CHAPTER 2

LITERATURE SURVEY

2.1 REVIEW OF LITERATURE

In 1965, Zadeh initiated the development of the modified set theory known as fuzzy set theory, which is a tool that makes possible the description of vague notions and manipulations with them. The concept of fuzzy derivative was first introduced by Chang & Zadeh (1972). Kandal & Byatt (1978) first introduced FDE and applied the concept of FDE to the analysis of dynamical problems ((Kandal 1980), Kandal & Byatt (1980)). Using Zadeh’s extension principle, Dubois & Prade (1982) defined the differentiation of ordinary functions at a fuzzy point and differentiation of fuzzy-valued functions at a non-fuzzy point. Puri & Ralescu (1983) proposed two definitions for the differential of fuzzy function using Radstrom embedding theorem, the first is based on the H-difference and is limited to a convex cone; the second is given over a whole Banach space. Goetschel & Voxman (1986) defined differentiation and integration of fuzzy-valued functions in ways that parallel closely the corresponding definitions for real differentiation and integration. Kaleva (1987) studied the differentiability and integrability concepts of fuzzy-set-valued mappings of a real variable whose values are normal, convex, upper semi continuous and compactly supported fuzzy sets in $\mathbb{R}^n$ and have stated conditions on existence and uniqueness of solution for FDE.
Seikkala (1987) considered the initial value problem \( y'(t) = f(t, y(t)), y(0) = y_0 \), with fuzzy initial value and with deterministic or fuzzy function \( f \). The author treated the fuzzy initial value problem (FIVP) with two different approaches. First the extension principle of Zadeh and the concept of fuzzy derivative are applied and showed that the fuzzy initial value problem has a unique fuzzy solution when \( f \) satisfies a generalized Lipschitz condition which guarantees a unique solution to the deterministic initial value problem. The second approach solves the problem by using minimal and maximal solutions. The local existence result analogous to Peano’s theorem is not valid for fuzzy differential equations, since \((E^n, d)\) is a metric space, which is not locally compact and hence mere continuity of \( f \) in equation (1.3) is not sufficient to guarantee local existence as in finite dimensions. Kaleva (1990b) studied the Cauchy problem for FDEs and showed that the Cauchy problem has a solution if and only if the subspace of normal, convex, upper semi continuous and compactly supported fuzzy sets in \( R^n \) is locally compact. However, if \( f \) is continuous and bounded, it can be proved an existence result (Nieto 1999). Wu (1988), Ouyang & Wu (1989), Kloden (1991), Congxin Wu et al (1996), Zhang Yue et al (1998) studied the fuzzy differential equation and the Cauchy problem.

But initial value problems studied through the concept of FDEs have many solutions that have an increasing length of support as the independent variable increases. Moreover, different formulations of the same FDE might lead to different solutions. According to Diamond (2000), the approach based on the Hukuhara derivative does not produce the variety of behaviours as in the case of ordinary differential equations. This shortcoming has been alleviated by Hullermeier (1997), who studied a FDE as a family of differential inclusions. But, the main shortcoming of Hullermeier’s approach is that it does not include a “fuzzification” of the differential operator. Bede et al(2007) also claim that the solution of a FDE is not necessarily a fuzzy
interval-valued function and have shown that in some situations, the approach based on Hullermeier’s interpretation also yields different solutions. The third interpretation was suggested by Buckley and Feuring (2000), who applied the extension principle to the crisp solution of ordinary differential equations (ODEs) in order to obtain a solution in fuzzy settings. Their work is important as it overcomes the existence of multiple definitions of the derivative of fuzzy functions. Moreover, a more general family of FDEs is faced from an analytical point of view.

Bede & Gal (2005) introduced a new concept of fuzzy derivatives called the generalized differentiability of fuzzy interval-valued functions. In this setting, the solution of a FDE may have a decreasing length of support as the independent variable increases. However, it depends on the selection of the fuzzy derivatives. Moreover, different formulations of the same FDE will lead to different solutions as well. Therefore, the uniqueness is not ensured. Bede (2008) presented characterization theorems for the solutions of FDE under the Hukuhara differentiability by an equivalent system of ODEs. So, in order to obtain numerical solutions of FDEs under Hukuhara differentiability, it is not necessary to rewrite the whole literature on numerical solutions of ODEs in the fuzzy setting, but instead any numerical method for the ODEs can directly be applied.

