CHAPTER 1

INTRODUCTION

1.1. An Introduction to Fuzzy Set Theory

In everyday life we use properties which cannot be dealt satisfactorily with yes or no basis. In such situation, many collections and categories do not exhibit their sharp boundaries. In 1965, L. A. Zadeh introduced the notion of weighted membership to eliminate the sharp boundary dividing members of the class from the nonmembers. He suggested that an individual could have a degree of membership which ranged over a continuum of values rather than being 0 or 1. An element may then belong more or less to a subset and there from, Zadeh [1965] introduced a fundamental concept, that of a fuzzy subset. The notion of fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respect the framework used in the case of ordinary sets, but is more general than the later potentially which may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables.

Since the inception of the notion of a fuzzy set by Zadeh [1965], which laid the foundation of fuzzy set theory, the literature on fuzzy set theory and its application has been growing rapidly amounting by now to several thousands of papers. These are widely scattered over many disciplines such as economics, psychology, network analysis etc.

The Japanese were the first to utilize fuzzy logic for practical applications. The first notable application was on the high-speed train in
Sendai, in which fuzzy logic was able to improve the economy, comfort, and precision of the ride. It has also been used in recognition of handwritten symbols in Sony pocket computers, Canon auto-focus technology, Omron auto-aiming cameras, earthquake prediction and modeling at the Institute of Seismology Bureau of Metrology in Japan, etc.

1.2. Motivation

A fuzzy set is a class of objects in which the transition from membership to non-membership is gradual rather than abrupt. Such a class is characterized by a membership function which assigns a grade or degree of membership to an element between 0 and 1. In classical mathematical systems, we deal with only those statements which are declarative in nature and which may be either true or false. Fuzzy mathematical systems, whose foundation was laid by Zadeh [1965], in his papers on theory of fuzzy subsets, deal with situations of interrogative, imperative, exclamatory and also declarative statements. So, fuzzy set theory is a generalization of classical set theory, which has been algebraically established in several sources. As a result, one of the most important motivations and aims in the theory of fuzzy algebra is to make more generalized the theory of algebra.

1.3. Contribution

The work in this thesis is on the applications of fuzzy set theory of Zadeh [1965] and intuitionistic fuzzy set theory by Atanassov [1989] in the theory of fuzzy algebra. It is asserted that the most important and potential generalization of fuzzy sets came in the form of Intuitionistic Fuzzy Sets (IFS) developed by Atanassov. The contribution in this thesis is in the areas of the theory of fuzzy rings, fuzzy modules and some kinds of fuzzy algebras such as fuzzy BG-algebra, fuzzy BE-algebra etc. Since the notion of fuzzy groups by Rosenfield [1971], a huge number of works have been reported in
this area. Theory of fuzzy algebra has been playing a great and significant role in different areas as in Computer Science, in Management Science, in Banking and Finance, in Social Science and in many more areas. It is expected that the work reported in this thesis will add an element of support to the existing theory of fuzzy algebra and to the theory of mathematical relations.

1.4. Thesis Organization

Chapter 1 is the introduction which includes an introduction to fuzzy set theory, motivation of the work in the thesis, contributions in the thesis and the present investigation in the thesis. We have discussed the historical developments of fuzzy algebra as well as the scopes, objectives and the motivations of the works reported in this thesis.

Chapter 2 includes the literature review of some earlier works related to the topic that were supported to my works in this thesis.

Chapter 3 consists of preliminaries on Fuzzy Superfluous Submodule the idea and the results of Fuzzy Superfluous Submodule, Fuzzy Radical and Fuzzy Indecomposable Submodule. In this chapter we have generalized the concept of superfluous submodules in fuzzy settings and got some interesting results. Also we have defined fuzzy indecomposable submodule and tried to investigate its relationship with fuzzy superfluous submodule and fuzzy radical of a module.

Chapter 4 consists of preliminaries on Divisible Submodule, Divisible Fuzzy Submodule, Pure Submodule and Pure Fuzzy Submodule as well as the definitions and results of Divisible Fuzzy Submodule and Pure Fuzzy Submodule and relation between Divisible and Pure FSMs of a
module M. In this Chapter, the concept of Divisible Fuzzy Submodules and Pure Fuzzy Submodules are introduced. The sum, union, intersection etc. of Divisible Fuzzy Submodules as well as Pure Fuzzy Submodules and several properties are discussed with suitable examples. Some new results of divisible and pure fuzzy submodules on commutative algebra are investigated and also some applications with level subsets of divisible and pure fuzzy submodules are given in this Chapter. Lastly, the relations between Divisible and Pure FSMs are discussed.

