6.1. Preliminaries on BG-algebra

Let us consider the set of integers $\mathbb{Z}$ and the ‘$-$’ be the binary operation ‘subtraction’. Then the algebraic structure $(\mathbb{Z}, -)$ obeys the following properties:

(i) $x - x = 0$    (ii) $x - 0 = x$    (iii) $(x - y) - (0 - y) = x, \forall x, y \in \mathbb{Z}$.

Also let $X$ be a non-empty set and $P(X)$, the power set of $X$ and let $\triangle$, the symmetric difference of two sets be, the binary operation. Then the algebraic structure $(P(X), \triangle)$ obeys the same properties as in the above example i.e.,

(i) $A \triangle A = \phi$    (ii) $A \triangle \phi = A$    (iii) $(A \triangle B) \triangle (\phi \triangle B) = A$ where $A$ & $B$ are subsets of $X$. In the above two examples, it seems that the role of $0$ and $\phi$ are identical. Such type of examples leads to provide the following definition.

6.1.1. Definition. A BG-algebra is a non-empty set $X$ with a constant $0$ and a binary operation $*$ satisfying the following axioms:

(i) $x * x = 0$

(ii) $x * 0 = x$

(iii) $(x * y) * (0 * y) = x$, for all $x, y \in X$

For brevity we also call $X$ a BG-algebra. A non-empty set $S$ of a BG-algebra $X$ is called sub algebra of $X$ if $x * y \in S$, for all $x, y \in S$. 
6.1.2. Example: Let \( X = \{0, 1, 2\} \) be a set with the following table:

\[
\begin{array}{ccc}
\ast & 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
1 & 1 & 0 & 1 \\
2 & 2 & 2 & 0
\end{array}
\]

Then \((X; \ast, 0)\) is a BG-algebra.

6.1.2. Definition. A nonempty subset \( I \) of a BG-algebra \( X \) is called an ideal of \( X \) if

(i) \( 0 \in I \)

(ii) \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \), for all \( x, y \in X \).

6.1.3. Example. Let \( X = \{0, 1, 2, 3\} \) be a set with the following table:

\[
\begin{array}{cccc}
\ast & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 1 & 1 \\
2 & 2 & 0 & 2 & 2 \\
3 & 3 & 3 & 3 & 0
\end{array}
\]
Then (X, *, 0) is a BG-algebra.

**6.1.4. Definition.** A fuzzy set \( \mu \) in a BG-algebra \( X \) is called a fuzzy ideal of \( X \) if

(i) \( \mu(0) \geq \mu(x) \)

(ii) \( \mu(x) \geq \min\{ \mu(x * y), \mu(y) \} \), for all \( x, y \in X \).

**6.1.5. Example.** Consider \( X = \{0, 1, 2, 3\} \) with * defined by the table

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Then (X, *, 0) is a BG-algebra. Define \( \mu: X \to [0, 1] \) by \( \mu(0) = \mu(1) = 0.5, \mu(2) = \mu(3) = 0.4 \). Then \( \mu \) is a fuzzy ideal of \( X \).

**6.2. (ε, εvq)-fuzzy ideals of BG-algebra.**

**6.2.1. Definition.** A fuzzy set \( \mu \) of the form

\[
\mu(y) = \begin{cases} 
  t & \text{if } y = x, \\
  0 & \text{if } y \neq x 
\end{cases}
\]

where \( t \in (0, 1] \), is called a fuzzy point with support \( x \) and value \( t \) and it is denoted by \( x_t \).
6.2.2. **Definition.** A fuzzy point $x_t$ is said to belongs to (respectively be quasi coincident with) a fuzzy set $\mu$, written as $x_t \in \mu$ (respectively $x_t q \mu$) if $\mu(x) \geq t$ (respectively $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q \mu$ then $x_t \in q \mu$.

6.2.3. **Definition.** A fuzzy subset $\mu$ of a BG-algebra $X$ is said to be a $(\in, \in q)$-fuzzy ideal of $X$ if $(x \ast y), y_s \in \mu \Rightarrow x_{m(t, s)} \in q \mu$ i.e. $\mu(x) \geq m(t, s)$ or $\mu(x) + m(t, s) > 1$, for all $x, y \in X$, where $m(t, s) = \min \{t, s\}$.

