Chapter 5

The Laminar Boundary Layer on a Moving Cylindrical rod with an Applied Magnetic Field

5.1 Introduction

Boundary layer flows over a moving or stretching surface are of great importance in view of their relevance to a wide variety of technical applications, especially in the manufacture of fibres in glass and polymer industries. Let us consider the steady laminar flow induced by the motion of a cylindrical rod issuing from an orifice into a fluid at rest. Due to hydrodynamic friction, the fluid adjacent to the rod’s surface is carried in the axial direction. At a short distance from the orifice, the flow takes the form of an axisymmetric boundary layer on the surface of the cylinder. The flow will be considered in frames of the laboratory reference system in which the velocity of the fluid on the rod’s surface, equal to the velocity of the rod, decreases in normal direction to zero. The problem set here takes its origin in the field of man-made fibres spinning. The moving cylinder corresponds to the filament extruded from the spinneret and drawn in the axial direction through the fluid. The ambient fluid motion has a great influence on the process of fibres formation and needs further investigation.
The first and foremost work regarding the boundary layer behaviour on moving surface in a quiescent fluid was considered by Sakiadis [163]. Jaffe and Okamura [164] studied the transverse curvature effect on the incompressible laminar boundary layer for a longitudinal flow over a cylinder. Boundary layer flow on a cylinder moving in a fluid at rest has been discussed by Crane [165]. Wang [166] reported a similarity reduction of the Navier-Stokes equations describing axisymmetric radial stagnation boundary layer flow normal to the cylinder. Kuiken [167] investigated the cooling of a heat resistance cylinder moving through a fluid. Indeed, in the manufacture of metal and polymer solid cylinders, the material is usually in a molten phase when thrust through an extrusion die and then cools and solidifies some distance away from the die. Experiments by Vleggaar [168] show that the velocity of the material is approximately proportional to the distance, so these systems are often modeled as linear stretching rods or cylinders. The axisymmetric nonsimilar stagnation flow on a moving cylinder was investigated by Gorla [169]. Zachara [170] examined the laminar boundary layer on a moving cylindrical rod. Recently the unsteady MHD viscous flow and heat transfer of newtonian fluids induced by an impulsively stretched plane surface is studied by Kumari and Nath [171]. Also, Hang and Wang [172] investigated annular axisymmetric stagnation flow on a moving cylinder. More recently, Brendan Redmond and David McDonnel [173] applied P.W. Hausen integral method to find the rate of heat loss of the fibre during manufacturing of polymer fibres. The objective of this chapter is to analyse the steady, nonsimilar laminar incompressible boundary layer flow and heat transfer on a moving cylindrical rod along with an applied magnetic field. In this problem the nonsimilarity is due to the transverse curvature of the moving cylindrical rod. The partial differential equations, governing the axisymmetric boundary layer flow and heat transfer, with two-point boundary conditions have been solved numerically using Kellar box method [174].
5.2 Governing Equations and Boundary Conditions

Let us consider the steady, axisymmetric laminar boundary layer forced convection flow of an incompressible fluid with an applied magnetic field on a continuous cylinder moving from an orifice in an axial direction at constant velocity through a fluid at rest. The fluid is assumed to be electrically conducting. The radius of the cylinder and its velocity are denoted by \( a \) and \( U \), respectively. A magnetic field of strength \( B_0 \) fixed relative to the fluid is applied in \( y \)-direction. [See Fig.5.1]. It is assumed that magnetic Reynolds number is small so that the induced magnetic field can be neglected. The effect of viscous dissipation is included in the analysis. The flow is considered in frames of the boundary layer coordinate system in which the \( x \)-axis, parallel to the axis of symmetry, is posed along the solid surface and the \( y \)-axis is normal to it. The origin of the coordinate system is put in the plane of the orifice. Under the aforementioned assumptions, the equations governing the above nonsimilar flow are [181]:

\[
\begin{align*}
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial y} (rv) &= 0 \quad (5.1) \\
ru \frac{\partial u}{\partial x} + rv \frac{\partial u}{\partial y} &= \nu \frac{\partial}{\partial y} \left( r \frac{\partial u}{\partial y} \right) - \frac{\sigma B_0^2}{\rho} (u - U) \quad (5.2) \\
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\sigma B_0^2}{\rho} u^2 \quad (5.3)
\end{align*}
\]

where \( r(y) = a + y \)

The boundary conditions are

\[
\begin{align*}
u(x, 0) &= U; \quad v(x, 0) = 0; \quad u(x, \infty) = 0 \\
T(x, 0) &= T_w; \quad T(x, \infty) = T_\infty
\end{align*}
\] (5.4)

The Eqns.(5.2) and (5.3) is transformed into nondimensional form using
the combination of the Mangler and Falkner - Skan transformations

\[ \eta = \left( \frac{U}{\nu x} \right)^{1/2} y \left( 1 + \frac{y}{2a} \right) \]
\[ \xi = \left( \frac{4\nu x}{Ua^2} \right)^{1/2} \]
\[ \psi(x, y) = a(\nu x)^{1/2} f(\xi, \eta) \]
\[ T = T_\infty + (T_w - T_\infty) G(\xi, \eta) \] (5.5)

which satisfies the continuity Eqn.(5.1). Consequently, the momentum Eqn.(5.2) and energy Eqn.(5.3) becomes

\[ (1 + \xi \eta) f''' + \xi f'' + \frac{f f''}{2} - M (f' - 1) = \frac{\xi}{2} (f' f'_\xi - f'' f_\xi) \] (5.6)
\[ (1 + \xi \eta) G'' + \frac{\xi G'}{2} + \frac{f G Pr}{2} + Ec Pr \left[ (1 + \xi \eta) f'' + \frac{M f'^2}{2} \right] = \frac{\xi Pr}{2} (f' G_\xi - G' f_\xi) \] (5.7)

where

\[ u = \frac{U}{x} \]
\[ v = -\frac{a}{r} \left( \frac{\nu}{4x} \right)^{1/2} \left[ f - \eta f' + \xi f_\xi \right] \]
\[ M = \frac{2x \sigma B_0^2}{\rho U} \]
\[ Ec = \frac{U^2}{c_p(T_w - T_\infty)} \] (5.8)

The transformed boundary conditions are:

\[ f(\xi, 0) = 0; \quad f'(\xi, 0) = 1; \quad f'(\xi, \infty) = 0 \]
\[ G(\xi, 0) = 1; \quad G(\xi, \infty) = 0 \] (5.9)

for \( \xi \geq 0 \).
The skin friction and heat transfer coefficients are defined as

\[ C_f = -\frac{a \tau(x, a)}{\mu U} = -\frac{2}{\xi} f''(0) \]  \hspace{1cm} (5.10)

\[ Nu = -\frac{a (\frac{\partial T}{\partial y})_{y=0}}{T_w - T_\infty} = -\frac{2}{\xi} G'(0) \]  \hspace{1cm} (5.11)

