Chapter 4

Symmetric two, three qubit gates

4.1 Introduction

Entanglement, considered to be the most nonclassical manifestation of quantum formalism is a valuable resource for quantum information processing. In the last few years there has been considerable increase in experimental activity [Fortschr.Phys. (2000)] aiming to create entangled quantum states which have potential applications in quantum information processing tasks. In practice, these states are created by some physical operations involving the interaction between several systems. Thus analyzing these operations with regard to the possibility of creating maximally entangled states from an initial unentangled one and characterization of entangling capabilities of quantum operators play an important role in quantum information theory. Considering two spin-$\frac{1}{2}$s in the symmetric subspace, one can then produce any entangled symmetric state by the time evolution of properly chosen Hamiltonian, for eg. NMR, NQR, quantum optics, Lipkin Hamiltonian [Lipkin & et al. (1965)] which is widely used in nuclear physics. As pointed out in Stockton et al. (2003), this necessarily does not lead to the most efficient way of creating a particular state. Knowing which states are prohibitively expensive to produce is an important experimental question. An interesting but difficult way to characterize this, is by quantifying the resources needed to create that state given a certain set of generators. Cartesian tensor operators $M_0, \ldots, M_8$ as defined in section 3.2 provide different logic gates for quantum computation. Since these two qubit symmetric gates are capable of producing entanglement, quantifying their entangling capability is very important. Makhlin (2002) has analyzed nonlocal properties of general two-qubit gates and also studied some basic properties of perfect entanglers which are defined as the unitary op-
operators that can generate maximally entangled states from some suitably chosen separable states. Zanardi et al. (2000) have explored the entangling power of quantum evolutions in terms of mean linear entropy produced when unitary operator acts on a given distribution of pure product states. Kraus & Cirac (2001), Rezakhani (2004) have given the tools to find the best separable two qubit input orthonormal product states such that some given unitary transformation can create maximally entangled quantum states. The entangling capability of a unitary quantum gate can be quantified by its entangling power $e_p(U)$ [Zanardi et al. (2000)].

We construct and study the properties of perfect entanglers acting on a symmetric subspace i.e., spin-1 operators. We show that the two qubit symmetric quantum gates expressed in terms of our newly defined basis set $M_4, \ldots, M_8$ belong to the class of perfect entanglers which can generate maximally entangled states from some suitably chosen product states. Here we also calculate the entangling power of two qubit symmetric gates following the simplified expression given by Balakrishnan & Sankaranarayanan (2010) in terms of local invariant $G_1$. Further we show that these symmetric two qubit gates belong to a family of special perfect entanglers under certain conditions. Also, the entangling property of Lipkin-Meshkov-Glick Hamiltonian [Pathak et al. (2008), Lipkin & et al. (1965)] is studied in the spin-1 subspace.

### 4.2 Symmetric two qubit gates

The algebraic study of symmetric two qubit gates is important not only for understanding fundamental properties of quantum circuits, but also to study possible experimental implementations in different physical systems. Hamiltonian evolution provides the hardware for quantum gates. i.e., the time evolution of the operators $M_k$’s provide various symmetric logic gates for quantum computation. The closed form expression for $e^{iM_k\theta}$ is given by

$$B_k = e^{iM_k\theta} = I + (\cos \theta - 1)M_k^2 + isin\theta M_k. \tag{4.1}$$
Here $k = 1, ..., 7$, $\theta = \zeta_k t$, where $\zeta_k$ is the coupling constant corresponding to each $M_k$ and $I$ is the $3 \times 3$ identity matrix. Following are the explicit forms of the gates $B_k$’s in the angular momentum basis $|11\rangle$, $|10\rangle$, $|1 - 1\rangle$:

$$B_1 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & -\frac{\sin \theta}{\sqrt{2}} & -\sin^2 \frac{\theta}{2} \\ -\frac{\sin \theta}{\sqrt{2}} & \cos \theta & -\frac{\sin \theta}{\sqrt{2}} \\ -\sin^2 \frac{\theta}{2} & \frac{\sin \theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & \sin \theta & \sin^2 \frac{\theta}{2} \\ -\frac{\sin \theta}{\sqrt{2}} & \cos \theta & \frac{\sin \theta}{\sqrt{2}} \\ \sin^2 \frac{\theta}{2} & -\frac{\sin \theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}, \quad B_4 = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix},$$

$$B_5 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & \frac{\sin \theta}{\sqrt{2}} & -\sin^2 \frac{\theta}{2} \\ -\frac{\sin \theta}{\sqrt{2}} & \cos \theta & \frac{\sin \theta}{\sqrt{2}} \\ -\sin^2 \frac{\theta}{2} & \frac{\sin \theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix}, \quad B_6 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & -\frac{\sin \theta}{\sqrt{2}} & \sin^2 \frac{\theta}{2} \\ -\frac{\sin \theta}{\sqrt{2}} & \cos \theta & \frac{\sin \theta}{\sqrt{2}} \\ \sin^2 \frac{\theta}{2} & \frac{\sin \theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix},$$

$$B_7 = \begin{pmatrix} \cos \theta & 0 & isin \theta \\ 0 & 1 & 0 \\ isin \theta & 0 & \cos \theta \end{pmatrix}, \quad B_8 = \begin{pmatrix} \frac{i\theta}{\sqrt{3}} & 0 & 0 \\ 0 & e^{-\frac{2i\theta}{\sqrt{3}}} & 0 \\ 0 & 0 & e^{\frac{i\theta}{\sqrt{3}}} \end{pmatrix}.$$
following the simplified expression given by Balakrishnan & Sankaranarayanan (2010) according to which the gate does not entangle if its entangling power

\[ e_p(B) = \frac{2}{9}(1 - |G_1|) \]  

is zero and is a perfect entangler for the range \( \frac{1}{8} \leq e_p \leq \frac{2}{3} \).

The local invariant \( G_1 \) in terms of symmetric, unitary matrix \( m \) is given by [Makhlin (2002), table II]

\[ G_1 = \frac{\text{tr}^2 m}{16 \text{det}[B]} \]  

Here

\[ m = B_{\text{Bell}}^T B_{\text{Bell}} \]  

where the gates in the Bell basis are given by \( B_{\text{Bell}} = U B U^\dagger \) and \( B = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \). \( U \) is a transformation matrix given by

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -\sqrt{2}i & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ -i & 0 & i & 0 \end{pmatrix} \],

which connects the angular momentum basis

\[ \{|1\rangle, |1\rangle, |1\rangle, |0\rangle\} \]

to the Bell basis

\[ \left\{ \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}, \frac{i(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}{\sqrt{2}}, \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}, \frac{i(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)}{\sqrt{2}} \right\} \].

The relation \( e_p(B) = \frac{2}{9}(1 - |G_1|) \) implies that gates having the same \( |G_1| \) must necessarily possess the same entangling power \( e_p \).

The gates \( B_1, B_2, B_3 \) do not produce entanglement as they represent rotations which is a local unitary transformation. For the above gates, \( \text{tr}(m) = 4 \) and \( \text{det}[B] = 1 \) and hence \( |G_1| \)
= 1 and \( e_p = 0 \). Interestingly, for the gates \( B_4, B_5, B_6 \) and \( B_7 \), \( \text{tr}(m) = 2(1 + \cos 2\theta) \) and hence \( tr^2(m) = 16\cos^4\theta \). Since \( \det(B) = 1 \), we have

\[
|G_1| = \cos^4(\theta).
\]

Since \( 0 \leq G_1 \leq 1 \), it is clear that \( 0 \leq e_p(B_{B_k}) \leq \frac{2}{9} \) (\( k = 4 \ldots 7 \)). The gates \( B_4 \ldots B_7 \) entangle for all values of \( \theta \) except when \( \theta = 0, \pi, 2\pi, 3\pi, \ldots \) and they are perfect entanglers for \( (2n + 1)\frac{\pi}{4} \leq \theta \leq (2n + 3)\frac{\pi}{4} \) where \( n = 0, 2, 4, 6, \ldots \). Similarly the gate \( B_8 \) has maximum entangling power i.e., \( e_p = 2/9 \) when \( \theta = \sqrt{3\pi}/2 \).

