CHAPTER V
SOME BIVARIATE EXTENSIONS

5.1 INTRODUCTION

Heavy tailed bivariate distributions with different tail index are used for modeling bivariate data. Considering this, Kozubowski et al. (2005) introduced marginal Laplace and Linnik distributions. A random vector $X = (X_1, X_2)$ is said to have marginal Laplace and Linnik distribution if it has the characteristic function

$$
\psi(t,s) = \frac{1}{1 + \lambda_1 t^2 + \lambda_2 |s|^\alpha}, \quad 0 < \alpha \leq 2, \quad \lambda_1, \lambda_2 > 0, \quad t, s \in R.
$$

Note that,

$$
\psi(t,0) = \frac{1}{1 + \lambda_1 t^2}.
$$

This Chapter is based on Mariamma Antony and Raju (2008b) and (2008c)

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and
\[ \psi(0, s) = \frac{1}{1 + \lambda_2 |s|^{\alpha}}. \]

Kozubowski et al. (2005) derived the representation of \( X \) as
\[ X \sim U^{1/2}X_1, U^{1/2}X_2 \]
where \( U \) is unit exponential, \( X_1 \) and \( X_2 \) are normal and \( \alpha \) stable random variables with respective characteristic functions
\[ \psi_1(t) = e^{-\lambda_1 t^2} \quad \text{and} \quad \psi_2(s) = e^{-\lambda_2 |s|^{\alpha}}. \]

The marginal Laplace and Linnik distribution can be generalized to include the asymmetry in the data as follows:

Consider a random vector \( X = (X_1, X_2) \) with characteristic function
\[ \psi(t, s) = \frac{1}{1 + \lambda_1 t^2 + \lambda_2 |s|^{\alpha} - i\mu t - i\nu s}, \lambda_1, \lambda_2 > 0, 0 < \alpha \leq 2, \mu, \nu \in \mathbb{R}. \]

Note that in this case, \( \psi(t, 0) = \frac{1}{1 + \lambda_1 t^2 - i\mu t} \) and \( \psi(0, s) = \frac{1}{1 + \lambda_2 |s|^{\alpha} - i\nu s}. \) We call the
distribution with characteristic function $\psi(t, s)$ as marginal asymmetric Laplace and asymmetric Linnik distribution.

In Section 2, we introduce geometric marginal asymmetric Laplace and asymmetric Linnik distribution and study its properties. In Section 3, we consider geometric marginal asymmetric Linnik and asymmetric Linnik distribution and its extensions. Geometric bivariate semi-$\alpha$-Laplace distribution are introduced and studied in Section 4.

5.2 GEOMETRIC MARGINAL ASYMMETRIC LAPLACE AND ASYMMETRIC LINNIK DISTRIBUTION

Kozubowskii et al. (2005) introduced and studied a class of multivariate distributions called operator geometric stable laws by generalizing operator stable and geometric stable laws. As a particular case, they studied a new class of bivariate distributions namely marginal Laplace and Linnik distributions. Kutyikrishnan and Jayakumar (2005) generalized this class of distributions and introduced and studied a class of bivariate distributions that contains marginal Laplace and Linnik distributions. The resulting class of bivariate distributions namely generalized marginal asymmetric Laplace and asymmetric Linnik (GeMALaAL) distributions have the characteristic function
\[
\phi(t,s) = \left[ \frac{1}{1 + \lambda_{1}t^2 + \lambda_{2}|s|^\alpha - i\mu t - \lambda \nu s} \right]^\tau,
\]
(5.2.1)

\[\lambda_{1}, \lambda_{2} > 0, -\infty < \mu, \nu < \infty, \tau \geq 0, \alpha \in (0, 2].\]

Let \(\psi(t,s)\) be the characteristic function of a geometrically infinitely divisible bivariate distribution given by the equation \(\phi(t,s) = e^{\left(1 - \frac{1}{\psi(t,s)}\right)}\) where \(\phi(t,s)\) is the characteristic function of an infinitely divisible bivariate distribution.