Here \( u \) and \( v \) are the velocity components along \( x \) and \( y \)-directions respectively; \( \xi \) and \( \eta \) are transformed coordinates; \( \psi \) and \( f \) are the dimensional and dimensionless stream functions, respectively; \( M \) is the nondimensional magnetic parameter; The subscript \( \xi \) denote partial derivative with respect to \( \xi \) and prime (‘) denote derivatives with respect to \( \eta \). It is worth mentioning here that when \( M = 0.0 \), Eqn.(5.6) reduces to

\[ (1 + \xi \eta) f''' + \xi f'' + \frac{f f''}{2} = \frac{\xi}{2} (f' f_\xi - f'' f_\xi) \]  \hspace{1cm} (5.12)

which is exactly same as that of A.Zachara [181] who has studied the laminar momentum boundary layer behaviour on a moving cylindrical rod in the absence of the magnetic field. In the present study, the analysis of heat transfer is included for investigation along with viscous dissipation in the presence of a transverse magnetic field.

5.3 Method of solution

The outline of Keller Box method is presented in Appendix D. The explanation of the method suitable for the problem under consideration is presented below. First, we introduce new dependent variable \( u(\xi, \eta) \) and \( v(\xi, \eta) \), such that

\[ f' = u, \quad u' = v, \quad G' = p \]  \hspace{1cm} (5.13)
so that the Eqn.(5.5) can be written as

\[(1 + \xi\eta)v' + \xi v + \frac{1}{2}fv - M(u - 1) = \frac{\xi}{2} \left( u \frac{\partial u}{\partial \xi} - v \frac{\partial f}{\partial \xi} \right) \quad (5.14)\]

\[(1 + \xi\eta)p' + \frac{1}{2}\xi p + \frac{1}{2}fpPr + EcPr \left[ (1 + \xi\eta)v + \frac{Mu^2}{2} \right] = \frac{\xi Pr}{2} \left( u \frac{\partial G}{\partial \xi} - p \frac{\partial f}{\partial \xi} \right) \quad (5.15)\]

We now consider the net rectangle in the $\xi$-$\eta$ plane and the net points defined as follows:

\[\xi^0 = 0, \quad \xi^n = \xi^{n-1} + k_n, \quad n = 1, 2, 3, \ldots N, \quad (5.16)\]

\[\eta_0 = \eta_j = \eta_{j-1} + h_j, \quad j = 1, 2, 3, \ldots N, \eta_j = \eta_\infty, \quad (5.17)\]

where $k_n$ is the $\Delta\xi$-spacing and $h_j$-is the $\Delta\eta$-spacing. Here $n$ and $j$ are just the sequence of numbers that indicate the coordinate location, not tensor indices or exponents.

The derivatives in the $\xi$-direction are replaced by finite differences. For example the finite difference form for any points are

\[(\partial u/\partial \xi)_j^{n-1/2} = \frac{1}{2} \left[ (\ )_j^n + (\ )_{j-1}^n \right] \quad (5.18)\]

\[(\partial u/\partial \eta)_j^{n-1/2} = \frac{1}{2} \left[ (\ )_j^n + (\ )_{j}^n \right] \quad (5.19)\]

\[\left( \frac{\partial u}{\partial \xi} \right)^{n-1/2}_{j-1/2} = \frac{u^n_j - u^{n-1}_{j-1/2}}{k_n} \quad (5.20)\]

\[\left( \frac{\partial u}{\partial \eta} \right)^{n-1/2}_{j-1/2} = \frac{u^{n-1/2}_j - u^{n-1/2}_{j-1}}{h_j} \quad (5.21)\]

We start by writing the finite difference form for the midpoint $(\xi^n, \eta_{j-1/2})$ of the segment $P_1P_2$ using centered difference derivatives (See Fig.D.1 in Appendix D). This process is called “centering about $(\xi^{n-1/2}, \eta_{j-1/2})$"."
We get

\[ f_j^n - f_{j-1}^n - \frac{1}{2} h_j \left( u_j^n + u_{j-1}^n \right) = 0 \] (5.22)

\[ u_j^n - u_{j-1}^n - \frac{1}{2} h_j \left( v_j^n + v_{j-1}^n \right) = 0 \] (5.23)

\[ G_j^n - G_{j-1}^n - \frac{1}{2} h_j \left( p_j^n + p_{j-1}^n \right) = 0 \] (5.24)

The left hand side of (5.14) denoted by \( L_1 \), so that

\[
\frac{1}{2} \left[ (L_1)_{j-1/2}^n + (L_1)_{j-1/2}^{n-1} \right]
= \frac{\xi^{n-1/2}}{2} \left[ \left( \frac{u_{j-1/2}^n + u_{j-1/2}^{n-1}}{2} \right) \left( \frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} \right) \right]
- \frac{\xi^{n-1/2}}{2} \left[ \left( \frac{v_{j-1/2}^n + v_{j-1/2}^{n-1}}{2} \right) \left( \frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n} \right) \right]
\] (5.25)

where

\[
(L_1)_{j-1/2}^n = \left[ (1 + \xi \eta) v' + \xi v + \frac{1}{2} f v - M(u - 1) \right]_{j-1/2}^n
= (1 + \xi^n \eta)_{j-1/2} \left( \frac{v_j^n - v_{j-1}^n}{h_j} \right) + (\xi v)_{j-1/2}^n
+ \frac{1}{2} f_{j-1/2}^n v_{j-1/2}^n - M(u_{j-1/2}^n - 1)
\] (5.26)

\[
(L_1)_{j-1/2}^{n-1} = \left[ (1 + \xi \eta) v' + \xi v + \frac{1}{2} f v - M(u - 1) \right]_{j-1/2}^{n-1}
= (1 + \xi^n \eta)_{j-1/2} \left( \frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j} \right) + (\xi v)_{j-1/2}^{n-1}
+ \frac{1}{2} f_{j-1/2}^{n-1} v_{j-1/2}^{n-1} - M(u_{j-1/2}^{n-1} - 1)
\] (5.27)
substituting (5.25)-(5.27) into (5.14), we get

\[
(1 + \xi \eta)^n_{j-1/2} \left( \frac{v^n_j - v^n_{j-1}}{h_j} \right) + \xi^n_{j-1/2}v^n_{j-1/2} + \\
\alpha_1(fv)^n_{j-1/2} + \alpha \left( v^n_{j-1/2}f^n_{j-1/2} - f^n_{j-1/2}v^n_{j-1/2} \right) - \\
Mu^n_{j-1/2} - \alpha(u^n_{j-1/2})^2 = R^n_{j-1/2}
\]