6.2.4. **Theorem.** A fuzzy subset $\mu$ of a BG algebra $X$ is a fuzzy ideal if and only if $\mu$ is a $(\in, \in)$-fuzzy ideal.

**Proof:** Let $\mu$ be a fuzzy ideal of $X$. Let $x, y \in X$ such that $(x \ast y), y_s \in \mu$, where $t, s \in (0, 1]$. Then $\mu(x \ast y) \geq t$ and $\mu(y) \geq s$.

Now $\mu(x) \geq \min \{\mu(x \ast y), \mu(y)\} \geq \min \{t, s\} = m(t, s) \Rightarrow x_{m(t, s)} \in \mu$.

Conversely let $\mu$ be a $(\in, \in)$-fuzzy ideal. Let $x, y \in X$ and $t = \mu(x \ast y)$ and $s = \mu(y)$. Then $\mu(x \ast y) \geq t$ and $\mu(y) \geq s$, i.e., $(x \ast y)_t, y_s \in \mu$.

By the given condition, $x_{m(t, s)} \in \mu$.

$\Rightarrow \mu(x) \geq m(t, s) = \min \{\mu(x \ast y), \mu(y)\}$. Hence $\mu$ is a fuzzy ideal.

6.2.5. **Theorem.** If $\mu$ is $(q, q)$-fuzzy ideal of a BG algebra $X$ then it is also a $(\in, \in)$-fuzzy ideal of $X$.

**Proof:** Let $\mu$ be a $(q, q)$-fuzzy ideal of a BG-algebra $X$. Let $x, y \in X$ be such that $(x \ast y), y_s \in \mu$. Then $\mu(x \ast y) \geq t$ and $\mu(y) \geq s$. If $\delta$ be an arbitrarily small positive number then we get

$\mu(x \ast y) + \delta > t$ and $\mu(y) + \delta > s$

$\Rightarrow \mu(x \ast y) + (1 + \delta - t) > 1$ and $\mu(y) + (1 + \delta - s) > 1$
\[(x * y)(1 + \delta - t) \in \mu \quad \text{and} \quad y(1 + \delta - s) \in \mu.
\]

Since \(\mu\) is \((q, q)\)-fuzzy ideal of \(X\) so we get \(x_m(1 + \delta - t, \ 1 + \delta - s) \in \mu\)

\[\Rightarrow \mu(x) + m(1 + \delta - t, \ 1 + \delta - s) > 1\]

\[\Rightarrow \mu(x) + 1 + \delta - M(t, s) > 1, \text{ where } M(t, s) = \max\{t, s\}.\]

\[\Rightarrow \mu(x) > M(t, s) - \delta.\]

Since \(\delta\) is arbitrary so \(\mu(x) \geq M(t, s) \geq m(t, s)\). This shows that \(x_{m(t, s)} \in \mu\) and hence \(\mu\) is a \((\epsilon, \epsilon)\)-fuzzy ideal of \(X\).

\[6.2.6. \textbf{Remark.} \text{ However the converse of the above theorem is not true for example consider the BG algebra } X = \{0, 1, 2, 3\} \text{ with } * \text{ defined by the table}\]

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and let us define \(\mu: X \rightarrow [0, 1]\) by \(\mu(0) = \mu(1) = 0.45, \mu(2) = \mu(3) = 0.37\).

Then \(\mu\) is a \((\epsilon, \epsilon)\)-fuzzy ideal but it is not \((q, q)\)-fuzzy ideal of \(X\). Because if \(x = 2, y = 1, t = 0.7, s = 0.6\), then \((x * y) \in \mu\) and \(y \in \mu\)

\[\text{But } \mu(x) + m(t, s) = \mu(2) + m(0.7, 0.6) = 0.37 + 0.6 = 0.97 < 1.\]
6.2.7. **Theorem.** A fuzzy subset $\mu$ of a BG algebra $X$ is a $(\in, \in\vee q)$-fuzzy ideal of $X$ if and only if $\mu(x) \geq m(\mu(x \ast y), \mu(y), 0.5)$ for all $x, y \in X$.

**Proof:** First let $\mu$ be a $(\in, \in\vee q)$-fuzzy ideal of $X$.

**Case-I:** Let $m(\mu(x \ast y), \mu(y)) < 0.5$, for all $x, y \in X$.

Then $m(\mu(x \ast y), \mu(y), 0.5) = m(\mu(x \ast y), \mu(y))$.