This generalisation was further studied by Chalco-Cano & Roman-Flores (2008). Chalco-Cano & Roman-Flores (2008) introduced the concept of lateral fuzzy H-derivatives for fuzzy mappings which is the enlargement of the class of differentiable fuzzy mappings. Subsequently, by using the lateral H-derivatives, there are two different interpretations of a fuzzy differential equation generating new solutions for a fuzzy differential equation. Chalco-Cano & Roman-Flores (2009) studied the class of FDEs where the dynamics
is given by a continuous fuzzy mapping which is obtained via Zadeh’s extension principle and obtained fuzzy solution for this class of FDEs.

Recently many research papers are focused on numerical solutions of FIVPS. Numerical Solutions of fuzzy differential equations has been introduced by Ming et al (1999) through Euler method. The authors used the embedding approach (Goetschel & Voxman (1986) and Wu Congxin & Ma Ming (1991)) and replaced the FDE in Banach space by two parametric ODEs. The new system which consists of two classic ODEs with initial conditions was solved numerically and showed that the numerical solution converges to the unique solution in Banach space. Although the work of Ming et al (1999) is significant, it has the disadvantage that, when examining the convergence of their Euler method, the authors practically work on the convergence of the ODEs system that occurs when solving numerically.

Javad Shokri (2006) used Modified Euler’s method to find the numerical solution of first order FDEs. Saberi et al (2011) obtained the numerical solutions of FIVPs by two methods, one is by the standard Euler method (Ming et al (1999)) and the other is the method proposed by Chalco-Cano & Roman-Flores (2009). On comparison of results, authors state that the second method is more efficient than the first one. Based on Zadeh’s extension principle, Ahmad & Hasan (2011) reformulate the classical Euler’s method, which takes into account the dependency problem that arise in fuzzy setting and used the new fuzzy version of Euler’s method to find the numerical solution of FIVP and haven shown the efficiency of the proposed by solving several linear and non-linear differential equations with fuzzy initial values.

Abbasbandy & Allahviranloo (2002b) studied numerical solution of FDEs by Taylor method. Abbasbandy & Allahviranloo(2004) developed four-stage- order Runge-Kutta methods for FDEs. However, the authors shared the
same problems as by Ming et al (1999) and concentrated exclusively on four-stage methods. Following the results of Buckley & Feuring (2000), Palligkinis et al (2009) applied Runge-Kutta methods for a more general category of problems and have given a convergence definition as well as error definitions that are in accordance with FDEs. The authors proved the convergence of s-stage Runge-Kutta methods.

Hybrid systems incorporate both continuous components, usually called plants, which are governed by differential equations, and also digital components such as digital computers, sensors and actuators controlled by programs. Lakshmikantham & Vatsala (2002) developed and investigate hybrid dynamic systems on time scales providing sufficient conditions for practical stability of the system in terms of Lyaponav-like functions and the comparison principle. [The notion of practical stability is more useful when the desired system may be mathematically unstable and the system may oscillate sufficiently near the state and yet its performance is acceptable].

The differential equation containing fuzzy valued functions and interactions with discrete time controllers can be named as hybrid fuzzy differential equations. Sambandham (2003) developed hybrid fuzzy dynamic systems on time scales and discussed practical stability by the application of Lyaponav-like functions and the comparison principle.

Pederson & Sambandham (2007, 2008) initiated the numerical solution of HFDE by an application of Euler method with the proof of convergence and also developed Runge-Kutta method for HFDE along with the convergence result. Pederson & Sambandham (2009) studied hybrid fuzzy differential equations initial value problems, and obtained numerical solutions by using characterization theorem of Bede for fuzzy differential equations. They extended Bede’s characterization theorem to HFDE and then used the result to numerically solve these systems by any suitable method for ODE’s.
Ezzati & Siah mansouir (2009) developed numerical solution of HFDEs by the improved predictor corrector method, in which the explicit three step method is used as a predictor and implicit two step method is used as a corrector. Prakash & Kalaiselvi (2009) have developed numerical solution of hybrid fuzzy differential equations by predictor corrector method in which Adams-Bashforth three step method is used as predictor and Adams-Moulton two step method is used as a corrector.

Allahviranloo & Salahshour (2010) determine the Euler method for HFDE under both cases of H-differentiability. They determined that the convergence in the sense of (i) differentiability established by Pederson & Sambandham (2007) does not have sufficient efficiency to be applied. So they extended the Euler method based on the generalized Hukuhara differentiability and discussed on the convergence of the Euler method.