(5.28)

Similarly, if the left hand side of (5.15) denoted by \(L_2\), then

\[
\frac{1}{2} \left[ (L_2)^n_{j-1/2} + (L_2)^{n-1}_{j-1/2} \right] = \\
\frac{\xi^{n-1/2} Pr}{2} \left[ \left( \frac{u^n_{j-1/2} + u^{n-1}_{j-1/2}}{2} \right) \left( \frac{G^n_{j-1/2} - G^{n-1}_{j-1/2}}{k_n} \right) \right] - \\
\frac{\xi^{n-1/2} Pr}{2} \left[ \left( \frac{p^n_{j-1/2} + p^{n-1}_{j-1/2}}{2} \right) \left( \frac{f^n_{j-1/2} - f^{n-1}_{j-1/2}}{k_n} \right) \right]
\]

(5.29)

where

\[
(L_2)^n_{j-1/2} = \left[ (1 + \xi \eta)p' + \frac{1}{2}\xi p + \frac{1}{2}fpPr \right]_{j-1/2}^n \\
+ \left[ EcPr \left( (1 + \xi \eta)v + M^2 u^{2n} \right) \right]_{j-1/2}^n \\
= (1 + \xi \eta)^n_{j-1/2} \left( \frac{p^n_j - p_{j-1}^n}{h_j} \right) + \frac{1}{2}(\xi p)^n_{j-1/2} \\
+ \frac{1}{2}f^n_{j-1/2}p^n_{j-1/2}Pr \\
+ EcPr \left[ (1 + \xi \eta)^n_{j-1/2}v^n_{j-1/2} + \frac{M(u^2)^n_{j-1/2}}{2} \right]
\]

(5.30)
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\[(L_2)^{n-1}_{j-1/2} = \left[ (1 + \xi \eta) p' + \frac{1}{2} \xi p + \frac{1}{2} f p Pr + Ec Pr (1 + \xi \eta) v + \frac{Mu^2}{2} \right]_{j-1/2}^{n-1}
\]

\[= (1 + \xi \eta)^{n-1}_{j-1/2} \left( \frac{p^n_{j-1} - p^n_{j-1}}{h_j} \right) + \frac{1}{2} (\xi p)^{n-1}_{j-1/2}
\]

\[+ \frac{1}{2} f_{j-1/2}^{n-1} p^n_{j-1/2} Pr
\]

\[+ Ec Pr \left[ (1 + \xi \eta)^{n-1}_{j-1/2} v^{n-1}_{j-1/2} + \frac{M(u^2)^{n-1}_{j-1/2}}{2} \right] \]

(5.31)

substituting (5.29)-(5.31) into (5.15), we get

\[\begin{align*}
(1 + \xi \eta)^{n-1}_{j-1/2} & \left( \frac{p^n_{j-1} - p^n_{j-1}}{h_j} \right) + \frac{1}{2} \xi^{n-1}_{j-1/2} p^n_{j-1/2} + \\
\alpha_1 Pr (fp)^n_{j-1/2} - \alpha Pr \left( (uG)^n_{j-1/2} - G^n_{j-1/2} u^n_{j-1/2} + u^{n-1}_{j-1/2} G^n_{j-1/2} \right) - \\
\alpha Pr \left( f^n_{j-1/2} p^n_{j-1/2} - p^n_{j-1/2} f^n_{j-1/2} \right) + Ec Pr \left[ (1 + \xi \eta)^{n-1}_{j-1/2} v^{n-1}_{j-1/2} + \frac{M(u^2)^{n-1}_{j-1/2}}{2} \right] = N^{n-1}_{j-1/2}
\end{align*}
\]

(5.32)

where

\[\alpha = \frac{\xi^{n-1/2}}{2k_n}\]

\[\alpha_1 = \frac{1}{2} + \alpha\]

\[R^n_{j-1/2} = -(L_1)^{n-1}_{j-1/2} + \alpha \left( (fv)^{n-1}_{j-1/2} - (u^2)^{n-1}_{j-1/2} \right) - M\]

\[N^n_{j-1/2} = -(L_2)^{n-1}_{j-1/2} + \alpha Pr \left( (fp)^{n-1}_{j-1/2} - (uG)^{n-1}_{j-1/2} \right)\]

Eqns.(5.22)-(5.26) and (5.30) are imposed for \(j = 1, 2, 3, \ldots, J\) at given \(n\), and the transformed boundary layer thickness, \(\eta_j\), is to be sufficiently large so that it beyond the edge of the boundary layer, (See Cebeci and Bradshaw [185]). The boundary conditions as yield at \(\xi = \xi^n\) are

\[f^n_0 = 0, \quad u^n_0 = 1, \quad u^n_j = 0\]

\[G^n_0 = 1, \quad G^n_j = 0\]

(5.33)
To linearize the nonlinear system of Eqns.(5.22)-(5.24), (5.28) and (5.32) using Newton’s method, we introduce the following iterates:

\[
\begin{align*}
  f_j^{(i+1)} &= f_j^{(i)} + \delta f_j^{(i)} \\
  u_j^{(i+1)} &= u_j^{(i)} + \delta u_j^{(i)} \\
  v_j^{(i+1)} &= v_j^{(i)} + \delta v_j^{(i)} \\
  G_j^{(i+1)} &= G_j^{(i)} + \delta G_j^{(i)} \\
  p_j^{(i+1)} &= p_j^{(i)} + \delta p_j^{(i)}
\end{align*}
\] (5.34)

Here, for simplicity, the unknown parameter \((f_n^j, u_n^j, v_n^j, G_n^j, p_n^j)\) at \(\xi = \xi^n\) are written by \((f_j, u_j, v_j, G_j, p_j)\) in above expressions. Substituting (5.35) into (5.22)-(5.24), (5.28) and (5.32), then drop the quadratic and higher order terms in, \(\delta f_j^{(i)}, \delta u_j^{(i)}, \delta v_j^{(i)}, \delta G_j^{(i)}\) and \(\delta p_j^{(i)}\) this procedure yields the following linear tridiagonal system:

\[
\begin{align*}
  \delta f_j - \delta f_{j-1} - \frac{1}{2} h_j (\delta u_j + \delta u_{j-1}) &= (r_1)_{j-1/2} \\
  \delta u_j - \delta u_{j-1} - \frac{1}{2} h_j (\delta v_j + \delta v_{j-1}) &= (r_4)_{j-1/2} \\
  \delta G_j - \delta G_{j-1} - \frac{1}{2} h_j (\delta p_j + \delta p_{j-1}) &= (r_5)_{j-1/2} \\
  (a_1)_j \delta v_j + (a_2)_j \delta v_{j-1} + (a_3)_j \delta f_j + (a_4)_j \delta f_{j-1} +
  (a_5)_j \delta u_j + (a_6)_j \delta u_{j-1} + (a_7)_j \delta G_j + (a_8)_j \delta G_{j-1} &= (r_2)_{j-1/2} \\
  (b_1)_j \delta p_j + (b_2)_j \delta p_{j-1} + (b_3)_j \delta f_j + (b_4)_j \delta f_{j-1} +
  (b_5)_j \delta u_j + (b_6)_j \delta u_{j-1} + (b_7)_j \delta G_j + (b_8)_j \delta G_{j-1} +
  (b_9)_j \delta v_j + (b_{10})_j \delta v_{j-1} &= (r_3)_{j-1/2}
\end{align*}
\] (5.35)\text{--}(5.39)

where

\[
\begin{align*}
  (r_1)_j &= f_{j-1} - f_j + h_j u_{j-1/2} \\
  (r_4)_{j-1} &= u_{j-1} - u_j + h_j v_{j-1/2} \\
  (r_5)_{j-1} &= G_{j-1} - G_j + h_j p_{j-1/2}
\end{align*}
\]
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\[
(a_1)_j = \frac{(1 + \xi \eta)_{j-1/2}}{h_j} + \frac{\alpha_1}{2} f_j - \frac{\alpha}{2} f_{j-1/2} + \frac{\xi_{j-1/2}}{2}
\]

\[
(a_2)_j = -\frac{(1 + \xi \eta)_{j-1/2}}{h_j} + \frac{\alpha_1}{2} f_j - \frac{\alpha}{2} f_{j-1/2} + \frac{\xi_{j-1/2}}{2}
\]

\[
(a_3)_j = \frac{\alpha_1}{2} v_j + \frac{\alpha}{2} v_{j-1/2}^{n-1}
\]

\[
(a_4)_j = \frac{\alpha_1}{2} v_{j-1} + \frac{\alpha}{2} v_{j-1/2}^{n-1}
\]

\[
(a_5)_j = -\frac{M}{2} - \alpha u_j
\]

\[
(a_6)_j = -\frac{M}{2} - \alpha u_{j-1}
\]

\[
(a_7)_j = 0.0
\]

\[
(a_8)_j = 0.0
\]

\[
(r_2)_j = R_{j-1/2}^{n-1} - \left[ (1 + \xi \eta)_{j-1/2} \frac{v_j - v_{j-1}}{h_j} + (\xi v)_{j-1/2} \right]
\]

\[- \left[ \alpha_1 (f v)_{j-1/2} - M u_{j-1/2} \right]
\]

\[- \left[ \alpha (v_{j-1/2}^{n-1} f_{j-1/2} - f_{j-1/2}^{n-1} v_{j-1/2}) - \alpha (u^2)_{j-1/2} \right]
\]

\[
(b_1)_j = \frac{(1 + \xi \eta)_{j-1/2}}{h_j} + \frac{\alpha_1 Pr}{2} f_j - \frac{\alpha Pr}{2} f_{j-1/2}^{n-1}
\]

\[
(b_2)_j = -\frac{(1 + \xi \eta)_{j-1/2}}{h_j} + \frac{\alpha_1 Pr}{2} f_j - \frac{\alpha Pr}{2} f_{j-1/2}^{n-1}
\]

\[
(b_3)_j = \frac{\alpha_1 Pr}{2} p_j + \frac{\alpha Pr}{2} p_{j-1/2}^{n-1}
\]

\[
(b_4)_j = \frac{\alpha_1 Pr}{2} p_{j-1} + \frac{\alpha}{2} p_{j-1/2}^{n-1}
\]

\[
(b_5)_j = -\frac{\alpha Pr}{2} G_j + \frac{\alpha Pr}{2} G_{j-1/2}^{n-1} + \frac{EcM Pr}{2} u_j
\]

\[
(b_6)_j = -\frac{\alpha Pr}{2} G_{j-1} + \frac{\alpha Pr}{2} G_{j-1/2}^{n-1} + \frac{EcM Pr}{2} u_{j-1}
\]

\[
(b_7)_j = -\frac{\alpha Pr}{2} u_{j-1} - \frac{\alpha Pr}{2} u_{j-1/2}^{n-1}
\]

\[
(b_8)_j = -\frac{\alpha Pr}{2} u_{j-1} - \frac{\alpha Pr}{2} u_{j-1/2}^{n-1}
\]

\[
(b_9)_j = \frac{(1 + \xi \eta)_{j-1/2} EcPr}{2}
\]

\[
(b_{10})_j = -\frac{(1 + \xi \eta)_{j-1/2} EcPr}{2}
\]
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\[(r_3)_j = N_{j-1/2}^{n-1} - \left[ (1 + \xi \eta)_{j-1/2} \frac{p_j - p_{j-1}}{h_j} + \frac{1}{2} (\xi p)_{j-1/2} + \alpha_1 Pr (fp)_{j-1/2} \right] \]

\[ - \alpha Pr \left[ (uG)_{j-1/2} - G_{j-1/2}^{n-1} u_{j-1/2} + u_{j-1/2}^{n-1} G_{j-1/2} \right] \]

\[ - \alpha Pr \left[ f_{j-1/2}^{n-1} p_{j-1/2} - p_{j-1/2}^{n-1} f_{j-1/2} \right] \]

\[ - EcPr \left[ (1 + \xi \eta)_{j-1/2} v_{j-1/2} + \frac{M}{2} (u^2)_{j-1/2} \right] \]

\[(5.40)\]

To complete the system (5.35)-(5.39) we use the boundary conditions (5.34), which can be satisfied exactly with no iteration. Therefore, in order to maintain these correct values in all the iterate, we take \( \delta f_0 = 0, \delta u_0 = 0, \delta G_0 = 0, \delta u_J = 0 \) and \( \delta G_J = 0 \).