As an example, we consider the direct product state \( |\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \), of two spinors in the qubit basis \( |\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \).

\[
|\psi_{12}\rangle = \begin{pmatrix} \cos \frac{\alpha_1}{2} \\ \sin \frac{\alpha_1}{2} e^{i\phi_1} \end{pmatrix} \otimes \begin{pmatrix} \cos \frac{\alpha_2}{2} \\ \sin \frac{\alpha_2}{2} e^{i\phi_2} \end{pmatrix} = \begin{pmatrix} \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \\ \cos \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} e^{i\phi_2} \\ \sin \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} e^{i\phi_1} \\ \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} e^{i(\phi_1 + \phi_2)} \end{pmatrix},
\]

where \( 0 \leq \alpha_{1,2} \leq \pi \), \( 0 \leq \phi_{1,2} < 2\pi \). Note that a separable state in the angular momentum basis \( |11\rangle, |10\rangle, \) and \( |1 - 1\rangle \) has the form

\[
|\psi_{12}\rangle_{\text{sym}} = \begin{pmatrix} \cos \frac{\alpha}{2} \\ \sqrt{2} \sin \frac{\alpha}{2} \cos \frac{\phi}{2} e^{i\phi} \\ \sin \frac{\alpha}{2} e^{2i\phi} \end{pmatrix}, \tag{4.5}
\]

where \( \alpha = \frac{\alpha_1 + \alpha_2}{2} = \alpha \) and \( \phi = \frac{\phi_1 + \phi_2}{2} = \phi \).

It is a well known fact that for a pure state of two qubits

\[
|\psi\rangle = a |\uparrow\rangle + b |\uparrow\downarrow\rangle + c |\downarrow\uparrow\rangle + d |\downarrow\downarrow\rangle,
\]

the expression for concurrence \([\text{Wootters (1998)}]\) is

\[
C(\psi) = 2|ad - bc|.
\]

78
Chapter 4. Symmetric two, three qubit gates

For a maximally entangled quantum state concurrence $C = 1$. We have observed that under the action of the gates $B_4, B_7$ and $B_8$ (with $e_p$ being maximum i.e., $2/9$), $|\psi_{12}\rangle_{\text{sym}}$ becomes maximally entangled state when $\alpha = \frac{\pi}{2}$. i.e.,

$$B_4|\psi_{12}\rangle_{\text{sym}} \theta = \frac{\pi}{2} \rightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} |\psi_{12}\rangle_{\text{sym}} \alpha = \frac{\pi}{2} \rightarrow \begin{pmatrix} -\frac{1}{2}e^{2i\phi} \\ \frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{1}{2} \end{pmatrix},$$

$$B_7|\psi_{12}\rangle_{\text{sym}} \theta = \frac{\pi}{2} \rightarrow \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix} |\psi_{12}\rangle_{\text{sym}} \alpha = \frac{\pi}{2} \rightarrow \begin{pmatrix} i^2e^{2i\phi} \\ \frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{i}{2} \end{pmatrix},$$

$$B_8|\psi_{12}\rangle_{\text{sym}} \theta = \sqrt{3}\frac{\pi}{2} \rightarrow \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} |\psi_{12}\rangle_{\text{sym}} \alpha = \frac{\pi}{2} \rightarrow \begin{pmatrix} 0 \\ \frac{1}{2}e^{2i\phi} \\ \frac{i}{2}e^{2i\phi} \end{pmatrix},$$