Ghazanfari & Shakerami (2011) have used extended Runge-Kutta-like formulae of order 4 to find the numerical solutions of FDEs. Authors used the extended Runge-Kutta-like formulae in order to enhance the order of accuracy of the solutions using evaluations of both $f$ and $f'$, instead of the evaluations of $f$ only.

Nirmala & Chenthur Pandian (2011) proposed solution for first order FDE by Runge Kutta method of order two with new parameters and harmonic mean in the main formula in order to enhance the order of accuracy of the solution.

Akbarzadeh Ghanai & Mohseni Moghadam (2011) interpreted a FDE by the strongly generalized differentiability concept (Chalco-Cano & Roman-Flores (2008)). Then the authors showed that by that concept, a FDE can be transformed to a system of ODEs. By solving the associated ODEs, they found out two solutions for FDE. The authors extended classical second
order Runge–Kutta method under strongly generalized differentiability and then used for solving FDE numerically. They have showed the advantage of their method by considering radioactive decay problem.

Khastan & Ivaz (2009) introduced Nyström methods for finding numerical solution of FDES. Authors developed the Nyström methods by using the concept of fuzzy interpolation. The midpoint rule is obtained as a particular case of these methods. The convergence and stability of the methods have been proved.

Solaymani Fard (2009) presented a numerical procedure, using power series expansion, to solution of FDEs. In this procedure, the solution of given FDE is assumed in terms of the dependent variable and the assumption is substituted in the given FDE. By neglecting higher powers of the dependent variable, a system of linear equations is obtained. Solving the system, the coefficients of dependent variable are obtained. By repeating the procedure for higher order terms, power series solution of given FDE can be obtained in any arbitrary order.

Solaymani Fard & Vahidian Kamyan (2011) studied the numerical solution of FIVP by modified k-step method which is a method in which one non-step point is taken to improve the order of stable k-step method. By taking k-step method as a predictor and a modified k-step method as a corrector, the numerical solutions of FIVPs have been studied. The Convergence and the stability of the proposed method have also been proved.

Dahaghin & Mohseni Moghadam (2010) proposed a modified two-step Simpson method of order two to FIVPS and has presented the convergence theorem for the proposed method.
Using the concept of generalized differentiability, Seif & Khashan (2011) generalized Trapezoidal method for solving FDEs and showed that the proposed method gave better approximation than Euler method.

Balooch Shahryari & Salashour (2012) used improved predictor-corrector method for solving FDES under generalized differentiability concept. The convergence and stability of the proposed methods have been given and the method is illustrated with numerical examples.


Sedaghatfar et al (2013) presented a new approach for solving first-order FDE with fuzzy initial value under strongly generalized H-differentiability. The solutions of FIVPs in 0-cut and 1-cut cases are first found under H-differentiability and then the convex combination of these two cuts is taken. This combination is the solution of FDE.

Omid Solaymani fard & Nima Ghal-Eh (2011) have discussed variational iterative method to obtain approximate-analytical solutions for the linear systems of first order FDEs. Numerical experiments have been done to show the efficiency of the proposed method.

Gasilov et al (2011) proposed a new method to solve a system of linear differential equations with real co efficient and with an initial condition described by a vector of fuzzy intervals. The proposed method is based on properties of linear transformations. However, the authors considered a fuzzy set of real vector functions rather than a fuzzy vector function. In order to solve FDEs with fuzzy co-efficients, fuzzy initial values and fuzzy forcing
functions, Akin et al (2013) proposed a new algorithm based on an analysis of the crisp solution. Ahmad et al (2013) proposed a new fuzzification of the classical Euler method and then incorporate an unconstrained optimization technique. The authors claim that this combination offers a powerful tool to tackle uncertainty in any numerical method and have provided an efficient computational algorithm to guarantee the convexity of fuzzy solutions on the time domain.

The IFS is being studied and used in different fields of science. Since, in the IFS proposed by Atanassov (1986), all the elements of IFS are independent from time. So, it cannot be used to represent parameters that depend upon time, for example, the reliability of industrial systems depends upon time, which cannot be represented by IFS proposed by Atanassov (1986). To handle such situations, Kumar et al (2011) extended the concept of time-dependent fuzzy set (Aliev & Kara (2004)) by time-dependent IFS. Since then, for analyzing the intuitionistic fuzzy reliability (IFR) by using Markov model, there is a need to solve time-dependent intuitionistic fuzzy Kolmogorov’s differential equations. Several authors (Cheng et al (2009), Mahapatra & Mahapatra (2010), Mahapatra & Roy (2009), Verma et al (2012), Shu et al (2006)) have extended the concept of fuzzy fault tree by intuitionistic fuzzy fault free.

