Chapter 5

Nonclassicality breaking channels

5.1 Introduction

Two notions that have been particularly well explored in the context of quantum information of continuous variable states are *nonclassicality* [81, 82] and *entanglement* [331, 332]. The 'older' notion of entanglement has become one of renewed interest in recent decades for its central role and applications in (potential as well as demonstrated) quantum information processes [181, 333], while the concept of nonclassicality, which emerges directly from the *diagonal representation* [81, 82] had already been well explored in the quantum optical context [334–336], even before the emergence of the present quantum information era. A fundamental distinction between these two notions may be noted: *While nonclassicality can be defined even for states of a single mode of radiation, the very notion of entanglement requires two or more parties*. Nevertheless, it turns out that the two notions are not entirely independent of one another; they are rather intimately related [89, 337–340]. In fact, nonclassicality is a prerequisite for entanglement [338–340]. Since a nonclassical bipartite state whose nonclassicality can be removed by local unitaries could not be entangled, one can assert, at least in an intuitive sense, that *entanglement is nonlocal nonclassicality*.

An important aspect in the study of nonclassicality and entanglement is in regard of their evolution under the action of a channel. A noisy channel acting on a state can degrade its nonclassical features [341–349]. Similarly, entanglement can be degraded by channels acting locally on the constituent parties or modes [77, 96, 316, 317, 350–356]. We have seen earlier (1.120), that *entanglement breaking* channels are those that render every bipartite state separable by action on one of the subsystems [77, 96, 357].

In this Chapter, we address the following issue : *which channels possess the property of ridding every input state of its nonclassicality?* Inspired by the notion of entanglement breaking channels, we may call such channels *nonclassicality breaking channels*. The close connection between nonclassicality and entanglement alluded to earlier raises a related second issue : *what is the connection, if any, between entanglement breaking channels and nonclassicality breaking channels?* To appreciate the nontriviality of the second issue, it suffices to simply note that the very definition of entanglement breaking refers to bipartite states whereas the notion of nonclassicality breaking makes no such reference. We show that both these issues can be completely answered in the case of bosonic Gaussian channels : nonclassicality breaking channels are enumerated, and it is shown that the set of all nonclassicality breaking channels is essentially the same as the set of all entanglement breaking channels.

We hasten to clarify the caveat 'essentially'. Suppose a channel Γ is nonclassicality breaking as well as entanglement breaking, and let us follow the action of this channel with a local unitary $\mathcal{U}$. The composite $\mathcal{U}\,\Gamma$ is clearly entanglement breaking. But local unitaries can create nonclassicality, and so $\mathcal{U}\,\Gamma$ need not be nonclassicality breaking. We say Γ is *essentially nonclassicality breaking* if there exists a fixed unitary $\mathcal{U}$ dependent on Γ but independent of the input state on which Γ acts, so that $\mathcal{U}\,\Gamma$ is nonclassicality breaking. We may stress that this definition is not vacuous, for given a collection of states *it is generically the case that there is no single unitary which would render the entire set nonclassical*. [This is not necessarily a property of the collection : given a nonclassical mixed state ρ, it is possibly not guaranteed that there exists an unitary $\mathcal{U}$ such that $\hat{\rho}' = \mathcal{U}\hat{\rho}\,\mathcal{U}^{\dagger}$ is classical.] It is thus reasonable to declare the set of entanglement breaking channels to be the same as the set of nonclassicality breaking channels if at all the two sets indeed turn out to be the same, modulo this 'obvious' caveat or provision.

We recall that Gaussian channels are physical processes that map Gaussian states to Gaussian states and their systematic analysis was presented in [76, 90–93, 95, 280, 358–360].