or in the qubit basis

$$B_4|\psi_{12}\rangle_{\text{sym}} \alpha = \frac{\pi}{2} \rightarrow - \frac{1}{2}e^{2i\phi} |\uparrow\uparrow\rangle + \frac{1}{2}e^{i\phi} |\uparrow\downarrow\rangle + \frac{1}{2}e^{i\phi} |\downarrow\uparrow\rangle + \frac{1}{2} |\downarrow\downarrow\rangle,$$

$$B_7|\psi_{12}\rangle_{\text{sym}} \alpha = \frac{\pi}{2} \rightarrow \frac{i}{2}e^{2i\phi} |\uparrow\uparrow\rangle + \frac{1}{2}e^{i\phi} |\uparrow\downarrow\rangle + \frac{1}{2}e^{i\phi} |\downarrow\uparrow\rangle + \frac{i}{2} |\downarrow\downarrow\rangle,$$

$$B_8|\psi_{12}\rangle_{\text{sym}} \alpha = \frac{\pi}{2} \rightarrow - \frac{i}{2} |\uparrow\uparrow\rangle + \frac{1}{2}e^{i\phi} |\uparrow\downarrow\rangle + \frac{1}{2}e^{i\phi} |\downarrow\uparrow\rangle + \frac{i}{2}e^{2i\phi} |\downarrow\downarrow\rangle.$$

Similarly, the gates $B_5, B_6$ acting on the symmetric separable state transform it into maximally entangled one when $\alpha = 0, \pi$. For eg:

$$B_5|\psi_{12}\rangle_{\text{sym}} \theta = \frac{\pi}{2} \rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} |\psi_{12}\rangle_{\text{sym}} \alpha = 0 \rightarrow \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix},$$

79
Chapter 4. Symmetric two, three qubit gates

\[
B_6 |\psi_{12}\rangle_{\text{sym}} \theta = \frac{\pi}{2} \left( \begin{array}{cccc}
\frac{1}{2} & -i\sqrt{2} & \frac{1}{2} \\
-i\sqrt{2} & 0 & \frac{i}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & \frac{1}{2}
\end{array} \right) |\psi_{12}\rangle_{\text{sym}} \alpha = 0 \rightarrow \left( \begin{array}{c}
\frac{1}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} \\
\frac{1}{2}
\end{array} \right).
\]

\[
B_5 |\psi_{12}\rangle_{\text{sym}} \alpha = 0 \rightarrow \frac{1}{2} |\uparrow\uparrow\rangle - \frac{1}{2} |\uparrow\downarrow\rangle - \frac{1}{2} |\downarrow\uparrow\rangle - \frac{1}{2} |\downarrow\downarrow\rangle.
\]

\[
B_6 |\psi_{12}\rangle_{\text{sym}} \alpha = 0 \rightarrow \frac{1}{2} |\uparrow\uparrow\rangle - \frac{i}{2} |\uparrow\downarrow\rangle - \frac{i}{2} |\downarrow\uparrow\rangle + \frac{1}{2} |\downarrow\downarrow\rangle.
\]

It is found that concurrence \( C = 1 \) in all these cases. Here the operators \( B_8 \) and \( B_4 \) produce spin squeezing resulting from a single axis twisting and two axis counter twisting respectively [Kitagawa & Ueda (1991), Kitagawa & Ueda (1993)]. Also, possibility of physical realization of these spin squeezing operators are to be found in Pathak et al. (2008).

