Chapter 3

EoS of QGP in Finite Volume

3.1 Introduction

The study of QGP employing relativistic heavy ion collisions is one of the largest current activities in the field of high energy physics. Recent studies on quark stars, neutron stars and hybrid stars indicate the presence of QGP in these objects [1]. There are many questions that need answers on the formation and evolution of QGP. One such question is the formulation of the exact confinement potential and derivation of the EoS. In the present study EoS for gluon plasma and quark gluon plasma are developed using a statistical thermodynamics model in the framework of grand canonical ensemble with relativistic harmonic oscillator potential [2-4].
3.2 Statistical Mechanics

From statistical mechanics the EoS for non-interacting Bose and Fermi systems are given by [5]

\[
\ln Z = -g_I \sum_{p,r} \ln\left(1 - e^{-\frac{H(p,r)}{T}}\right) \quad (3.1)
\]

\[
\ln Z = +g_I \sum_{p,r} \ln\left(1 + e^{-\frac{H(p,r)}{T}}\right) \quad (3.2)
\]

respectively where \( Z \) is the grand partition function, \( g_I \) is the internal degrees of freedom and \( T \) is the absolute temperature. For gluons \( g_I \) is equal to 16 and for quarks it is equal to \( 12n_f \) where \( n_f \) is the number of flavours. The finite size correction to the partition function is small and hence can be neglected for any classical system[6]. In the Stefan-Boltzmann limit which is the high temperature limit, we will express pressure by \( P \equiv P_s \), energy density \( \epsilon \equiv \epsilon_s \) and number density \( n \equiv n_s \). Throughout this thesis we use natural units where the velocity of light \( c \), the Dirac constant \( \hbar \) and Boltzmann constant \( k \) are all equal to unity.

3.3 EoS for Gluon Plasma

Using Eq (2.18) \( H(p, r) = \sqrt{p^2 + C_g^4 r^2} \), logarithm of the partition function becomes

\[
\ln Z = -\frac{2g_I}{\pi} \int_0^R r^2 dr \int_0^\infty p^2 dp \ln \left(1 - e^{-\frac{\sqrt{p^2 + C_g^4 r^2}}{T}}\right) \quad (3.3)
\]
in natural units for finite gluon plasma. Taking $A(r) = C_g^2 r$ and $p^2 = A(r)^2 x$, we get the logarithm of the partition function as

$$\ln Z = \frac{g_I}{3\pi T} \int_0^R C_g^8 r^6 dr \sum_{l=1}^{\infty} \int_0^\infty x^\frac{3}{2} e^{-\frac{lA(r)^2}{4T}} \frac{dx}{\sqrt{1 + x}}$$

(3.4)

Using the value for the standard integral [7]

$$\int_0^\infty x^{\nu-1}e^{-\beta \sqrt{1+x}} \frac{dx}{\sqrt{1 + x}} = \frac{2}{\sqrt{\pi}} \left(\frac{\beta}{2}\right)^{\frac{1}{2}-\nu} \Gamma(\nu)K_{\frac{1}{2}-\nu}(\beta)$$

(3.5)

and the expression for pressure [5]

$$P = T \frac{\partial (\ln Z)}{\partial V}$$

we get the pressure for gluon plasma as

$$P_g = \frac{g_I T^2 C_g^4 R^2}{2\pi^2} \sum_{l=1}^{\infty} \frac{1}{l^2} K_2 \left(\frac{lC_g^2 R}{T}\right)$$

(3.6)

where $K_2 \left(\frac{lC_g^2 R}{T}\right)$ is the Bessel function of the second kind. Similarly the EoS for energy density is derived using the energy $U = -\frac{\partial (\ln Z)}{\partial \beta}$ and energy density $\varepsilon = \frac{\partial U}{\partial V}$. Then using the recurrence relation for the Bessel function

$$z \frac{\partial (K_\nu(z))}{\partial z} = - (\nu K_\nu(z) + zK_{\nu-1}(z))$$

where $z$ is the variable and $\nu$ is the order of the Bessel function, we get the energy density as

$$\varepsilon_g = \frac{3}{2} \frac{g_I T^2 C_g^4 R^2}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{l^2} K_2 \left(\frac{lC_g^2 R}{T}\right) + \frac{g_I}{2\pi^2} TC_g^6 R^3 \sum_{l=1}^{\infty} \frac{1}{l} K_1 \left(\frac{lC_g^2 R}{T}\right)$$