Substituting (5.2.1) in the equation \(\phi(t,s) = e^{\left(1 - \frac{1}{\psi(t,s)}\right)}\), we obtain

\[\psi(t,s) = \frac{1}{1 + \tau \ln \left(1 + \lambda_{1}t^2 + \lambda_{2}|s|^\alpha - i\mu t - \lambda \nu s\right)}.
\]
(5.2.2)

\[\lambda_{1}, \lambda_{2} > 0, \tau \geq 0, -\infty < \mu, \nu < \infty, \alpha \in (0, 2].\]

Hence \(\psi(t,s) = \frac{1}{1 + \tau \ln \left(1 + \lambda_{1}t^2 + \lambda_{2}|s|^\alpha - i\mu t - \lambda \nu s\right)}\) is the characteristic function of a geometrically infinitely divisible bivariate distribution.

A bivariate distribution with characteristic function (5.2.2) is called Type I generalized geometric marginal asymmetric Laplace and asymmetric Linnik GeGMALaAL\(_{1}\) distribution with parameters \(\alpha, \mu, \nu, \lambda_{1}, \lambda_{2}, \tau\). Note that when
\( \mu = \nu = 0 \) and \( \tau = 1 \), this becomes geometric marginal Laplace and Linnik distribution.

If \((X,Y)\) is bivariate random vector with characteristic function \((5.2.2)\), we represent it as \((X,Y) \overset{d}{=} GeGMALaAL_{1}(\alpha, \lambda_1, \lambda_2, \mu, \nu, \tau)\). An asymptotic property of \(GeGMALaAL_{1}\) distribution is given in the following theorem.

**THEOREM 5.2.1**

The \(GeGMALaAL_{1}\) distribution is the limit distribution of the geometric sums of \(GeMALaAL\) random variables.

The theorem can be proved using the argument similar to the Proof of Theorem 3.3.1.

Now it is useful to develop a bivariate time series model using the \(GeGMALaAL_{1}\) marginal distribution. A one parameter autoregressive model equivalent to \(TEAR(1)\) structure of Lawrance and Lewis (1981) can be constructed corresponding to the set of bivariate time series data as follows:

Let \(\{(\varepsilon_n, \eta_n), n \geq 1\}\) be a sequence of independent and identically distributed bivariate random vectors and let \((X_0, Y_0) \overset{d}{=} GeGMALaAL_{1}(\alpha, \lambda_1, \lambda_2, \mu, \nu, \tau)\).
Define \( \{(X_n, Y_n), n \geq 1\} \) as
\[
X_n = \begin{cases} 
\varepsilon_n & \text{w.p. } p \\
X_{n-1} + \varepsilon_n & \text{w.p. } 1-p 
\end{cases}
\]

and
\[
Y_n = \begin{cases} 
\eta_n & \text{w.p. } p \\
X_{n-1} + \eta_n & \text{w.p. } 1-p 
\end{cases} \tag{5.2.3}
\]

where \( 0 < p < 1 \).

Let \( \Phi_{X_n, Y_n}(t, s) \) and \( \Phi_{\varepsilon_n, \eta_n}(t, s) \) be the characteristic functions of \( (X_n, Y_n) \) and \( (\varepsilon_n, \eta_n) \) respectively. Then (5.2.3) gives
\[
\Phi_{(\varepsilon_n, \eta_n)}(t, s) = \frac{\Phi_{(X_n, Y_n)}(t, s)}{p + (1-p)\Phi_{(X_{n-1}, Y_{n-1})}(t, s)}. \tag{5.2.4}
\]

If \( \{(X_n, Y_n)\} \) is a stationary sequence with \( GeGMaLaAL_4(\alpha, \lambda_1, \lambda_2, \mu, \nu, \tau) \) marginal distribution, then from (5.2.4) we get
\[
\Phi_{(\varepsilon_n, \eta_n)}(t, s) = \frac{1}{1 + p\tau \ln \left( 1 + \lambda_1 t^2 + \lambda_2 |s|^\alpha - i\mu t - \kappa s \right)}. \tag{5.2.5}
\]

Hence \( (\varepsilon_n, \eta_n) \approx GeGMaLaAL_4(\alpha, \lambda_1, \lambda_2, \mu, \nu, p\tau) \).
Also it can be verified that if \((X_0, Y_0) \sim GeGMLaAL_4(\alpha, \lambda_1, \lambda_2, \mu, \nu, \tau)\) and \(\{(\varepsilon_n, \eta_n), n \geq 1\}\) is an independent and identically distributed sequence of bivariate random variables with characteristic function given by (5.2.5), the first order autoregressive process (5.2.3) is stationary with \(GeGMLaAL_4(\alpha, \lambda_1, \lambda_2, \mu, \nu, \tau)\) marginal distribution.