### 4.2.1 Special perfect entanglers

Rezakhani (2004) has analyzed the perfect entanglers and found that some of them have the unique property of maximally entangling a complete set of orthonormal product vectors. Such operators for which \( e_p = \frac{2}{9} \) belong to a well known family of special perfect entanglers. A study of using such special perfect entanglers as the building blocks of the most efficient universal gate simulation is also given in Rezakhani (2004). We study the conditions under which the perfect entanglers can be classified as special perfect entanglers. When \( e_p = \frac{2}{9} \), \( B_4, \ldots, B_8 \) in the qubit basis \( |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \) are given by \( U(B)_{jm}U^\dagger = B_{\text{qubit}} \), where \( B_{jm} \) is in the basis \{ |11\rangle, |10\rangle, |1 - 1\rangle, |00\rangle \}. Explicitly,

\[
B_4 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad B_5 = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & 1 & 1 & 1
\end{pmatrix}.
\]
Chapter 4. Symmetric two, three qubit gates

\[
B_6 = \frac{1}{2} \begin{pmatrix}
1 & -i & -i & 1 \\
-i & 1 & -1 & i \\
-i & 1 & 1 & i \\
1 & i & i & 1
\end{pmatrix},
B_7 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
i & 0 & 0 & 0
\end{pmatrix},
B_8 = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & i
\end{pmatrix}.
\]

Following Rezakhani (2004), the most general separable basis (upto general phase factors for each vector) is

\[
|\psi_1\rangle = (a|\uparrow\rangle + b|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle),
\]

\[
|\psi_2\rangle = (-b^*|\uparrow\rangle + a^*|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle),
\]

\[
|\psi_3\rangle = (e|\uparrow\rangle + f|\downarrow\rangle) \otimes (-d^*|\uparrow\rangle + c^*|\downarrow\rangle),
\]

\[
|\psi_4\rangle = (-f^*|\uparrow\rangle + e^*|\downarrow\rangle) \otimes (-d^*|\uparrow\rangle + c^*|\downarrow\rangle),
\]

where \(|a|^2 + |b|^2 = |c|^2 + |d|^2 = |e|^2 + |f|^2 = 1\).

The gates \(B_4\), \(B_7\) and \(B_8\) acting on \(|\psi_1\rangle\) behave as perfect entanglers provided \(C = 4|abcd| = 1\).

\[
[B_{4,7,8}|\psi_1\rangle = -bd|\uparrow\uparrow\rangle + ad|\uparrow\downarrow\rangle + bc|\downarrow\uparrow\rangle + ac|\downarrow\downarrow\rangle.\]

Thus the above gates transform the orthonormal states \(|\psi_1\rangle\), \(|\psi_2\rangle\), \(|\psi_3\rangle\) and \(|\psi_4\rangle\) into maximally entangled ones if

\[
|abcd| = |cdef| = \frac{1}{4}.
\]

Similarly, for the gates \(B_5\) and \(B_6\), condition for finding a full set of orthonormal product states is

\[
|(a^2 + b^2)(c^2 + d^2)| = |(e^2 + f^2)(c^2 + d^2)| = 1.
\]
Chapter 4. Symmetric two, three qubit gates

As a realistic example, we consider Lipkin-Meshkov-Glick interaction Hamiltonian for detailed analysis.

### 4.3 Lipkin-Meshkov-Glick interaction Hamiltonian

Lipkin-Meshkov-Glick interaction Hamiltonian [Lipkin & et al. (1965); Pathak et al. (2008)] which is widely used in nuclear physics is given by

\[
H_L = G_1 (J_+^2 + J_-^2) + G_2 (J_+ J_- + J_- J_+) .
\]  

(4.6)

Here \(G_1\) and \(G_2\) are the coupling constants. In terms of our operators \(M_k\)'s,

\[
H_L = 2G_1 M_7 + \frac{2}{\sqrt{3}} G_2 (\sqrt{8} M_0 - M_8) ,
\]

(4.7)

or

\[
H_L = G'_1 M_7 + G'_2 (\sqrt{8} M_0 - M_8) ,
\]

(4.8)

where \(G'_1 = 2G_1\) and \(G'_2 = \frac{2}{\sqrt{3}} G_2\). And taking the time evolution of \(H_L\),

\[
\exp(iH_lt) = \exp(i(G'_1 M_7 + G'_2 (\sqrt{8} M_0 - M_8))t) .
\]