5.2 Nonclassicality breaking channels

Any density operator $\hat{\rho}$ representing some state of a single mode of radiation field can always be expanded as

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi}\, \phi_\rho(\alpha)|\alpha\rangle\langle\alpha|, \tag{5.1}$$

where $\phi_\rho(\alpha) = W_1(\alpha;\rho)$ is the diagonal 'weight' function, $|\alpha\rangle$ being the coherent state. This *diagonal representation* is made possible because of the over-completeness property of the coherent state 'basis' [81, 82]. The diagonal representation (5.1) enables the evaluation, *in a classical-looking manner*, of ensemble averages of normal-ordered operators, and this is important from the experimental point of view [361].

An important notion that arises from the diagonal representation is the *classicality-nonclassicality divide*. If $\phi_\rho(\alpha)$ associated with density operator $\hat{\rho}$ is pointwise nonnegative over $\mathcal{C}$, then the state is a convex sum, or ensemble, of coherent states. Since coherent states are the most elementary of all quantum mechanical states exhibiting classical behaviour, any state that can be written as a convex sum of these elementary classical states is deemed classical. We have,

$$\phi_\rho(\alpha) \geq 0 \ \text{ for all } \ \alpha \in \mathcal{C} \Leftrightarrow \hat{\rho} \text{ is classical.} \tag{5.2}$$

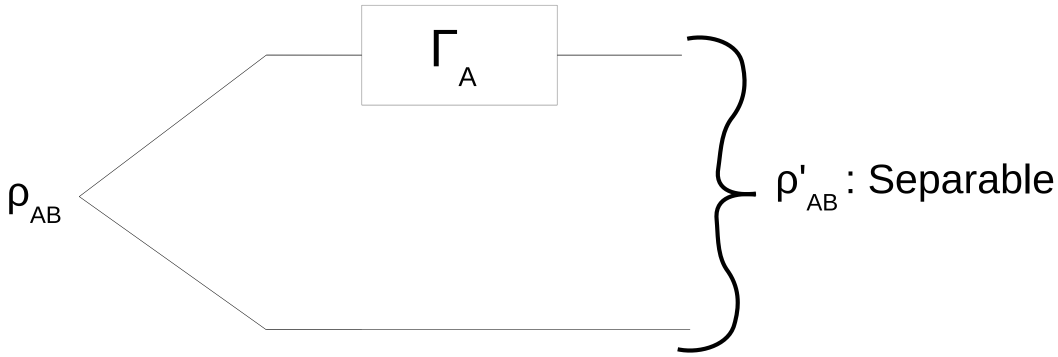

Figure 5.1: A schematic diagram depicting the notion of entanglement breaking channels.

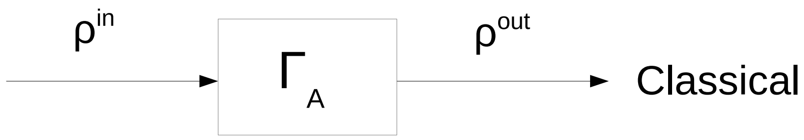

Figure 5.2: Showing the notion of nonclassicality breaking channels.

Any state which cannot be so written is declared to be nonclassical. Fock states $|n\rangle\langle n|$, whose diagonal weight function $\phi_{|n\rangle\langle n|}(\alpha)$ is the n$^{\text{th}}$ derivative of the delta function, are examples of nonclassical states. [All the above considerations generalize from one mode to *n*-modes in a painless manner, with $\alpha, \xi \in \mathcal{R}^{2n} \sim C^n$.]

This classicality-nonclassicality divide leads to the following natural definition, inspired by the notion of entanglement breaking (See Fig. 5.1) :

Definition : A channel Γ is said to be *nonclassicality breaking* if and only if the output state $\hat{\rho}_{\text{out}} = \Gamma(\hat{\rho}_{\text{in}})$ is classical *for every* input state $\hat{\rho}_{\text{in}}$, i.e., if and only if the diagonal function of every output state is a genuine probability distribution (See Fig. 5.2).

