X, \tau)\) be a topological space and it is said to be:

(i) \(\text{GRW-}T_0\), if for each pair of points \(x, y \in X\), \(x \neq y\), there is a \(\text{GRW-}\) open set containing one of the points, but not the other one.

(ii) \(\text{GRW-}T_1\), if for each pair of distinct points \(x, y \in X\) there exist a pair of \(\text{GRW-}\) open sets, one of them containing \(x\) but not \(y\) and the other one containing \(y\) but not \(x\).

(iii) \(\text{GRW-}T_2\), if for each pair of distinct points \(x, y \in X\) there exist disjoint \(\text{GRW-}\) open sets \(U\) and \(V\), in \(X\) such that \(x \in U\) and \(y \in V\).

(iv) \(T_{GRW}\) if ever \(\text{GRW-}\) closed set is a closed set in \(X\).

**Theorem 3.7.2**: Let \((X, \tau)\) be a topological space. Then \(X\) is a \(\text{GRW-}T_0\) space if and only if for any \(x, y \in X\) such that \(x \neq y\), \(\text{cl}^{GRW}(\{x\}) \neq \text{cl}^{GRW}(\{y\})\).

**Proof**: Suppose that \(X\) is a \(\text{GRW-}T_0\) space, then for any pair of distinct points \(x, y \in X\) there exists a \(\text{GRW-}\) open set \(U\), such that \(x \in U\) and \(y \notin U\) or \(y \in U\) and \(x \notin U\). It follows that \(\text{cl}^{GRW}(\{x\}) \neq \text{cl}^{GRW}(\{y\})\). Suppose that \(x, y \in X\) and \(x \neq y\), then \(\text{cl}^{GRW}(\{x\}) \neq \text{cl}^{GRW}(\{y\})\). It follows that, given \(x \neq y\), there is a point \(z \in X\) such that \(z \in \text{cl}^{GRW}(\{y\})\) and \(z \notin \text{cl}^{GRW}(\{x\})\) or \(z \in \text{cl}^{GRW}(\{x\})\) and \(z \notin \text{cl}^{GRW}(\{y\})\). If \(z \in \text{cl}^{GRW}(\{y\})\) and \(z \notin \text{cl}^{GRW}(\{x\})\), there exist a \(\text{GRW-}\) open set \(V\) such that \(y \in V\) and \(V \cap \{x\} = \phi\); If \(z \in \text{cl}^{GRW}(\{x\})\) and \(z \notin \text{cl}^{GRW}(\{y\})\), there exist a \(\text{GRW-}\) open set \(U\) such that \(x \in U\) and \(U \cap \{y\} = \phi\); This shows that \(X\) is \(\text{GRW-}T_0\) space.

**Theorem 3.7.3**: Let \((X, \tau)\) be a topological space. For the topological space \((X, \tau)\), the followings conditions are equivalent:
(a) X is a GRW- $T_1$ space.

(b) Each singleton set $\{x\}$, $x \in X$, is a GRW- closed set.

(c) Each subset of X is the intersection of all super GRW- open sets containing it.

**Proof:** (a) implies (b). Let X be a GRW- $T_1$ space. Given $y \in X - \{x\}$, then $y \neq x$, by hypothesis there are GRW- open sets $U, V \subseteq X$ such that $x \in U$, $y \notin U$, and $y \in V, x \notin V$. Therefore, $y \in V \cap X - \{x\}$, since $V \cap \{x\} = \emptyset$; It follows that $X - \{x\}$ is a GRW-open set and, therefore, $\{x\}$ is a GRW-closed set. (b) Implies (c). Suppose that each $\{x\}, x \in X$, is a GRW- closed set. Given $A \subseteq X$ and define the set $D(A)$ as follows: $D(A) = \bigcap \{S : A \subseteq S \text{ and } S \text{ is a GRW-open set} \}$. It will be proved that $A = D(A)$. In general, $A \subseteq D(A)$. Suppose that $x \notin A$. Then $A \subseteq X - \{x\}$ and $X - \{x\}$ is a GRW-open because $\{x\}$ is GRW-closed. Therefore, $x \notin D(A)$ and hence $D(A) \subseteq A$. Consequently $A = D(A)$. (c) implies (a) let $x, y \in X$ and $y \neq x$, by the assumption $\{x\} = \bigcap \{S : \{x\} \subseteq S \text{ and } S \text{ is a GRW-open set} \}$, hence there exists GRW-open sets $U$ and $V$ such that $x \in U$, $y \notin U$, and $y \in V, x \notin V$, which implies X is a GRW-$T_1$ space.