(4.9)

Since \([M_7, M_8] = 0\), we have the gate \(B_L\) in spin-1 subspace as

\[
e^{iH_L t} = B_L = \begin{pmatrix}
 e^{\sqrt{3}i\beta\cos\xi} & 0 & ie^{\sqrt{3}i\beta\sin\xi} \\
 0 & e^{2\sqrt{3}i\beta} & 0 \\
 ie^{\sqrt{3}i\beta\cos\xi} & 0 & e^{\sqrt{3}i\beta\cos\xi}
\end{pmatrix} .
\]

(4.10)
Chapter 4. Symmetric two, three qubit gates

Here $\xi = G'_1 t = 2G_1 t$ and $\beta = G'_2 t = \frac{2}{\sqrt{3}} G_2 t$. The gate $B_L$ in Bell basis is given by

$$B_L = \begin{pmatrix}
 e^{\sqrt{3}i\beta} e^{i\xi} & 0 & 0 & 0 \\
 0 & e^{2\sqrt{3}i\beta} & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & e^{\sqrt{3}i\beta} e^{-i\xi}
\end{pmatrix}.$$  \hspace{1cm} (4.11)

Then

$$m = (B_L)^T (B_L)_B = \begin{pmatrix}
 e^{2\sqrt{3}i\beta} (e^{i\xi})^2 & 0 & 0 & 0 \\
 0 & e^{4\sqrt{3}i\beta} & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & e^{2\sqrt{3}i\beta} (e^{-i\xi})^2
\end{pmatrix}.$$  \hspace{1cm} (4.12)

We have

$$e_p = \frac{2}{9} \left[ 1 - \left| (G_1)_L \right| \right],$$

where

$$(G_1)_L = \frac{Tr^2 m}{16 \det(B_L)}.$$

$$e_p = \frac{2}{9} \text{ for } 4G_2 t = \pi + 4G_1 t.$$
Chapter 4. Symmetric two, three qubit gates

As an example we consider the action of this gate on a most general separable state in spin-1 subspace as given in equation (4.5). Under the action of this gate (with $e_p = \frac{2}{9}$), the separable state $|\uparrow\uparrow\rangle (|\downarrow\downarrow\rangle)$ becomes entangled for all values of $t$ except when $t = \frac{n\pi}{4G_1}$, $n=0,1,2,...$ and maximally entangled when

$$4G_1 t = (2n + 1)\frac{\pi}{2}.$$ 

For eg.,

$$B_L|\psi_{12}\rangle_{sym}^{\alpha=0} \rightarrow cos(2G_1 t) |\uparrow\uparrow\rangle + isin(2G_1 t) |\downarrow\downarrow\rangle.$$ 

We have considered unitary evolutions of two spin-$\frac{1}{2}$ states in angular momentum subspace ($j=1$) and constructed physically realizable logic gates using $(2j+1)$ dimensional representation of the above set of basis matrices. Entangling properties of these gates have been studied in terms of their entangling power $e_p$. $e_p$ is found to be maximum ($2/9$) for $B_4, ..., B_8$ under certain conditions which is the signature for special perfect entanglers. These logic gates are obtained by the exponentiation of the quadratic form of angular momentum operators $J_x, J_y, J_z$. As an example we have taken the well known Lipkin-Meshkov-Glick Hamiltonian and studied its entangling properties in spin-1 subspace. Further, we have shown that precisely at what time the initial separable state becomes maximally entangled under the action of perfect entanglers which consists of one-axis twisting and two axis twisting Hamiltonians that produce spin squeezing.