5.3 Nonclassicality-based canonical forms for Gaussian channels

The canonical forms for Gaussian channels have been described by Holevo [91, 92] and Werner and Holevo [90]. Let $\mathcal{S}$ denote an element of the symplectic group $Sp(2n, R)$ of linear canonical transformations and $\mathcal{U}(\mathcal{S})$ the corresponding unitary (metaplectic) operator [78]. One often encounters situations wherein the aspects one is looking for are invariant under local unitary operations, entanglement being an example. In such cases a Gaussian channel Γ is 'equivalent' to $\mathcal{U}(\mathcal{S}')\,\Gamma\,\mathcal{U}(\mathcal{S})$, for arbitrary symplectic group elements $\mathcal{S}, \mathcal{S}'$. The orbits or double cosets of equivalent channels in this sense are the ones classified and enumerated by Holevo and collaborators [90–92] and recalled in Table 1.1.

While the classification of Holevo and collaborators is entanglement-based, as just noted, the notion of nonclassicality breaking has *a more restricted invariance*. A nonclassicality breaking Gaussian channel Γ preceded by any Gaussian unitary $\mathcal{U}(\mathcal{S})$ is nonclassicality breaking if and only if Γ itself is nonclassicality breaking. In contradistinction, the nonclassicality breaking aspect of Γ and $\mathcal{U}(\mathcal{S})\,\Gamma$ [Γ followed the Gaussian unitary $\mathcal{U}(\mathcal{S})$] are not equivalent in general; they are equivalent if and only if $\mathcal{S}$ is in the intersection $Sp(2n, R) \cap SO(2n, R) \sim U(n)$ of 'symplectic phase space rotations' or passive elements [78, 279]. In the single-mode case this intersection is just the rotation group $SO(2) \subset Sp(2, R)$ described in Eq. (1.131). We thus need to classify single-mode Gaussian channels Γ into orbits or double cosets $\mathcal{U}(\mathcal{R})\,\Gamma\,\mathcal{U}(\mathcal{S})$, $\mathcal{S} \in Sp(2, R)$, $\mathcal{R} \in SO(2) \subset Sp(2, R)$. Equivalently, we need to classify (X, Y) into orbits $(\mathcal{S}\,X\,\mathcal{R},\ \mathcal{R}^T\,Y\,\mathcal{R})$. It turns out that there are three distinct canonical forms, and the type into which a given pair (X, Y) belongs is fully determined by $\det X$.

First canonical form : $\det X > 0$.

A real 2×2 matrix X with $\det X = \kappa^2 > 0$ is necessarily of the form $\kappa\,\mathcal{S}_X$ for some

$\mathcal{S}_X \in Sp(2, R)$. Indeed we have $\mathcal{S}_X = (\det X)^{-1/2} X$ Choose $\mathcal{R} \in SO(2)$ so as to diagonalise $Y > 0$: $\mathcal{R}^T Y \mathcal{R} = \mathrm{diag}(a, b)$. With such an $\mathcal{R}$, the choice $\mathcal{S} = \mathcal{R}^T \mathcal{S}_X^{-1} \in Sp(2, R)$ takes (X, Y) to the canonical form $(\kappa 1\!\!1, \mathrm{diag}(a, b))$, where $\kappa = \sqrt{\det X} > 0$, and a, b are the eigenvalues of Y.

Second canonical form : $\det X < 0$.

Again choose $\mathcal{R}$ so that $\mathcal{R}^T Y \mathcal{R} = \mathrm{diag}(a, b)$. Since $\det X < 0$, X is necessarily of the form $\kappa \mathcal{S}_X \sigma_3$, for some $\mathcal{S}_X \in Sp(2, R)$: $\mathcal{S}_X = (\det X\sigma_3)^{-1/2} X\sigma_3$. Since $\mathcal{R}\sigma_3\mathcal{R} = \sigma_3$ for every $\mathcal{R} \in SO(2)$, it is clear that the choice $\mathcal{S} = \mathcal{R}\mathcal{S}_X^{-1} \in Sp(2, R)$ takes (X, Y) to the canonical form $(\kappa\sigma_3, \mathrm{diag}(a, b))$ in this case, with $\kappa = \sqrt{\det X\sigma_3}$, and the parameters a, b being the eigenvalues of Y.