(3.7)
The number density for gluon plasma is obtained using the equation for number of particles [5]

\[ N = \sum_{p=0, r=0}^{\infty} e^{-\frac{H(p,r)}{T}} \]

and number density \[ n = \frac{\partial N}{\partial V} \]. The equation for \( N \) is

\[ N = \frac{2g_I}{\pi} \int_0^R r^2 dr \int_0^\infty p^2 dp \sum_{l=0}^{\infty} e^{-\frac{\sqrt{C_g^2 r^2 + p^2}}{T}} \quad (3.8) \]

Making substitutions as in earlier cases and using equation(7)

\[ N = \frac{2g_I T}{\pi} \int_0^R C_g^4 r^4 \sum_{l=1}^{\infty} \frac{1}{l} K_2 \left( \frac{lC_g^2 R}{T} \right) \quad (3.9) \]

Figure 3.1: Plots of \( \frac{P}{T^4} \) as a Function of \( T \) in Classical RHO Potential and Lattice Results (dots) for Gluon Plasma
The number density of gluon plasma is now given by

\[ n_g = \frac{g I}{2\pi^2} T C_g^4 R^2 \sum_{l=1}^{\infty} \frac{1}{l^2} K_2 \left( \frac{l C_g^2 R}{T} \right) \]  

(3.10)

At very high temperature we have

\[ K_2 \left( \frac{l C_g^2 R}{T} \right) \approx \frac{2T^2}{l^2 C_g^4 R^2} \]

Using

\[ \sum_{l=1}^{\infty} \frac{1}{l^4} = \frac{\pi^4}{90} \]

pressure and energy density become

\[ P_{gs} = \frac{g I}{90} \pi^2 T^4 \]

\[ \varepsilon_{gs} = \frac{3g I}{90} \pi^2 T^4 \]

### 3.4 EoS of Quark-Antiquark Plasma

Using the Eq (2.17) Hamiltonian for quarks \( H(p, r) = \sqrt{(M_q^2 + p^2 + \Omega_q^2 r^2)} \)

and the logarithm of the partition function becomes

\[ \ln Z = \frac{2g_I}{\pi} \int_0^R r^2 dr \int_0^\infty p^2 dp ln \left( 1 + e^{-\sqrt{(M_q^2 + p^2 + \Omega_q^2 r^2)} T} \right) \]  

(3.11)

for finite quark-antiquark plasma. The EoS for quark-antiquark plasma with zero chemical potential are obtained as below. Taking \( A^2(r) = \Omega_q^2 r^2 + M_q^2 \)

and \( p^2 = A(r)^2 x \) and performing calculations as in the case of gluon plasma
the equations for pressure, energy density and number density are obtained as

\[ P_{q\bar{q}} = \frac{g_1}{2\pi^2} T^2 \left( M_q^2 + \Omega_q^2 R^2 \right) \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^2} K_2 \left( \frac{l \sqrt{M_q^2 + \Omega_q^2 R^2}}{T} \right) \] (3.12)

\[ \epsilon_{q\bar{q}} = \frac{3g_1}{2\pi^2} T^2 \left( M_q^2 + \Omega_q^2 R^2 \right) \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^2} K_2 \left( \frac{l \sqrt{M_q^2 + \Omega_q^2 R^2}}{T} \right) \]

\[ + \frac{g_1}{2\pi^2} T^2 \left( M_q^2 + \Omega_q^2 R^2 \right)^\frac{3}{2} \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^2} K_1 \left( \frac{l \sqrt{M_q^2 + \Omega_q^2 R^2}}{T} \right) \] (3.13)

\[ n_{q\bar{q}} = \frac{g_1}{2\pi^2} T \left( M_q^2 + \Omega_q^2 R^2 \right) \sum_{l=1}^{\infty} \frac{(-1)^l}{l} K_2 \left( \frac{l \sqrt{M_q^2 + \Omega_q^2 R^2}}{T} \right) \] (3.14)

In the high temperature limit

\[ K_2 \left( \frac{l \sqrt{\Omega_q^2 R^2 + M_q^2}}{T} \right) \approx \frac{2T^2}{l^2 (\Omega_q^2 R^2 + M_q^2)} \]