Chapter 5 consists of preliminaries and some results on intuitionistic fuzzy ideals with threshold $(\alpha,\beta)$. In this chapter, some results of intuitionistic fuzzy ideals with thresholds $(\alpha,\beta)$ of a ring have been established. Here role of thresholds $(\alpha,\beta)$ on an intuitionistic fuzzy ideals of a ring is discussed.

In chapter 6, there are preliminaries on $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of BG-algebra, definitions and results of $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of BG-algebra and homomorphism of BG-algebra and fuzzy ideals. Herein we have defined $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of a BG-algebra and investigated some of its properties.

In chapter 7, it consist of preliminaries on $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of a BE-algebra, definitions and results of $(\varepsilon, \varepsilon \vee q)$- fuzzy ideal of a BE-algebra. Here the concept of $(\varepsilon, \varepsilon \vee q)$- fuzzy ideal of a BE-algebra is introduced. A necessary and sufficient condition for a fuzzy set to be a $(\varepsilon, \varepsilon \vee q)$- fuzzy ideal is stated and several properties are investigated. Also images and inverse images of a $(\varepsilon, \varepsilon \vee q)$-fuzzy ideals under a homomorphism are studied.
1.5. Preliminaries.

1.5.1. Definition: A fuzzy subset $\mu$ of a set $E$ is a function $\mu: E \rightarrow [0, 1]$. $\mu$ is called the membership function.

1.5.2. Definition. If $A = \{< x, \mu_A(x), \nu_A(x) > | x \in E\}$ and $B = \{< x, \mu_B(x) > | x \in E\}$ be any two FS of a set $E$ then,

$A \subseteq B$ if and only if for all $x \in E$, $\mu_A(x) \leq \mu_B(x)$;

$A = B$ if and only if for all $x \in E$, $\mu_A(x) = \mu_B(x)$;

$A \cap B = \{< x, (\mu_A \cap \mu_B)(x) > | x \in E\}$,

where $(\mu_A \cap \mu_B)(x) = \mu_A(x) \land \mu_B(x)$;

$A \cup B = \{< x, (\mu_A \cup \mu_B)(x) > | x \in E\}$,

where $(\mu_A \cup \mu_B)(x) = \mu_A(x) \lor \mu_B(x)$.

Also we see that a fuzzy set has the form $\{< x, \mu_A(x), \mu_A^c(x) > | x \in E\}$, where $\mu_A^c(x) = 1 - \mu_A(x)$.

5.1.3. Definition. An FS $A = \{< x, \mu_A(x), \nu_A(x) > | x \in R\}$ of a ring $R$ is said to be an Fuzzy Ideal (in short FI) of $R$ if for all $x, y \in R$,

(i) $\mu_A(x - y) \geq \mu_A(x) \land \mu_A(y)$

(ii) $\mu_A(xy) \geq \mu_A(x) \lor \mu_A(y)$

1.5.4. Theorem. If $A = \{< x, \mu_A(x) > | x \in R\}$ is an FI of $R$ then

$\mu_A(0) \geq \mu_A(x)$, $\mu_A(-x) = \mu_A(x)$ for all $x \in R$.

1.5.5. Definition. Let $A = \{< x, \mu_A(x) > | x \in R\}$ and $B = \{< x, \mu_B(x) > | x \in R\}$ be two FI’s of a ring $R$ then their sum $A + B$ is defined as
A + B = \{< x, (\mu_A + \mu_B)(x) > | x \in R}\}

where \((\mu_A + \mu_B)(x) = \bigvee_{x=a+b} \{\mu_A(a) \land \mu_B(b)\}\).

1.5.6. Proposition. If \(\mu\) and \(\sigma\) be two FSMs of the left R-module M than the sum of two FSMs is again a FSM of M.

1.5.7. Definition. Let \(A = \{< x, \mu_A(x) > | x \in R\}\) and \(B = \{< x, \mu_B(x) > | x \in R\}\) be two FI’s of a ring R then their product \(AB\) is defined as

\[AB = \{< x, (\mu_A \mu_B)(x) > | x \in R\}\]

where \((\mu_A \mu_B)(x) = \bigvee_{x=\sum_{i=1}^n a_i} \{\land_{i \in \mathbb{N}} (\mu_A(a_i) \land \mu_B(b_i))\}\).