Third canonical form : $\det X = 0$.

Let κ be the singular value of X; choose $\mathcal{R}'$, $\mathcal{R} \in SO(2)$ such that $\mathcal{R}' X \mathcal{R} = \mathrm{diag}(\kappa, 0)$. It is clear that the choice $\mathcal{S}_X = \mathrm{diag}(\kappa^{-1}, \kappa)\mathcal{R}'^T \in Sp(2, R)$ along with $\mathcal{R} \in SO(2)$ takes (X, Y) to the canonical form $(\mathrm{diag}(1, 0), Y_0 = \mathcal{R}^T Y \mathcal{R})$. Y_0 does not, of course, assume any special form. But if $X = 0$, then $\mathcal{R} \in SO(2)$ can be chosen so as to diagonalise Y: in that case $Y_0 = (a, b)$, a, b being the eigenvalues of Y.

5.4 Nonclassicality breaking Gaussian channels

Having obtained the nonclassicality-based canonical forms of (X, Y), we now derive the necessary and sufficient conditions for a single-mode Gaussian channel to be nonclassicality breaking. We do it for the three canonical forms in that order.

First canonical form : $(X, Y) = (\kappa 1\!\!1, \mathrm{diag}(a, b))$.

There are three possibilities: $\kappa = 1$, $\kappa < 1$, and $\kappa > 1$. We begin with $\kappa = 1$; it happens that the analysis extends quite easily to the other two cases and, indeed, to the other two canonical forms as well. The action on the normal-ordered characteristic function in this case is

$$\begin{aligned}\chi_N^{\text{in}}(\xi_1,\, \xi_2; \rho) &\rightarrow \chi_N^{\text{out}}(\xi_1, \xi_2; \rho)\\ &= \exp\left[-\frac{a\,\xi_1^2}{2} - \frac{b\,\xi_2^2}{2}\right]\chi_N^{\text{in}}(\xi_1,\, \xi_2; \rho).\end{aligned} \tag{5.3}$$

[For clarity, we shall write the subscript of χ explicitly as N, W, or A in place of 1, 0, or -1]. It should be appreciated that *for this class of Gaussian channels* ($\kappa = 1$) the above input-output relationship holds even with the subscript N replaced by W or A uniformly. Let us assume $a,\, b > 1$ so that $a = 1 + \epsilon_1$, $b = 1 + \epsilon_2$ with $\epsilon_1,\, \epsilon_2 > 0$. The above input-output relationship can then be written in the form

$$\chi_N^{\text{out}}(\xi_1, \xi_2; \rho) = \exp\left[-\frac{\epsilon_1\,\xi_1^2}{2} - \frac{\epsilon_2\,\xi_2^2}{2}\right]\chi_W^{\text{in}}(\xi_1,\, \xi_2; \rho).$$

Note that the subscript of χ on the right hand side is now W and not N.

Define $\lambda > 0$ through $\lambda^2 = \sqrt{\epsilon_2/\epsilon_1}$, and rewrite the input-output relationship in the suggestive form

$$\begin{aligned}\chi_N^{\text{out}}(\lambda\xi_1, \lambda^{-1}\xi_2; \rho) &= \exp\left[-\frac{1}{2}(\sqrt{\epsilon_1\epsilon_2}\,\xi_1^2 - \sqrt{\epsilon_1\epsilon_2}\,\xi_2^2)\right]\\ &\quad\times\chi_W^{\text{in}}(\lambda\xi_1,\, \lambda^{-1}\xi_2; \rho).\end{aligned} \tag{5.4}$$