**Theorem 3.7.4:** Every $T_{GRW}$- topological Space is a $T_{\frac{1}{2}}$-space.

**Proof:** Let $(X, \tau)$ be a $T_{GRW}$- topological space. And let $A$ be a g-closed set in $X$. Every g-closed set is GRW-closed set and hence it is closed in $X$ so, $X$ is a $T_{\frac{1}{2}}$-space.

**Remark 3.7.5:** The converse of the above theorem need not be true from the following example.
**Example 3.7.6:** The topological space \((X, \tau)\), where \(X=\{\phi, \{a\}, \{a, b\}, X\}\) is \(T_{\frac{1}{2}}\), but not \(T_{GRW}\)-space, since \(\{b\}\) is GRW-closed but not closed in \(X\).

**Theorem 3.7.7:** Every \(T_0\), \(T_1\), and \(T_2\) Spaces is respectively \(GRW-T_0\), \(GRW-T_1\), \(GRW-T_2\) spaces.

**Proof:** Since every open set in \(X\) is GRW-open, proof follows from the definitions.

**Remarks 3.7.8:** The converse of the above theorem need not be true.

**Example 3.7.9:** The topological space \((X, \tau)\), where \(X=\{a, b, c\}, \tau=\{\{a, b\}\), \(X, \phi\}\) is \(GRW-T_0\)-space, but it is not \(T_0\)-space.

**Example 3.7.10:** The topological space \((X, \tau)\), where \(X=\{a, b, c\}, \tau=\{\phi, \{a, b\}, \{c, d\}, X\}\), is \(GRW-T_1\)-space, but it is not \(T_1\)-space.

**Example 3.7.11:** The topological space \((X, \tau)\), where \(X=\{a, b, c\}, \tau=\{\phi, \{a, b\}, X, \phi\}\), is \(GRW-T_2\)-space, but it is not \(T_2\)-space.

**Remark 3.7.12:** The implications shown in Figure 3.3 are true from the above discussions.

**Figure 3.3 Relationships of GRW-\(T_i\)-spaces**
**Theorem 3.7.13** Let \((X, \tau)\) be a topological spaces. \(X\) is \(T_{GRW}\)-space if and only if for every \(x \in X\), \(\{x\}\) is open or \(\{x\}^c\) is RSO and \(cl^*(A) = A\) in \(X\).

**Proof:** (Necessity) Let \(X\) be \(T_{GRW}\)-space. For every \(x \in X\), the set \(X-\{x\}\) is GRW-closed in \(X\) or RSO in \(X\) by the theorem 3.2.8. Case (i) If \(X-\{x\}\) is GRW-closed, then \(\{x\}^c\) is GRW-closed, hence \(\{x\}\) is open. Case (ii) If \(X-\{x\}\) is RSO. Since every g-closed set is GRW-closed and \(X\) is \(T_{GRW}\)-space, \(cl^*(A) = cl(A)\).

(Sufficiency) Suppose for every \(x \in X\), \(\{x\}\) is open or \(\{x\}^c\) is RSO in \(X\) and \(cl^*(A) = cl(A)\), if \(A\) is a GRW-closed set in \(X\), if \(x \in cl(A)\), then \(x \in A\) whenever \(\{x\}\) is open. If \(\{x\}^c\) is RSO and \(x \notin A\), then \(A \subseteq X-\{x\}\) and \(cl^*(A) \subseteq X-\{x\}\), since \(A\) is GRW-closed in \(X\), but \(cl(A) = cl^*(A)\) is open, which contradicts the fact that \(x \in cl(A)\), hence \(x \in A\), \(cl(A) = A\) and \(A\) is closed in \(X\).

### 3.8 CONCLUSION

The concept of GRW-closed sets in topological spaces has been introduced, their relationships with few other existing sets are studied further, some separation axioms are defined and some of their properties have been investigated. A closure operator of a set in a topological space denoted by \(cl^{GRW}\) is defined and a topological space using GRW-closed sets is introduced. In chapter 4 the set defined in this chapter is used to study some generalized continuity, closed, open, and irresolute and homeomorphism functions in topological spaces.