1.5.8. Theorem. If A and B are two FI’s of a ring R then A + B and AB are also FI’s of R.

1.5.9. Definition. Let R and R’ be two rings and \(f : R \to R’\) be a homomorphism, and \(A = \{< x, \mu_A(x) > | x \in R\}\), \(A’ = \{< y, \mu_A(y) > | y \in R’\}\) be fuzzy subsets of R and R’ respectively, then the image \(f(A)\) and the inverse image \(f^{-1}(A’)\) are defined as follows:

\[f(A) = \{< y, f(\mu_A)(y) > | y \in R’\}\]

and \(f^{-1}(A) = \{< x, f^{-1}(\mu_A)(x) > | x \in R\}\), where

\[(f(\mu_A))(y) = \begin{cases} \bigvee \{\mu_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases},\]

And \(f^{-1}(\mu_A)(x) = \mu_A(f(x))\).

1.5.10. Definition. Let I be an ideal of a ring R. If \(A = \{< x, \mu_A(x) > | x \in R\}\) is an FI of R then the fuzzy subset of R/I defined by \(\bar{A} = \{< x + I, \mu_A(x + I) > | x \in R\}\) is an FI of R/I, where \(\mu_A(x + I) = \bigvee \{\mu_A(x + a) | a \in I\}\).
1.5.11. **Definition.** A fuzzy subset $\mu$ of $M$ is said to be a fuzzy submodule (in short FSM) if

(i) $\mu(0) = 1$

(ii) $\mu(x-y) \geq \Lambda\{\mu(x), \mu(y)\}$ and

(iii) $\mu(rx) \geq \mu(x)$, for all $x, y \in M, r \in R$.

1.5.12. **Definition.** If $\mu$ is a FSM of $M$ then we define

$\text{exc}=\{x \in M : \mu(x) = 1\}$

1.5.13. **Proposition.** If $\mu$ is an FSM then $\text{exc}$ is a submodule of $M$.

1.5.14. **Definition.** We define two fuzzy subsets $\Omega$ and $\mu_M$ of $M$ as $\Omega(0) = 1$ and $\Omega(x) = 0$ for all $x \neq 0$ and $\mu_M(x) = 1$ for all $x \in M$. Then $\Omega$ and $\mu_M$ are FSMs of $M$, which are actually fuzzy equivalent of $\{0\}$ and $M$ in module theory.

1.5.15. **Proposition.** If $\mu$ is a FSM of $M$ then $\text{exc} = M$ if and only if $\mu = \mu_M$. Also $\mu \subseteq \sigma \Rightarrow \text{exc} \subseteq \sigma$.

1.5.16. **Proposition.** If $\mu$ and $\sigma$ be two FSMs then $(\mu \cap \sigma)_* = \mu_* \cap \sigma_*$, $(\mu \cup \sigma)_* = \mu_\cup \sigma_*$. These results can be extended to infinite intersections and unions. Further if $\mu$ and $\sigma$ have finite images then $(\mu + \sigma)_* = \mu_* + \sigma_*$, where the sum of two FSMs is defined as usual as $(\mu + \sigma)(x) = \bigvee_{x=a+b}\Lambda\{\mu(a), \sigma(b)\}$.

**Proof:** The first two proofs are trivial. Also for the last part we have,

$\mu, \sigma \subseteq \mu + \sigma$, so $\mu_* \cup \sigma_* \subseteq (\mu + \sigma)_*$ and hence $\mu_* + \sigma_* \subseteq (\mu + \sigma)_*$.

Next if $x \in (\mu + \sigma)_*$ then $(\mu + \sigma)(x) = 1$

$\Rightarrow \bigvee_{x=a+b}\Lambda\{\mu(a), \sigma(b)\} = 1$
⇒ \Lambda \{\mu(a), \sigma(b)\} = 1 \text{ for some } x = a + b \text{ [as } \mu \text{ and } \sigma \text{ have finite images]}

⇒ \mu(a) = 1 \text{ and } \sigma(b) = 1

⇒ a \in \mu_\ast \text{ and } b \in \sigma_\ast

⇒ x = a + b \in \mu_\ast + \sigma_\ast.

Hence the equality holds.