But $\chi_W^{\text{in}}(\lambda\xi_1,\, \lambda^{-1}\xi_2; \rho)$ is simply the Weyl-ordered or Wigner characteristic function of a (single-mode-) squeezed version of $\hat{\rho}$, for every $\hat{\rho}$. If $\mathcal{U}_\lambda$ represents the unitary (metaplectic) operator that effects this squeezing transformation specified by squeeze parameter λ, we have

$$\chi_W^{\text{in}}(\lambda\xi_1,\, \lambda^{-1}\xi_2; \rho) = \chi_W^{\text{in}}(\xi_1,\, \xi_2; \mathcal{U}_\lambda\,\rho\,\mathcal{U}_\lambda^\dagger), \tag{5.5}$$

so that the right hand side of the last input-output relationship, *in the special case* $\epsilon_1\epsilon_2 = 1$, reads

$$\chi_W^{\rm out}(\lambda\xi_1,\, \lambda^{-1}\xi_2;\rho) = \chi_A^{\rm in}(\xi_1,\, \xi_2; \mathcal{U}_\lambda\, \rho\, \mathcal{U}_\lambda^\dagger). \tag{5.6}$$

This special case would transcribe, on Fourier transformation, to

$$\begin{aligned} \phi^{\rm out}(\lambda\alpha_1,\, \lambda^{-1}\alpha_2;\rho) &= Q^{\rm in}(\alpha_1,\, \alpha_2; \mathcal{U}_\lambda\, \rho\, \mathcal{U}_\lambda^\dagger) \\ &= \langle\alpha|\mathcal{U}_\lambda\, \hat{\rho}\, \mathcal{U}_\lambda^\dagger|\alpha\rangle \geq 0,\ \ \forall\ \alpha,\ \ \forall\ \hat{\rho}. \end{aligned} \tag{5.7}$$

That is, the output diagonal weight function evaluated at $(\lambda\alpha_1,\, \lambda^{-1}\alpha_2)$ equals the input Q-function evaluated at $(\alpha_1,\, \alpha_2)$, and hence is nonnegative for all $\alpha \in \mathcal{C}$. Thus the output state is classical for every input, and hence the channel is nonclassicality breaking. It is clear that if $\epsilon_1\epsilon_2 > 1$, the further Gaussian convolution corresponding to the additional multiplicative factor $\exp\left[-(\sqrt{\epsilon_1\epsilon_2} - 1)(\xi_1^2 + \xi_2^2)/2\right]$ in the output characteristic function will only render the output state even more strongly classical. We have thus established this *sufficient condition*

$$(a-1)(b-1) \geq 1, \tag{5.8}$$

or, equivalently,

$$\frac{1}{a} + \frac{1}{b} \leq 1. \tag{5.9}$$

Having derived a sufficient condition for nonclassicality breaking, we derive a necessary condition by looking at the signature of the output diagonal weight function *for a particular input state* evaluated at *a particular phase space* point at the output. Let the input be the Fock state $|1\rangle\langle 1|$, the first excited state of the oscillator. Fourier transforming the input-output relation (5.3), one readily computes the output diagonal weight function to be

$$\phi^{\text{out}}(\alpha_1,\, \alpha_2; |1\rangle\langle 1|) = \frac{2}{\sqrt{ab}}\, \exp\left[-\frac{2\alpha_1^2}{a} - \frac{2\alpha_2^2}{b}\right] \\ \times\left(1 + \frac{4(\alpha_1+\alpha_2)^2}{a^2} - \frac{1}{a} - \frac{1}{b}\right). \tag{5.10}$$

An obvious necessary condition for nonclassicality breaking is that this function should be nonnegative everywhere in phase space. Nonnegativity at the single phase space point $\alpha = 0$ gives the necessary condition $1/a + 1/b \leq 1$ which is, perhaps surprisingly, the same as the sufficiency condition established earlier! That is, *the sufficient condition (5.8) is also a necessary condition for nonclassicality breaking*. Saturation of this inequality corresponds to the boundary wherein the channel is 'just' nonclassicality breaking. *The formal resemblance in this case with the law of distances in respect of imaging by a thin convex lens* is unlikely to miss the reader's attention.