If possible let $\mu(x) < m(\mu(x \ast y), \mu(y))$. Choose a real number $t$ such that $\mu(x) < t < m(\mu(x \ast y), \mu(y))$.

Then $(x \ast y)_t$ and $y_t \in \mu$.

But $\mu(x) < t$ i.e., $x_t \not\in \mu$ and $\mu(x) + t < t + t = 2t < 2m(\mu(x \ast y), \mu(y)) < 1$.

This contradicts the fact that $\mu$ is a $(\in, \in\vee q)$-fuzzy ideal of $X$.

Hence $\mu(x) \geq m(\mu(x \ast y), \mu(y)) = m(\mu(x \ast y), \mu(y), 0.5)$.

**Case-II:** Let $m(\mu(x \ast y), \mu(y)) \geq 0.5$, for all $x, y \in X$.

Then $m(\mu(x \ast y), \mu(y), 0.5) = 0.5$. If possible let $\mu(x) < m(\mu(x \ast y), \mu(y), 0.5) = 0.5$. Then $\mu(x \ast y) \geq 0.5$ and $\mu(y) \geq 0.5$, so $(x \ast y)_{0.5}, y_{0.5} \in \mu$.

But $\mu(x) < 0.5$ i.e., $x_{0.5} \not\in \mu$ and $\mu(x) + 0.5 < 0.5 + 0.5 = 1$, which is again a contradiction.

Hence we must have $\mu(x) \geq 0.5 = m(\mu(x \ast y), \mu(y), 0.5)$.

Conversely let $\mu(x) \geq m(\mu(x \ast y), \mu(y), 0.5)$. Let $x, y \in X$ such that $(x \ast y)_t, y_s \in \mu$. Then $\mu(x \ast y) \geq t$ and $\mu(y) \geq s$. So $m(\mu(x \ast y), \mu(y)) \geq m(t, s)$.
By the given condition $\mu(x) \geq m(\mu(x \ast y), \mu(y), 0.5) \geq m(t, s, 0.5)$. If $m(t, s) \leq 0.5$, then $m(t, s, 0.5) = m(t, s)$ and so $\mu(x) \geq m(t, s)$ i.e., $x_{m(t,s)} \in \mu$.

Again if $m(t, s) > 0.5$, then $\mu(x) \geq 0.5$ and so $\mu(x) + m(t, s) > 0.5 + 0.5 = 1$ i.e., $x_{m(t,s)} \in q \mu$. Hence $x_{m(t,s)} \in \circ \mu$ and consequently $\mu$ is a $(\epsilon, \circq)$-fuzzy ideal of $X$.

6.2.8. Remark. A $(\epsilon, \circ)$-fuzzy ideal is always a $(\epsilon, \circq)$-fuzzy ideal but not conversely. For example consider the BG algebra $X = \{0, 1, 2, 3\}$ with $\ast$ defined by the table

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and let us define $\mu$: $X \rightarrow [0, 1]$ by $\mu(0) = \mu(1) = \mu(2) = 0.8$, $\mu(3) = 0.6$. Then $\mu$ is a $(\epsilon, \circq)$-fuzzy ideal by theorem 6.2.7. But it is not a $(\epsilon, \circ)$-fuzzy ideal because $2_{0.8} = (3 \ast 1)_{0.8}$, $1_{0.8} \in \mu$ but $3_{0.8} \notin \mu$.

6.2.9. Theorem. If a fuzzy subset $\mu$ of a BG algebra $X$ is a $(\epsilon, \circq)$-fuzzy ideal of $X$ and $\mu(x) < 0.5$, for all $x \in X$ then $\mu$ is also a $(\epsilon, \circ)$-fuzzy ideal of $X$. 

Proof: Let $\mu$ be a $(\epsilon, \epsilon \vee q)$-fuzzy ideal of $X$ and $\mu(x) < 0.5$, for all $x \in X$.

Let $(x \ast y)_h \in \mu$ and $y_s \in \mu$. Then $t \leq \mu(x \ast y) < 0.5$ and $s \leq \mu(y) < 0.5$ and so $m(t, s) < 0.5$. Also $\mu(x) < 0.5$. Thus $\mu(x) + m(t, s) < 1$. But $\mu$ is a $(\epsilon, \epsilon \vee q)$-fuzzy ideal of $X$, so we must have $(x)_{m(t, s)} \in \mu$. This shows that $\mu$ is a $(\epsilon, \epsilon)$-fuzzy ideal of $X$.