1.5.9. Definition. If \mu \text{ and } \sigma \text{ are two FSMs of } M, \text{ then the sum } \mu + \sigma \text{ is called the direct sum of } \mu \text{ and } \sigma \text{ if } \mu \cap \sigma = \Omega \text{ and we denote it by } \mu \oplus \sigma.

1.5.17. Proposition. \((\mu \oplus \sigma)_\ast = \mu_\ast \oplus \sigma_\ast\), \text{ if } \mu \text{ and } \sigma \text{ have finite images.}

1.5.18. Definition. A FSM \mu \text{ of } M \text{ is said to be fuzzy maximal if for any FSM } \sigma \text{ of } M, \mu \subseteq \sigma \Rightarrow \text{ either } \sigma = \mu_M \text{ or } \mu_\ast = \sigma_\ast.

1.5.19. Proposition. A FSM \mu \text{ of } M \text{ is fuzzy maximal if and only if } \text{Im} \mu = \{1, t\}, \text{ } t \neq 1 \text{ and } \mu_\ast \text{ is a maximal submodule of } M.

1.5.20. Definition. Let \(x, x_\alpha \in M\), where \(\alpha \in \Lambda\). By a sum \(x = \Sigma_{\alpha \in \Lambda} x_\alpha\), we mean all but finitely many \(x_\alpha\) are 0. If \(\{\mu_\alpha | \alpha \in \Lambda\}\) be a collection of FSMs of M then the fuzzy subset \(\Sigma_{\alpha \in \Lambda} \mu_\alpha\) of M is defined by \((\Sigma_{\alpha \in \Lambda} \mu_\alpha)(x) = \vee_{x = \Sigma_{\alpha \in \Lambda} x_\alpha}\{\Lambda(\mu_\alpha(x_\alpha))\}\)

1.5.21. Proposition. If \(\{\mu_\alpha | \alpha \in \Lambda\}\), be a collection of FSMs of M then

\(\Sigma_{\alpha \in \Lambda} \mu_\alpha\) is again a FSM of M and \(\Sigma_{\alpha \in \Lambda}(\mu_\alpha)_\ast \Rightarrow (\Sigma_{\alpha \in \Lambda} \mu_\alpha)_\ast\).

1.5.22. Definition. A nonzero module M is said to be indecomposable if \(\{0\}\) and M are the only direct summands of M.
1.5.23. **Definition.** Let $\mu$ and $\sigma$ be two FSMS of $M$ such that $\mu \subseteq \sigma$. Then $\mu_*$ is a submodule of $\sigma_*$. Also $\sigma$ restricted to $\sigma_*$ is FSM of $\sigma_*$. The fuzzy subset $\sigma$ of $M/\mu_*$ defined by $\sigma(x + \mu_*) = \bigvee \{ \sigma(x + y) \mid y \in \mu_* \}$ is a FSM of $M/\mu_*$. We denote this FSM $\sigma$ by $\sigma/\mu_*$.

1.5.24. **Proposition.** If $\mu$ and $\sigma$ are two FSMS of $M$ such that $\mu \subseteq \sigma$ then

$$(\sigma/\mu)_* = \sigma_*/\mu_*.$$ 

**Proof:** Let $x + \mu_* \in \sigma/\mu_*$. Then $(\sigma/\mu)(x + \mu_*) = 1$

$$\Rightarrow \bigvee \{ \sigma(x + y) \mid y \in \mu_* \} = 1.$$ 

Since $\sigma$ has finite image so there exists $y \in \mu_*$ such that $\sigma(x + y) = 1$

$$\Rightarrow x + y \in \sigma_*.$$ 

Also $y \in \mu_* \subseteq \sigma_*$, so $x \in \sigma_*$. Hence $x + \mu_* \in \sigma_*/\mu_*$.

Conversely let $x + \mu_* \in \sigma_*/\mu_*$, then $x \in \sigma_* \Rightarrow \sigma(x) = 1$

$$\Rightarrow \bigvee \{ \sigma(x + y) \mid y \in \mu_* \} \geq \sigma(x) = 1$$

$$\Rightarrow \bigvee \{ \sigma(x + y) \mid y \in \mu_* \} = 1 \Rightarrow (\sigma/\mu)(x + \mu_*) = 1$$

$$\Rightarrow x + \mu_* \in (\sigma/\mu)_*.$$ 

Hence the result follows.

1.5.25. **Proposition.** If $\mu$ and $\sigma$ be two FSMS of the left $R$-module $M$ than the sum of two FSMS is again a FSM of $M$. 