The linearized difference Eqns. (5.35)-(5.39) has a block tridiagonal structure consists of variables or constants, but here it consists of block matrices. The elements of the matrices are defined as follows:

\[
\begin{bmatrix}
[A_1] & [C_1] \\
[B_2] & [A_2] & [C_2] \\
& \vdots & \\
& \vdots & \\
& \vdots & \\
[B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\
[B_J] & [A_J]
\end{bmatrix}
\begin{bmatrix}
[\delta 1] \\
[\delta 2] \\
\vdots \\
\vdots \\
\vdots \\
[\delta_{J-1}]
\end{bmatrix}
= 
\begin{bmatrix}
[r_1] \\
[r_2] \\
\vdots \\
\vdots \\
\vdots \\
[r_{J-1}]
\end{bmatrix}
\]

\[(5.41)\]

That is

\[A \delta = r \]

\[(5.42)\]
where A is the tridiagonal matrix.

\[
[A_0] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_0 & \alpha_1 \\
0 & -1 & d & 0 & 0 \\
0 & 0 & 0 & -1 & d
\end{bmatrix}
\] (5.43)

\[
[A_J] = \begin{bmatrix}
1 & d & 0 & 0 & 0 \\
(a_3)_j & (a_5)_j & (a_1)_j & (a_7)_j & 0 \\
(b_3)_j & (b_5)_j & (b_9)_j & (b_7)_j & (b_1)_j \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\] (5.44)

\[
[A_j] = \begin{bmatrix}
1 & d & 0 & 0 & 0 \\
(a_3)_j & (a_5)_j & (a_1)_j & (a_7)_j & 0 \\
(b_3)_j & (b_5)_j & (b_9)_j & (b_7)_j & (b_1)_j \\
0 & -1 & d & 0 & 0 \\
0 & 0 & 0 & -1 & d
\end{bmatrix}, 1 \leq j \leq J - 1
\] (5.45)

\[
[B_j] = \begin{bmatrix}
-1 & d & 0 & 0 & 0 \\
(a_4)_j & (a_6)_j & (a_2)_j & (a_8)_j & 0 \\
(b_4)_j & (b_6)_j & (b_{10})_j & (b_8)_j & (b_2)_j \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, 1 \leq j \leq J
\] (5.46)
\[
[C_j] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & d & 0 & 0 \\
0 & 0 & 0 & 1 & d \\
\end{bmatrix}, \quad 0 \leq j \leq J - 1
\] (5.47)

where

\[
d = -\frac{h_j}{2}
\] (5.48)

\[
[\delta_j] = \begin{bmatrix}
\delta f_j \\
\delta u_j \\
\delta v_j \\
\delta G_j \\
\delta p_j
\end{bmatrix}, \quad 0 \leq j \leq J
\] (5.49)

\[
[r_0] = \begin{bmatrix}
0 \\
0 \\
0 \\
(r_4)_1 \\
(r_5)_1
\end{bmatrix}
\] (5.50)

\[
[r_J] = \begin{bmatrix}
(r_1)_J \\
(r_2)_J \\
(r_3)_J \\
0 \\
0
\end{bmatrix}
\] (5.51)
Moving Cylindrical Rod............

and

\[
[r_j] = \begin{bmatrix}
(r_1)_j \\
(r_2)_j \\
(r_3)_j \\
(r_4)_j \\
(r_5)_j
\end{bmatrix}, \quad 1 \leq j \leq J - 1
\]  \hfill (5.52)

To solve (5.35)-(5.39), we assume that the matrix \( A \) is nonsingular and it can be factored into

\[
A = LU
\]  \hfill (5.53)

where

\[
L = \begin{bmatrix}
[\alpha_1] \\
[B_2] & [\alpha_2] \\
& \ddots \\
& & [\alpha_{J-1}] \\
& & [B_J] & [\alpha_J]
\end{bmatrix}
\]  \hfill (5.54)

and

\[
U = \begin{bmatrix}
[I] & [\Gamma_1] \\
[I] & [\Gamma_2] \\
& \ddots \\
[I] & [\Gamma_{J-1}] \\
[I]
\end{bmatrix}
\]  \hfill (5.55)

where \([I]\) is the identity matrix of order 3 and \([\alpha_i]\), and \([\Gamma_i]\) are 3x3 matrices in which elements are determined by the following equations:

\[
[\alpha_1] = [A_1]
\]  \hfill (5.56)
[A_1][\Gamma_1] = [C_1] \quad (5.57)

[\alpha_j] = [A_j] - [B_j][\Gamma_{j-1}], \, j = 2, 3, \ldots J \quad (5.58)

[\alpha_j][\Gamma_j] = [C_j], \, j = 2, 3, \ldots, J - 1 \quad (5.59)

Eqn.(5.53) can now be substituted into (5.42) and we get

$$LU\delta = r \quad (5.60)$$

If we define

$$U\delta = W \quad (5.61)$$

then (5.60) becomes

$$LW = r \quad (5.62)$$

where

$$W = \begin{bmatrix}
[W_1] \\
[W_2] \\
\vdots \\
[W_{J-1}] \\
[W_J]
\end{bmatrix} \quad (5.63)$$

and the [W_j] are 3x1 column matrices. The elements in W can be solved from (5.62):

$$[\alpha_1][W_1] = [r_1] \quad (5.64)$$

$$[\alpha_j][W_j] = [r_j] - [B_j][W_{j-1}], \, 2 \leq j \leq J \quad (5.65)$$

The step in which \(\Gamma_j, \alpha_j\) and \(W_j\) are calculated is usually referred to as the forward sweep. Once the elements of W are found, (5.61) then gives the solution \(\delta\) in the so called backward sweep, in which the elements are obtained by the following relations:

$$[\delta_j] = [W_j] \quad (5.66)$$
\[ [\delta_j] = [W_j] - [\Gamma_j][\delta_{j+1}], \quad 1 \leq j \leq J - 1 \] (5.67)

These calculations are repeated until some convergence criterion is satisfied and calculations are stopped when
\[ |\delta v_0^{(i)}| < 10^{-5}. \] (5.68)

### 5.4 Results and Discussion

The partial differential Eqns.(5.6) and (5.7) along with boundary condition (5.9) has been solved numerically employing Kellar box method, as explained in previous section. The detailed description of the method is presented in [185], as well as in Appendix D. The step sizes \( \Delta \eta \) and \( \Delta \xi \) are taken in \( \xi \) and \( \eta \)-directions, respectively. For computation purpose \( \Delta \eta = 0.2, \Delta \xi = 0.1 \) and \( \eta_\infty \) (edge of the boundary layer) = 8.0 have been adopted. In order to assess the accuracy of our method of solution, we have compared our skin friction coefficient \( (C_f) \) for \( M = 0.0 \) (i.e., without magnetic field) with that of Zachara [181] by solving the Eqn.(5.9). [See Fig.5.2]. Our results are found to be in excellent agreement with the above study.

The effect of magnetic field \( (M) \) on skin friction \( (C_f) \) and heat transfer \( (Nu) \) coefficients are displayed in Fig.5.3. As the magnetic field \( (M) \) increases the skin friction \( (C_f) \) and heat transfer \( (Nu) \) coefficients are found to increase for all streamwise locations \( \xi \). The percentage of increase of \( C_f \) from \( M = 0.0 \) to \( M = 0.25 \) is about 38.5 % and it is about 33.8% in the range of \( M \) \((0.25 \leq M \leq 0.50)\) at an arbitrary value of \( \xi \) \((\xi = 0.5)\). The heat transfer coefficient increases uniformly with the increase of magnetic field.