6.3. Homomorphism of BG-algebra and fuzzy ideals.

5.3.1. Definition. Let $X$ and $X'$ be two BG algebras. Then a mapping $f : X \to X'$ is said to be a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$.

5.3.2. Theorem. Let $X$ and $X'$ be two BG algebras and $f : X \to X'$ be a homomorphism. If $\mu'$ be a $(\epsilon, \epsilon \vee q)$-fuzzy ideal of $X'$ then $f^{-1}(\mu')$ is also a $(\epsilon, \epsilon \vee q)$-fuzzy ideal of $X$.

Proof: Recall that $f^{-1}(\mu')$ is defined as $(f^{-1}(\mu'))(x) = \mu'(f(x))$ for all $x \in X$. Let $\mu'$ be a $(\epsilon, \epsilon \vee q)$-fuzzy ideal of $X'$. Let $x, y \in X$ such that $(x \ast y)_h, y_s \in f^{-1}(\mu')$.

Then $(f^{-1}(\mu'))(x \ast y) \geq t$ and $(f^{-1}(\mu'))(y) \geq s \Rightarrow \mu'(f(x \ast y)) \geq t$ and $\mu'(f(y)) \geq s$

$\Rightarrow (f(x \ast y))_h, f(y)_s \in \mu' \Rightarrow (f(x)_h \ast f(y)_s), f(y)_s \in \mu'$.

But $\mu'$ is $(\epsilon, \epsilon \vee q)$-fuzzy ideal of $X'$, so we have $(f(x))_{m(t, s)} \in \mu'$ or $\mu'(f(x)) + m(t, s) > 1$.

$\Rightarrow \mu'(f(x)) \geq m(t, s)$ or $\mu'(f(x)) + m(t, s) > 1$

$\Rightarrow ((f^{-1}(\mu'))(x) \geq m(t, s)$ or $(f^{-1}(\mu'))(x) + m(t, s) > 1$
\( x_{m(s,t)} \in f^{-1}(\mu') \) or \( x_{m(t,s)} \) or \( f^{-1}(\mu') \).

Hence \( f^{-1}(\mu') \) is a \((\epsilon, \in \vee \mu)\)-fuzzy ideal of \( X \).

5.3.3. **Theorem.** Let \( X \) and \( X' \) be two BG algebras and \( f: X \to X' \) be an onto homomorphism. If \( \mu' \) be a fuzzy subset of \( X' \) such that \( f^{-1}(\mu') \) is a \((\epsilon, \in \vee \mu)\)-fuzzy ideal of \( X \) then \( \mu' \) is also a \((\epsilon, \in \vee \mu)\)-fuzzy ideal of \( X' \).

**Proof:** Let \( u, v \in X' \) such that \((u * v)_s, v_s \in \mu'\), where \(0 \leq t, s \leq 1\). Then \( \mu'(u * v) \geq t \) and \( \mu'(v) \geq s \). Since \( f \) is onto so there exists \( x, y \in X \) such that \( f(x) = u \) and \( f(y) = v \). Also \( f \) is a homomorphism so \( f(x * y) = f(x) * f(y) = u * v \).

So we get \( \mu'(f(x * y)) \geq t \) and \( \mu'(f(y)) \geq s \)

\( \Rightarrow (f^{-1}(\mu'))(x * y) \geq t \) and \( (f^{-1}(\mu'))(y) \geq s \). But \( f^{-1}(\mu') \) is a \((\epsilon, \in \vee \mu)\)-fuzzy ideal of \( X \), so we get \( (x)_{m(t,s)} \in \vee \mu f^{-1}(\mu') \)

\( \Rightarrow (f^{-1}(\mu'))(x) \geq m(t, s) \) or \( (f^{-1}(\mu'))(x) + m(t, s) > 1 \)

\( \Rightarrow \mu'(f(x)) \geq m(t, s) \) or \( \mu'(f(x)) + m(t, s) > 1 \)

\( \Rightarrow \mu'(u) \geq m(t, s) \) or \( \mu'(u) + m(t, s) > 1 \)

\( \Rightarrow u_{m(t,s)} \in \vee \mu \).

Hence \( \mu' \) is a \((\epsilon, \in \vee \mu)\)-fuzzy ideal of \( X' \).