The above proof for the particular case of classical noise channel ($\kappa = 1$) gets easily extended to noisy beamsplitter (attenuator) channel ($\kappa < 1$) and noisy amplifier channel ($\kappa > 1$). The action of the channel $(\kappa 1\!\!1,\, \text{diag}(a, b))$ on the normal-ordered characteristic function follows from that on the Wigner characteristic function given in (1.170) :

$$\begin{aligned} \chi_N^{\text{out}}(\xi;\rho) &= \exp\left[-\frac{\tilde{a}\,\xi_1^2}{2} - \frac{\tilde{b}\,\xi_2^2}{2}\right]\chi_N^{\text{in}}(\kappa\,\xi;\rho), \\ \tilde{a} &= a + \kappa^2 - 1, \;\; \tilde{b} = b + \kappa^2 - 1. \end{aligned} \tag{5.11}$$

This may be rewritten in the suggestive form

$$\chi_N^{\text{out}}(\kappa^{-1}\xi;\rho) = \exp\left[-\frac{\tilde{a}\,\xi_1^2}{2\kappa^2} - \frac{\tilde{b}\,\xi_2^2}{2\kappa^2}\right]\chi_N^{\text{in}}(\xi;\rho). \tag{5.12}$$

With this we see that the right hand side of (5.12) to be the same as right hand side of (5.3) with $\tilde{a}/\kappa^2$, $\tilde{b}/\kappa^2$ replacing a, b. The case $\kappa \neq 1$ thus gets essentially reduced to the case $\kappa = 1$, the case of classical noise channel, analysed in detail above. This leads to the following *necessary and sufficient condition for nonclassicality breaking*

$$\frac{1}{a+\kappa^2-1}+\frac{1}{b+\kappa^2-1}\leq\frac{1}{\kappa^2}$$
$$\Leftrightarrow\ (a-1)(b-1)\geq\kappa^4, \tag{5.13}$$

for all $\kappa > 0$, thus completing our analysis of the first canonical form.

Second canonical form : $(\mathbf{X},\mathbf{Y}) = (\kappa\,\boldsymbol{\sigma}_3,\ \mathbf{diag}(\mathbf{a},\mathbf{b}))$. The noisy phase conjugation channel with canonical form $(\kappa\,\sigma_3,\ \text{diag}(a,b))$ acts on the

normal-ordered characteristic function in the following manner, as may be seen from its action on the Weyl-ordered characteristic function (1.170) :

$$\chi_N^{\text{out}}(\xi;\rho) = \exp\left[-\frac{\tilde{a}\,\xi_1^2}{2}-\frac{\tilde{b}\,\xi_2^2}{2}\right]\chi_N^{\text{in}}(\kappa\,\sigma_3\,\xi;\rho), \tag{5.14}$$

with $\tilde{a} = a+\kappa^2-1,\ \ \tilde{b} = b+\kappa^2-1$ again, and $\kappa\,\sigma_3\,\xi$ denoting the pair $(\kappa\,\xi_1, -\kappa\,\xi_2)$. As in the case of the noisy amplifier/attenuator channel, we rewrite it in the form

$$\chi_N^{\text{out}}(\kappa^{-1}\,\sigma_3\,\xi;\rho) = \exp\left[-\frac{\tilde{a}\,\xi_1^2}{2\kappa^2}-\frac{\tilde{b}\,\xi_2^2}{2\kappa^2}\right]\chi_N^{\text{in}}(\xi;\rho), \tag{5.15}$$

the right hand side of (5.15) has the same form as (5.3), leading to the *necessary and sufficient nonclassicality breaking condition*

$$\frac{1}{\tilde{a}}+\frac{1}{\tilde{b}}\leq\frac{1}{\kappa^2}\ \ \Leftrightarrow\ \ (a-1)(b-1)\geq\kappa^4. \tag{5.16}$$