Fig.5.4 shows the effect of magnetic field \( (M) \) on velocity \( (F) \) and temperature \( (G) \) profiles at an arbitrary value of \( \xi \) \((\xi = 0.5)\). The magnitude of the velocity and temperature decreases with the increase of magnetic field. It is true for all values of \( \xi \). This results in the reduction
of momentum and thermal boundary layer thickness. In fact, the thickness of momentum boundary layer decrease about 3.2% from $M = 0.0$ to $M = 0.25$ and is about 2.9% in the range of $M (0.25 \leq M \leq 0.50)$ near $\eta = 2.0$. Further, the thermal boundary layer thickness decreases uniformly.

Viscous dissipation ($Ec$) is the transformation of energies from internal energy to kinetic energy. Its effect on both heat transfer ($Nu$) and temperature profile ($G$) in the presence of magnetic field ($M = 0.5$) are shown in Fig.5.5. It is found that heat transfer decreases with the increase of $Ec$ ($Ec = 0.5$) whereas, it increases with the decrease of $Ec$ ($Ec = -0.5$) [See Fig.5.5(a)]. In fact, the percentage of decrease of heat transfer when $Ec = 0.5$ is 49.44% and the percentage of increase in $Nu$ when $Ec = -0.5$ is 105.09% at the streamwise location $\xi = 0.5$. This shows that the heat transfer is strongly depends on viscous dissipation. Also, it is observed that the thermal boundary layer is significantly affected by the viscous dissipation parameter [See Fig.5.5(b)]. As $Ec$ increases the thickness of the thermal boundary layer decreases whereas, the thickness of the thermal boundary layer increases with the decrease of viscous dissipation parameter.

### 5.5 Conclusions

The steady, laminar incompressible boundary layer flow of an electrically conducting fluid on a moving cylindrical rod with an applied magnetic field has been studied. Nonsimilar solutions of the problem have been obtained using Keller Box method.
The numerical results indicate that, the skin friction and heat transfer coefficients increases with the increase of magnetic field. The thickness of momentum and thermal boundary layers found to decrease at all axial distances. Further, the heat transfer increases with the decrease of viscous dissipation parameter whereas, it decreases with the increase of viscous dissipation parameter. Also, it is found that the thickness of thermal boundary layer strongly depends on viscous dissipation.
Figure 5.1: Flow model and coordinate system
Figure 5.2: Comparison of skin friction coefficient ($C_f$) with that of Zachara [5]
Figure 5.3: Effect of magnetic field on (a) skin friction ($C_f$) and (b) heat transfer ($Nu$) coefficients
Figure 5.4: Effect of magnetic field on (a) velocity and (b) temperature profiles
Figure 5.5: Effect of viscous dissipation on (a) heat transfer coefficient and (b) temperature profiles
Appendix A

Introduction to Magnetohydrodynamics

Magnetofluid dynamics (MFD) is the study of flow of electrically conducting fluids in electric and magnetic fields. It unifies in a common framework the electromagnetic and fluid dynamic theories, to yield a description of the concurrent effects of the magnetic field on the flow and, the flow on the magnetic field. Magnetofluid dynamics deals with an electrically conducting fluid, whereas its subtopics, magnetohydrodynamics (MHD) and magnetogasdynamics (MGD), are specially concerned with electrically conducting liquids and ionized compressible gases. However, the term MHD generally used for incompressible electrically conducting fluids including conducting liquids and gases.

For an incompressible hydromagnetic fluid flow, the basic equations are

\[
\rho \frac{D\mathbf{U}}{Dt} = \nabla \cdot \tau + \mathbf{J} \times \mathbf{B} \quad \text{ (A.1)}
\]

\[
\nabla \cdot \mathbf{U} = 0 \quad \text{ (A.2)}
\]
\[ \nabla \cdot \mathbf{B} = 0 \quad \text{(Magnetic field continuity)} \quad (A.3) \]
\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Flux continuity)} \quad (A.4) \]
\[ \nabla \times \mathbf{B} = \mu_m \mathbf{J} + \frac{1}{c_0^2} \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere’s law)} \quad (A.5) \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday’s law)} \quad (A.6) \]

where \( \rho \) is the density of the fluid, \( \mathbf{U} \) the velocity field, \( D/Dt \) is the time derivative (in which \( \partial \mathbf{U}/\partial t \) is the local acceleration and \( (\mathbf{U} \cdot \nabla) \mathbf{U} \) is the advection term), \( (\nabla \cdot) \) is the divergence and \( \tau_i = -\delta \mathbf{I} + \mu \gamma \) is the Newtonian stress in which \( -\delta \mathbf{I} \) is the indeterminate part of the spherical stress and

\[ \gamma = \nabla \mathbf{U} + (\nabla \mathbf{U})^T \quad (A.7) \]

is the rate of deformation tensor [175]. Further, \( \varepsilon_0 \) represents the permittivity, \( c_0 \) is the speed of light, \( \mathbf{J} \) is the current density, \( \mu_m = 1/\varepsilon_0 c_0^2 \) the magnetic field permeability, \( \mathbf{E} \) the total electric field current, \( \mathbf{B} \) is the magnetic field.

To precede the discussion the following assumptions are made [176-178]: The density \( \rho \), magnetic permeability \( \mu_m \) are considered constant throughout the flow field region. The electric field conductivity \( \sigma \) is also considered as constant and assumed to finite. Total magnetic field \( \mathbf{B} \) is perpendicular to the velocity field \( \mathbf{U} \) and the induced magnetic field \( \mathbf{b} \) is negligible compared with the applied magnetic field \( \mathbf{B}_0 \) so that the magnetic Reynolds number is small [188,189].