Using

\[ \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^4} = \frac{7\pi^4}{720} \]

pressure and energy density now become

\[ P_{q\bar{q}s} = g_1 \frac{7}{720} \pi^2 T^4 \]

\[ \epsilon_{q\bar{q}s} = g_1 \frac{21}{720} \pi^2 T^4 \]
3.5 EoS for (2+1) Flavour Quark-Gluon Plasma

In this section we derive the EoS for (2+1) flavour QGP. (2+1) flavour plasma is a QGP [8] with two light quarks (u) and one heavy quark (s) along with gluons. For (2+1) flavour QGP combining Eq(3.6) and Eq(3.12), the expression for pressure becomes

\[ P_{\text{qgp}} = \frac{12T^2}{\pi^2} \left( M_u^2 + \Omega_q^2 R^2 \right) \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^2} K_2 \left( \frac{l \sqrt{M_u^2 + \Omega_q^2 R^2}}{T} \right) \]

\[ + \frac{6T^2}{\pi^2} \left( M_s^2 + \Omega_q^2 R^2 \right) \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^2} K_2 \left( \frac{l \sqrt{M_s^2 + \Omega_q^2 R^2}}{T} \right) \]

\[ + \frac{8T^2 C_g^2 R^2}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{l^2} K_2 \left( \frac{l C_g^2 R}{T} \right) \quad (3.15) \]

where \( M_u \) is the mass of up quark and \( M_s \) is the mass of strange quark. We have taken \( n_f = 2 \) for up quarks and \( n_f = 1 \) for strange quarks. Similarly we can find the EoS for energy density and number density for the plasma.

At high temperature, pressure and energy density for QGP become

\[ P_{\text{gs}} = \left( \frac{8}{45} \pi^2 + \frac{7}{20} \pi^2 \right) T^4 \]

\[ \varepsilon_{\text{gs}} = \left( \frac{24}{45} \pi^2 + \frac{21}{20} \pi^2 \right) T^4 \]

Thus at high temperature according to RHO model QGP behaves as an ideal gas but as the temperature is lowered it shows non-ideal nature.
3.6 Results

It is interesting to see that the RHO model fits the lattice results [8-12] for large regions of data for the system taken, namely, the (2+1) flavour QGP. For the system R is adjusted so that we get a good fit to the lattice results.

![Figure 3.2: Plots of $\frac{P_T^4}{T^4}$ as a Function of T in Classical RHO Potential and Lattice Results(dots) for (2+1) Flavour Plasma](image)

In the case of (2+1) flavour system[8] the rest masses are taken as $\frac{M_u}{T} = 0.4$ and $\frac{M_s}{T} = 1.0$. With $C_g = 2.35(fm)^{-1}$ and $C_q = 1.164(fm)^{-1}$ (which are the values obtained from Hadron spectroscopy with the RHO model) for the best fit we got $R = 1.121$ fm which gives the radius of the (2+1) flavour plasma.
3.7 Discussion

1. The effective volume for a pp collision where Quark Gluon Plasma (QGP) is formed is a few cubic fm while the same in RHIC collision for charge conservation is a few dozen cubic fm. We got the confinementent radius of the (2+1) plasma equal to \( R = 1.121 \) fm.

2. The life time of the plasma formed during the heavy ion collision can be determined by the expansion time scale \( \tau_{\text{exp}} = \frac{R}{C_s} \) although details of its evolution is influenced by the other factors like the kinetic properties of the plasma and hadronisation mechanism which are still unknown. Our result \( \tau_{\text{exp}} \approx 3.4 fm/c \) for QGP agree well with the experimental observations.

3. Bulk thermodynamic observable such as pressure, show strong deviations from ideal gas behaviour even at \( \sim (2 - 3)T_c \) and approach the ideal gas limit only slowly which agrees well with the lattice QCD.

4. Thus our model with only a single parameter variable could explain why the size of QGP formed is small in heavy ion collision. For \( T \approx T_c \) our model is not anyway giving good results. This may be due to nonperturbative effects like hadronic resonances.

5. Our results clearly show that pressure and energy density have \( T^4 \) and number density has \( T^3 \) dependence at high temperature for both plasmas.

6. It should be noted that different models proposed to explain lattice results have their own merits and demerits. Some models like Strongly Coupled Quark Gluon Plasma (SCQGP) model [13] could explain the
nonideal behaviour of QGP near $T = T_c$. Quasiparticle model [14] was also successful near $T_c$, but the origin of these models from the first principle is yet to be worked out.
Bibliography