Hence, we have the following theorem.

**THEOREM 5.2.3**

Let \(\{(\varepsilon_n, \eta_n), n \geq 1\}\) be a sequence of independent and identically distributed \(GeGMLaAL_4(\alpha, \lambda_1, \lambda_2, \mu, \nu, p\tau)\) random vectors and \((X_0, Y_0) \sim GeGMLaAL_4(\alpha, \lambda_1, \lambda_2, \mu, \nu, \tau)\). Then the relation (5.2.3) defines a stationary bivariate time series with \(GeGMLaAL_4\) marginal distribution.

**5.3 GEOMETRIC MARGINAL ASYMMETRIC LINNIK AND ASYMMETRIC LINNIK DISTRIBUTION AND ITS EXTENSIONS**

In practice we come across bivariate random vectors where the components of the vectors have heavy tails than normal distribution and component distributions are asymmetric with steep peak.
DEFINITION 5.3.1

Let $X = (X_1, X_2)$ be a random vector with characteristic function

$$
\phi(t,s) = \frac{1}{1 + \lambda_1 |t|^{\alpha_1} + \lambda_2 |s|^{\alpha_2} - i\mu t - i\nu s},
$$

where $\lambda_1, \lambda_2 > 0$, $\tau \geq 0$, $-\infty < \mu, \nu < \infty$, $0 < \alpha_1, \alpha_2 \leq 2$.

Then we say that $X = (X_1, X_2)$ has generalized marginal asymmetric Linnik and asymmetric Linnik distribution and we denote $X$ by $X \sim \text{GeMALAL} (\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau)$.

The geometric version of this and its generalization are the subject of study in this Section.

DEFINITION 5.3.2

A random vector $X = (X_1, X_2)$ is said to have geometric marginal asymmetric Linnik and asymmetric Linnik distribution and write $X \sim \text{GMALAL} (\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu)$ distribution if it has the following characteristic function
\[
\phi(t,s) = \frac{1}{1 + \ln \left(1 + \lambda_1 \left| t \right|^\alpha_1 + \lambda_2 \left| s \right|^\alpha_2 - i \mu t - \lambda_2 s \right)},
\]

\[
\lambda_1, \lambda_2 > 0, \tau \geq 0, -\infty < \mu, \nu < \infty, 0 < \alpha_1, \alpha_2 \leq 2.
\]

Note that when \(\alpha_1 = 2\) and \(\alpha_2 = \alpha\), the geometric marginal asymmetric Linnik and asymmetric Linnik distribution turns out to be geometric marginal asymmetric Laplace and asymmetric Linnik distribution studied in Section 2.

**REMARK 5.3.1**

As in Chapters II and III, the two generalizations of the geometric marginal asymmetric Linnik asymmetric Linnik distribution are Type I generalized geometric marginal asymmetric Linnik and asymmetric Linnik distribution defined by the characteristic function

\[
\phi(t,s) = \left[ \frac{1}{1 + \tau \ln \left(1 + \lambda_1 \left| t \right|^\alpha_1 + \lambda_2 \left| s \right|^\alpha_2 - i \mu t - \lambda_2 s \right)} \right]^{(5.3.1)}
\]

denoted by \(X = (X_1, X_2) \sim GeGMALAL_1(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau)\) and Type II generalized geometric marginal asymmetric Linnik and asymmetric Linnik distribution defined by the characteristic function.
\[
\phi(t, s) = \left[ \frac{1}{1 + \ln \left( 1 + \lambda_1 \left| t \right|^{\alpha_1} + \lambda_2 \left| s \right|^{\alpha_2} - i\mu t - \nu s \right)} \right]^{\frac{s}{n}}
\]
denoted by \( X \overset{d}{\sim} \text{GeGMALAL}_2(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau) \).