We assume a situation where no energy is added or extracted from the fluid by the electric field, which implies that there is no electric field present in the fluid flow region.
Under these conditions, we readily obtain the simplified form of the MHD Eqns.(A.3)-(A.6), that is

\[ \mathbf{J} \times \mathbf{B} = \sigma (\mathbf{U} \times \mathbf{B}) \times \mathbf{B} = \sigma \mathbf{B}_0^2 \mathbf{U} \]  

(A.8)

where, \((\mathbf{J} \times \mathbf{B})\) is actually, the Lorentz force on the conducting fluid produced by the interaction of the current and the magnetic field. The velocity field is given by

\[ \mathbf{U} = [u(y, x), \mathbf{u}(x), 0] \]  

(A.9)

which follows from the continuity equation. The magnetic field usually applied in the y-direction normal to the boundary layer flow. The flow is in x-direction whose velocity component is \(u\). Thus \(J \times B\), representing Lorentz force becomes \(\sigma \mathbf{B}_0^2 \rho \mathbf{u}\). The negative sign in the term is because of retardation. The Eqn.(A.1) with Eqn.(A.8), in the absence of pressure gradient, becomes

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma \mathbf{B}_0^2}{\rho} \mathbf{u} \]  

(A.10)

where \(\nu = \mu/\rho\) is the kinematic viscosity of the fluid and the term \(\frac{\sigma \mathbf{B}_0^2}{\rho} \mathbf{u}\) describes the x-component of the magnetic field, where \(\frac{\sigma \mathbf{B}_0^2}{\rho}\) is the magnetic parameter which is the ratio of the electromagnetic force to the inertial force. In fact, the Eqn.(A.10) represents a typical unsteady, two-dimensional momentum boundary layer equation with an applied magnetic field.
Appendix B

Outline of the Quasilinearization Technique

The method of quasilinearization can be viewed as an extension of the Newton-Raphson’s method for the solution of algebraic equations to the solution of differential equations [100]. It is an iterative method combined with linear approximations. Monotonicity and quadratic convergence are the favourable features of this method. This method provides a sequence of functions which in general convergence rapidly to the solution of original equations, inspite of using uninspired initial guesses. In this method, instead of solving the nonlinear differential equations directly they are solved recursively by a series of, locally, linear differential equations. The existence and uniqueness of the solution by the quasilinearization technique has been very well established [101], and several applications of this technique have been discussed in [179-181].

Since a majority of the boundary-layer problems are of the two-point, nonlinear, boundary value type, a second order vector system with one independent variable has been considered in the domain \([a, b]\) to demonstrate the quasilinearization method. However, extension to the systems with more independent variables is simple and obvious.

Let us consider the vector equation

\[ Y'' = \Phi(x, Y, Y'), \quad a \leq x \leq b \quad (B.1) \]
subject to the boundary conditions

\[ Y(a) = A, \quad Y(b) = B \]  

(B.2)

where \( Y \) is a vector composed of \( n \) dependent variables in the system and \( x \) is the independent variable. The prime \((')\) denotes the differentiation with respect to \( x \). \( \Phi \) is a vector function which gives the derivatives of the dependent variable, and \( A \) and \( B \) are constant vectors whose values are known.

Applying the method of quasilinearization to the above system, we obtain a linear system of equations which, in vector notation, can be written as

\[
Y''^{(k+1)} = \Phi^{(k)} + \sum_{i=1}^{n} \frac{\partial^{(k)} \Phi}{\partial Y_{i}} \left[ Y_{i}^{(k+1)} - Y_{i}^{(k)} \right] \\
+ \sum_{i=1}^{n} \frac{\partial^{(k)} \Phi}{\partial Y_{i}} \left[ Y_{i}^{(k+1)} - Y_{i}^{(k)} \right]
\]  

(B.3)

where the subscript \( i \) denotes the \( i^{th} \) component of the vector, the subscript \( (k) \) denotes the \( k^{th} \) approximate and \( \frac{\partial^{(k)} \Phi}{\partial Y_{i}} \) denote the derivative expressed in terms of \( k^{th} \) approximates. Eqn.(B.3) can be rearranged and written as an explicit linear system in terms of the \((k + 1)^{th}\) approximates as

\[
Y''^{(k+1)} = P^{(k)}Y^{(k+1)} + Q^{(k)}Y^{(k+1)} + R^{(k)}
\]  

(B.4)

where the coefficients \( P, Q \) and \( R \) are matrices of functions of the \( k^{th} \) iterative and are therefore known.
Appendix C

Finite-Difference Methods

The method of finite differences are widely used in Computational Fluid Dynamics (CFD). In fact, these methods are extensively applied to the solution of both linear and nonlinear partial differential equations. Indeed, the region of integration is divided into a network of computational cells by a generally fixed orthogonal grid. Ordinary/partial derivatives of functions in various directions are replaced by finite-differences such as central or backward differences.

Application of the finite difference method to any physical problem consists of following three steps.

1. Division of spatial domain into an orthogonal computational grid. The grid levels are conveniently indexed in case of two dimensions by integers $i$ and $j$ which increase monotonically along the $x$- and $y$- directions, respectively. The intersections of the grid levels define a set of node points at which dependent variable is defined.

2. Discretization of the governing equation and boundary conditions in space and time to derive approximately equivalent algebraic equation for each node, using Taylor series expansion. The spatial discretization of the governing partial differential equation is performed over the finite difference grid, while time discretization is carried out by marching in time.
3. Solution of the discretized (algebraic) equations is obtained by a suitable matrix inversion along with an iterative technique.

For boundary layer equations which are parabolic in nature the central differences are used in \( \eta \)-direction (along the boundary layer thickness) and backward differences are used in streamwise direction \( \xi \) as well as in time direction \( t^* \) with constant step-size \( \Delta \eta, \Delta \xi \) and \( \Delta t^* \) in \( \eta, \xi \) and \( t^* \)-directions, respectively. The mesh-point diagram for finite difference scheme has been shown in Fig. C.1. Consequently, the relevant finite-differences are given by

\[
\begin{align*}
X &= \Theta X_{m,n,p} + (1 - \Theta) X_{m+1,n,p} \\
X' &= \left[ \Theta(X_{m,n,p+1} - X_{m,n,p-1}) + (1 - \Theta)(X_{m,n+1,p-1} - X_{m,n+1,p-1}) \right]/2\Delta \eta \\
X_{t^*} &= \left[ \Theta(X_{m,n,p} - X_{m-1,n,p}) + (1 - \Theta)(X_{m+1,n,p} - X_{m-1,n+1,p}) \right]/\Delta t^* \\
X_\xi &= \left[ \Theta(X_{m,n,p} - X_{m,n-1,p}) + (1 - \Theta)(X_{m+1,n,p} - X_{m+1,n-1,p}) \right]/\Delta \xi \\
X'' &= \left[ \Theta(X_{m,n,p+1} - 2X_{m,n,p} + X_{m,n,p-1}) + (1 - \Theta)(X_{m,n+1,p+1} - 2X_{m,n+1,p} + X_{m,n+1,p-1}) \right]/(\Delta \eta)^2
\end{align*}
\]  

(C.1)

The subscripts \( m, n \) and \( p \) denote particular locations corresponding to \( t^*, \xi \) and \( \eta \) respectively. However, the above finite differences are written with a parameter \( \Theta \) which will give the various finite difference schemes as indicated below:

\[
\Theta = \begin{cases} 
0 & \text{Explicit method} \\
\frac{1}{2} & \text{Crank-Nicolson method} \\
1 & \text{Implicit method}
\end{cases}
\]