Sneh Lata & Amit Kumar (2012) proposed nth –order time dependent intuitionistic fuzzy linear differential equations and used the proposed method to solve intuitionistic fuzzy Kolmogorov’s differential equations, obtained by Markov model of condensate system. The result obtained was used to analyze the IFR of condensate system.

2.2 SCOPE OF THE THESIS

From the literature survey it is noted that various numerical procedure are applied to find the numerical solution of HFDEs and FDEs.
Generalized differentiability has been defined for larger class of fuzzy valued function, than the H-derivative and the solution can be obtained for the problems which has decreasing length of support.

The research deals with different Runge-Kutta methods to find the numerical solution of HFDEs. For better accuracy and to reduce the function evaluations various Runge-Kutta methods are discussed, which also includes the derivative $f'$. Multistep Runge Kutta method of order two with new parameters is discussed which involves evaluations of both $f$ and $f'$. This method gives better solution than the Classical Runge Kutta method of order two. Extended Runge-Kutta like formula of order four is also applied to solve HFDEs numerically which requires only four evaluations of both $f$ and $f'$ whereas arbitrary classical Runge-Kutta methods of order three and four used together would require six evaluations of $f$ per step. Generalized differentiability is applied to the problems with decreasing length of support for RK3, RKN3 and ERK4 methods. Along with this hybrid intuitionistic fuzzy differential equations is developed and numerical algorithm using Euler method is obtained for HIFDEs.

2.3 RESEARCH PROBLEM

The problem of stabilizing a continuous plant governed by differential equation through the interaction with the discrete time controllers leads to the consideration of hybrid systems.

The hybrid fuzzy differential systems is given by

\[
\begin{align*}
\{ x(t) &= f(t, x(t), \lambda_k(x_k)), \quad t \in [t_k, t_{k+1}], \\
x(t_k) &= x_k 
\end{align*}
\]  \hfill (2.1)

Various numerical procedures are developed for equation (2.1) in order to enhance the accuracy of the approximated solution.
The physical problem given by the equation (2.1) consists of fuzzy-valued functions and interactions with discrete time controllers. Instead if it contains intuitionistic fuzzy valued function and interactions with discrete time controllers then hybrid intuitionistic fuzzy differential equations is obtained.

2.4 SOLUTION METHODOLOGY

The research work deals with the numerical solution of HFDEs by

- Third order Runge-Kutta and Runge-Kutta Nystrom method.
- Second order Runge-Kutta method in which new parameters based on harmonic mean is used in the main formula in order to increase the order of accuracy of the solution.
- Extended Runge-Kutta like formula of order four which enhance the order of accuracy of the solutions using evaluations of both $f$ and $f'$.

And finally HIFDEs is framed and it is numerically solved using Euler’s method.

2.5 ORGANIZATION OF THE THESIS

The thesis is divided into seven chapters and a brief content of each chapter is delineated below.

Chapter 1 gives a general introduction to fuzzy set theory, basic definitions and properties related to fuzzy set theory, measurability, integrability and differentiability of fuzzy functions. FIVP, HFDE and intuitionistic fuzzy initial value problem are also presented. This chapter also
briefs the numerical methods and the conditions for the consistency and the stability of the numerical methods.

**Chapter 2** reviews the literature presented by several authors along with this scope of the thesis, Research problem and solution methodology is also discussed.

**Chapter 3** contains Runge Kutta method and Runge Kutta Nystrom method of order three for solving HFDEs.

**Chapter 4** contains Second order Runge –Kutta method in which new parameters are taken in \( k_i \)'s and Harmonic mean of \( k_i \)'s is used in the main formula in order to increase the order of accuracy of the solution.

**Chapter 5** deals with Extended Runge-Kutta like formula of order four which enhance the order of accuracy of the solutions using evaluations of both \( f \) and \( f' \).

**Chapter 6** deals with HIFDE and a numerical procedure to solve it.

**Chapter 7** of the thesis focuses on the summary and general conclusion of the present work and future prospects of the work.