**THEOREM 5.3.1**

The \( \text{GeGMALAL}_4(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau) \) distribution is the limit distribution of geometric sums of \( \text{GeMALAL}(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \frac{\tau}{n}) \) random variables.

**PROOF**

Let \( \phi(t, s) \) be the characteristic function of \( \text{GeMALAL}(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \frac{\tau}{n}) \). Then

\[
\phi(t, s) = \left[ \frac{1}{1 + \lambda_1 \left| t \right|^{\alpha_1} + \lambda_2 \left| s \right|^{\alpha_2} - i\mu t - \nu s} \right]^{\frac{s}{n}}.
\]

Define \( \Theta(t, s) = \frac{1}{\phi(t, s)} - 1 \)

\[
= \left( 1 + \lambda_1 \left| t \right|^{\alpha_1} + \lambda_2 \left| s \right|^{\alpha_2} - i\mu t - \nu s \right)^{-\frac{s}{n}} - 1.
\]
Consider \( \phi_n(t,s) = \frac{1}{1 + n \left(1 + \lambda_1 |t|^{\alpha_1} + \lambda_2 |s|^{\alpha_2} - i\mu t + i\nu s \right)^{\frac{1}{n}} - 1} \).

\[
\lim_{n \to \infty} \phi_n(t,s) = \frac{1}{1 + \lim_{n \to \infty} n \left(1 + \lambda_1 |t|^{\alpha_1} + \lambda_2 |s|^{\alpha_2} - i\mu t + i\nu s \right)^{\frac{1}{n}} - 1}.
\]

\[
= \frac{1}{1 + \tau \ln \left(1 + \lambda_1 |t|^{\alpha_1} + \lambda_2 |s|^{\alpha_2} - i\mu t + i\nu s \right)}.
\]

Now we develop a bivariate time series model using GeGMALAL marginal distribution.

Let \( \{(\varepsilon_n, \eta_n), n \geq 1\} \) be a sequence of independent and identically distributed bivariate random vectors and let \( (X_0, Y_0) \) be GeGMALAL \((\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau) \) be a random vector with characteristic function (5.3.1). Define \( \{(X_n, Y_n), n \geq 1\} \) as

\[
X_n = \begin{cases} 
\varepsilon_n & \text{w.p. } p \\
X_{n-1} + \varepsilon_n & \text{w.p. } 1 - p
\end{cases} \quad (5.3.2)
\]

\[
Y_n = \begin{cases} 
\eta_n & \text{w.p. } p \\
X_{n-1} + \eta_n & \text{w.p. } 1 - p
\end{cases}
\]

where \( 0 < p < 1 \).
Let $\phi(x_n,y_n)(t,s)$ and $\phi(\varepsilon_n,\eta_n)(t,s)$ be the characteristic functions of $\{(X_n, Y_n)\}$ and $\{(\varepsilon_n, \eta_n)\}$ respectively.

Then (5.3.2) gives

$$\phi(\varepsilon_n, \eta_n)(t,s) = \frac{\phi(x_n,y_n)(t,s)}{p + (1-p)\phi(x_{n-1},y_{n-1})(t,s)}.$$  \hfill (5.3.3)

If $\{(X_n, Y_n)\}$ is a stationary sequence with $\text{GeGMALAL}_{4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau)$ marginal distribution, then from (5.3.3) we get

$$\phi(\varepsilon_n, \eta_n)(t,s) = \frac{1}{1 + p\tau \ln\left(1 + \lambda_1|t|^\alpha_1 + \lambda_2|s|^\alpha_2 - i\mu t - \nu s\right)}.$$  

Hence

$$\phi(\varepsilon_n, \eta_n) \sim \text{GeGMALAL}_{4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau).$$  \hfill (5.3.4)