Usually, in an explicit scheme each equation is allowed the determination of one unknown quantity in terms of known quantities, while an implicit scheme involves the solution of a set of simultaneous algebraic equations at each station. The explicit scheme is conditionally stable and requires a very small mesh size. Hence they are found to be unsuitable for engineering problems. On the other hand, implicit schemes are unconditionally stable. It is appropriate to remark here that, a numerical
Finite-difference method is considered to be convergent if the solution of the discretized equation tends to the exact solution of the differential equation as the grid spacing tends to zero. For initial value problems, the Lax equivalence theorem (Fletcher, 1988) states that “given a properly posed linear initial value problems and a finite difference approximation to it that satisfies the consistency condition, stability is necessary and sufficient condition for convergence”. The consistency aspect of finite difference scheme can be attributed to the discretisation of governing partial differential equations which becomes exact as the grid spacing tends to zero. The difference between the discretized equation and the exact is actually introduces the truncation error. It is usually estimated by replacing all the nodal values in the discrete approximation by a Taylor series expansion about a single point. As a result one recovers the original partial differential equations plus a remainder, which represent the truncation error.

For a method to be consistent, the truncation error must become zero when the mesh spacing become very very small. Also, a numerical solution is stable if it ensures a bounded solution, without magnifying the errors that appear in the course of numerical solution process. The Lax equivalence theorem is of great importance, since it is relatively easy to show stability of an algorithm and its consistency with the original partial differential equation, whereas it is usually very difficult to show convergence of its solution to that of the partial differential equation. In fact, for non-linear problems, which are strongly influenced by boundary conditions, the stability and convergence of a method are difficult to demonstrate. Therefore, convergence is usually checked using numerical experiments, i.e., repeating the calculation as a series of successive refined grids. If the discretization process are consistent, we usually find that the solution does converge to a grid independent solutions. The stability, convergence and consistency of the finite-difference schemes have been discussed in detail in references [182,110,88,183-185,114,186,187,111,188].
The quasilinearization technique explained in Appendix B is used along with finite-difference scheme. Indeed, quasilinearization technique makes the system of nonlinear partial differential equations to become linear, locally which helps in the faster convergence of the numerical solution. The difference equations written by using above finite-differences along with quasilinearization technique are expressed in the general form of matrix equation as:

\[ A_n W_{n-1} + B_n W_n + C_n W_{n+1} = D_n \quad (2 \leq n \leq \tilde{N} - 1) \]  

(C.2)

where the vectors and the coefficient matrices are of the form

\[ W_n = \begin{bmatrix} F_n \\ G_n \end{bmatrix}, \quad D_n = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \quad A_n = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \]

and \( B_n \) and \( C_n \) have similar expressions as \( A_n \). Also, \( F \) and \( G \) are dependent variables. The matrix Eqn.(C.2) can be solved by using Varga’s algorithm [133], to get the solution. The algorithm is as follows:

\[ W_n = -E_n W_{n+1} + J_{n+1}, \quad 1 \leq n \leq \tilde{N} - 1 \]

\[ E_n = (B_n - A_n E_{n-1})^{-1} C_n \]

where

\[ J_n = (B_n - A_n E_{n-1})^{-1} (D_n - A_n J_{n-1}) \quad 2 \leq n \leq \tilde{N} - 1 \]

\[ E_1 = E_{\tilde{N}} = 0, \quad J_1 = \begin{bmatrix} F(a) \\ G(a) \end{bmatrix}, \quad J_{\tilde{N}} = \begin{bmatrix} F(b) \\ G(b) \end{bmatrix} \]

The column matrices \( J_1 \) and \( J_{\tilde{N}} \) are known, since they represent the boundary conditions \( a \) and \( b \).
Figure C.1: Mesh point diagram for finite difference scheme
Appendix D

The Keller Box Method

The Box method reported by H.B. Keller [111] and also known as Keller Box method has become popular for obtaining similar and nonsimilar solutions for boundary layer problems [188]. The Box method, having desirable advantages that make it appropriate for the solution of all parabolic partial differential equations. The main features of this method are:

1. The method is simple, efficient and easy to program;
2. Only slightly more arithmetic to solve as compared to other finite-difference schemes;
3. The method is unconditionally stable and has second order accuracy with arbitrary (nonuniform) $x$ and $y$ spacings;
4. Allows very rapid $x$-direction (stream wise coordinate) variations;
5. The method enables one to compute extremely close to the point of boundary layer separation with no special precautions.

The general net rectangle in the $x - y$ plane is shown in Fig.D.1. The net points are defined as:

\begin{align}
  x^0 &= 0 & x^n &= x^{n-1} + k_n, & n = 1, 2, \ldots, J \\
  y_0 &= 0 & y_j &= y_{j+1} + h_j, & j = 1, 2, \ldots, J, & y_J &= y_\infty
\end{align}
where $k_n$ is the $\Delta x$-spacing and $h_j$ is the $\Delta y$-spacing. Here $n$ and $j$ are just the sequence of numbers that indicate the coordinate location, not tensor, indices or exponents. The finite-differences in this scheme are expressed as

$$
\left( \begin{array}{c} \frac{1}{2} \left[ ( \cdot )_j^n + ( \cdot )_{j-1}^n \right] \\
\frac{1}{2} \left[ ( \cdot )_j^{n-1} + ( \cdot )_{j-1}^{n-1} \right] \\
\end{array} \right)_{j-1/2} = \frac{1}{2} \left[ ( \cdot )_j^n + ( \cdot )_{j-1}^n \right] \\
\frac{1}{2} \left[ ( \cdot )_j^{n-1} + ( \cdot )_{j-1}^{n-1} \right] \\
$$

(D.3)

$$
\left( \frac{\partial u}{\partial x} \right)_{j-1/2}^{n-1/2} = \frac{u_j^n - u_{j-1}^{n-1/2}}{k_n} \\
\left( \frac{\partial u}{\partial y} \right)_{j-1/2}^{n-1/2} = \frac{u_j^{n-1/2} - u_{j-1}^{n-1/2}}{h_j} \\
$$

(D.4)

(D.5)

(D.6)

The solution of a differential equations by this method can be obtained by the following four steps:

1. Reduce the equation or equations to a first order system;
2. Write the difference equations using appropriate finite differences;
3. Linearize the resulting algebraic equations by Newton’s method (which makes nonlinear difference equations to linear at each iteration) and write them in matrix vector form;
4. Solve the linear system by block-tridiagonal-elimination technique.

The Keller Box method, being an elegant and lucid method, has been successfully used by several contemporary researchers.
Figure D.1: Net rectangle in $xy$-plane for Keller box method