4.4 Symmetric three qubit gates

The closed form expression for $e^{iM_k\theta}$ ($k = 4,..,7,11,15$) is

$$e^{iM_k\theta} = B_k = cos\theta + iM_k sin\theta.$$ 

84
Explicitly, in angular momentum basis $|\frac{3}{2}, \frac{3}{2}\rangle, |\frac{3}{2}, \frac{1}{2}\rangle, |\frac{3}{2}, -\frac{1}{2}\rangle, |\frac{3}{2}, -\frac{3}{2}\rangle$ we have

\[
B_4 = \begin{pmatrix}
\cos\theta & 0 & -\sin\theta & 0 \\
0 & \cos\theta & 0 & -\sin\theta \\
\sin\theta & 0 & \cos\theta & 0 \\
0 & \sin\theta & 0 & \cos\theta \\
\end{pmatrix}, \quad B_5 = \begin{pmatrix}
\cos\theta & -\sin\theta & 0 & 0 \\
-\sin\theta & \cos\theta & 0 & 0 \\
0 & 0 & \cos\theta & \sin\theta \\
0 & 0 & \sin\theta & \cos\theta \\
\end{pmatrix},
\]

\[
B_6 = \begin{pmatrix}
\cos\theta & \sin\theta & 0 & 0 \\
-\sin\theta & \cos\theta & 0 & 0 \\
0 & 0 & \cos\theta & -\sin\theta \\
0 & 0 & \sin\theta & \cos\theta \\
\end{pmatrix}, \quad B_7 = \begin{pmatrix}
\cos\theta & 0 & \sin\theta & 0 \\
0 & \cos\theta & 0 & \sin\theta \\
\sin\theta & 0 & \cos\theta & 0 \\
0 & \sin\theta & 0 & \cos\theta \\
\end{pmatrix},
\]

\[
B_{11} = \begin{pmatrix}
\cos\theta & 0 & \sin\theta & 0 \\
0 & \cos\theta & 0 & -\sin\theta \\
\sin\theta & 0 & \cos\theta & 0 \\
0 & -\sin\theta & 0 & \cos\theta \\
\end{pmatrix}, \quad B_{15} = \begin{pmatrix}
\cos\theta & 0 & -\sin\theta & 0 \\
0 & \cos\theta & 0 & \sin\theta \\
\sin\theta & 0 & \cos\theta & 0 \\
0 & -\sin\theta & 0 & \cos\theta \\
\end{pmatrix}.
\]

These gates acting on a separable state say, $|\uparrow\uparrow\uparrow\rangle$ or $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ give rise to W states when $\theta = \frac{\pi}{2}$. For example,

\[
B_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_5 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]

\[
B_6 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad B_7 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]
Chapter 4. Symmetric two, three qubit gates

\[
B_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Note: \(M_8\) and \(M_{14}\) are diagonal matrices.

Since the closed form expressions for \(B_9\) and \(B_{10}\) are complicated, let us consider the following to evaluate the final state. For example, action of \(B_9\) on separable initial state \(|\uparrow\uparrow\uparrow\rangle\) or \(|\downarrow\downarrow\downarrow\rangle\);

\[
|\psi_{\text{final}}\rangle = U^\dagger e^{i\theta t} U |\uparrow\uparrow\uparrow\rangle
\]
such that \(UM_9U^\dagger = M^d_9\). Here \(M^d_9\) is the diagonal form of \(M_9\). Therefore

\[
|\psi_{\text{final}}\rangle = U^\dagger e^{i\frac{3\pi}{4} t} \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |\text{GHZ}\rangle.
\]

The above condition holds good when \(\theta = \sqrt{2} t = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}\).

Here \(U = \begin{pmatrix} \frac{i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\).

Similarly,

\[
|\psi_{\text{final}}\rangle = U^\dagger e^{i\theta t} U |\downarrow\downarrow\downarrow\rangle
\]

\[
|\psi_{\text{final}}\rangle = U^\dagger e^{i\frac{3\pi}{4} t} \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |\text{GHZ}\rangle.
\]

It is of interest to study maximally entangled states generated by the action of these gates in terms of the local invariants, the study of which we will take up in the next chapter.