Also it can be verified that if

$$(X_0, Y_0) \sim \text{GeGMALAL}_{4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau)$$  and \{$(\varepsilon_n, \eta_n), n \geq 1$\} is independent and identically distributed sequence of bivariate random variables given by (5.3.4) then the first order autoregressive process (5.3.2) is stationary with $\text{GeGMALAL}_{4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu, \nu, \tau)$ marginal distribution.
5.4 GEOMETRIC BIVARIATE SEMI $\alpha$ LAPLACE DISTRIBUTION

A bivariate semi $\alpha$ Laplace distribution $X$ is defined by the characteristic function

$$\phi(t, s) = \frac{1}{1 + \delta(t, s)} \quad (5.4.1)$$

where $\delta(t, s)$ satisfies the functional equation

$$\delta(t, s) = \frac{1}{p} \delta\left(\frac{t}{\alpha_1 t}, \frac{s}{\alpha_2 s}\right), \quad 0 < p < 1, 0 < \alpha_1, \alpha_2 \leq 2. \quad (5.4.2)$$

A solution of the equation (5.4.2) is

$$\delta(t, s) = |t|^{\alpha_1} \delta_1(t) + |s|^{\alpha_2} \delta_2(s)$$

where $\delta_1(t)$ and $\delta_2(s)$ are periodic functions in $\ln|t|$ and $\ln|s|$ with periods $\frac{-2\pi\alpha_1}{\ln p}$ and $\frac{-2\pi\alpha_2}{\ln p}$ respectively.

The solution of (5.4.2) is not unique.

For example, the function

$$\delta(t, s) = \left[\frac{1}{2}(t, s)\Sigma(t, s)\right]^{\alpha/2}, \quad 0 < \alpha \leq 2 \quad (5.4.3)$$
and \( \Sigma \) is any non negative definite matrix, satisfy the functional equation (5.4.3) with \( \alpha_1 = \alpha_2 = \alpha \) and any \( p \in (0, 1) \). When
\[
\Sigma = \begin{bmatrix}
\sigma_1^2 & \rho_1 \sigma_1 \sigma_2 \\
\rho_1 \sigma_1 \sigma_2 & \sigma_2^2
\end{bmatrix}
\]
in (5.4.3) with \( \alpha = 2 \), then (5.4.1) becomes
\[
\phi(t, s) = \frac{1}{1 + \frac{\sigma_1^2 t^2}{2} + \frac{\sigma_2^2 s^2}{2} + \rho \sigma_1 \sigma_2 ts}, \sigma_1, \sigma_2 > 0, 0 < \rho \leq 1
\]
is the characteristic function of a bivariate Laplace distribution (for details, see Kuttykrishnan and Jayakumar (2008) and Kotz et al. (2001)). Kuttykrishnan and Jayakumar (2008) studied the properties of bivariate semi \( \alpha \) Laplace distribution in (5.4.1) and obtained some characterizations of the distribution.

In this Section, we introduce and study geometric bivariate semi \( \alpha \) Laplace distribution. A random vector \( \mathbf{X} \) is said to have geometric bivariate semi \( \alpha \) Laplace distribution if it has characteristic function
\[
\phi(t, s) = \frac{1}{1 + \ln \left(1 + \delta(t, s)\right)} \tag{5.4.4}
\]
where \( \delta(t, s) \) satisfies the equation (5.4.2). Note that the distribution we discuss here has the form
\[ \phi(t,s) = \frac{1}{1 + \ln\left(1 + \left|t\right|^{\alpha_1} \delta_1(t) + \left|s\right|^{\alpha_2} \delta_2(t)\right)} \]

where \( \delta_1(t) = \delta_1\left(p^{1/\alpha_1} t\right) \) and \( \delta_2(s) = \delta_2\left(p^{1/\alpha_2} s\right) \) for every \( 0 < \alpha_1, \alpha_2 \leq 2 \) and for some \( p \in (0,1) \). When \( \delta_1(t) = \delta_2(s) = 1 \) and \( \alpha_1 = 2 \), the geometric bivariate semi-\( \alpha \) Laplace distribution reduces to geometric marginal Laplace and Linnik distribution.