Remark : We note in passing that in exploiting the ‘similarity’ of Eqs. (5.12) and (5.15) with Eq. (5.3), we made use of the following two elementary facts : (1) An invertible linear change of variables $[f(x)\rightarrow f(A\,x),\ \det A\neq 0]$ on a multivariable function $f(x)$ reflects as a corresponding linear change of variables in its Fourier transform ; (2) A function $f(x)$ is pointwise nonnegative if and only if $f(A\,x)$ is pointwise nonnegative for every invertible

A. In the case of (5.12), the linear change A corresponds to uniform scaling, and in the case of (5.15) it corresponds to uniform scaling followed or preceded by mirror reflection.

Third canonical form: Singular $\boldsymbol{X}$. Unlike the previous two cases, it proves to be convenient to begin with the Weyl or symmetric-ordered characteristic function in this case of singular X:

$$\chi_W^{\text{out}}(\xi;\rho) = \exp\left[-\frac{1}{2}\xi^T\, Y_0\, \xi\right]\chi_W^{\text{in}}(\xi_1, 0;\rho). \tag{5.17}$$

Since we are dealing with symmetric ordering, $\chi_W^{\text{in}}(\xi_1, 0;\rho)$ is the Fourier transform of the marginal distribution of the first quadrature ('position' quadrature) variable. Let us assume that the input $\hat{\rho}$ is a (single-mode-) squeezed Gaussian pure state, squeezed in the position (or first) quadrature. For arbitrarily large squeezing, the state approaches a position eigenstate and the position quadrature marginal approaches the Dirac delta function. That is $\chi_W^{\text{in}}(\xi_1, 0;\rho)$ approaches a constant. Thus, the Gaussian $\exp\left[-(\xi^T\, Y_0\, \xi)/2\right]$ is essentially the Weyl-characteristic function of the output state, and hence corresponds to a classical state if and only if

$$Y_0 \geq 1\!\!1, \ \text{ or } \ a,\ b \geq 1, \tag{5.18}$$

a, b being the eigenvalues of Y.

We have derived this as a *necessary condition for nonclassicality breaking*, taking as input a highly squeezed state. It is clear that for any other input state the phase space distribution of the output state will be a convolution of this Gaussian classical state with the position quadrature marginal of the input state, rendering the output state more strongly classical, and thus proving that the condition (5.18) is *also a sufficient condition for nonclassicality breaking*.

In the special case in which $X = 0$ identically, we have the following input-output relation in place of (5.17) :

$$\chi_W^{\text{out}}(\xi;\rho) = \exp\left[-\frac{1}{2}\xi^T\, Y\, \xi\right]\chi_W^{\text{in}}(\xi = 0;\rho). \qquad (5.19)$$

Since $\chi_W^{\text{in}}(\xi = 0;\rho) = 1$ independent of $\hat{\rho}$, the output is an input-independent *fixed state*, and $\exp\left[-\frac{1}{2}\xi^T\, Y\, \xi\right]$ is its Weyl-characteristic function. But we know that this fixed output is a classical state if and only if $Y \geq 1\!\!1$. In other words, *the condition for nonclassicality breaking is the same for all singular X, including vanishing X.*

We conclude our analysis in this Section with the following, perhaps redundant, remark : Since our canonical forms are nonclassicality-based, rather than entanglement-based, if the nonclassicality breaking property applies for one member of an orbit or double coset, it applies to the entire orbit.

5.5 Nonclassicality breaking *vs* entanglement breaking

We are now fully equipped to explore the relationship between nonclassicality breaking Gaussian channels and entanglement breaking channels. In the case of the first canonical form the nonclassicality breaking condition reads $(a-1)(b-1) \geq \kappa^4$, the entanglement breaking condition reads $ab \geq (1+\kappa^2)^2$, while the complete positivity condition reads $ab \geq (1-\kappa^2)^2$. These conditions are progressively weaker, indicating that the family of channels which meet these conditions are progressively larger. For the second canonical form the first two conditions have the same formal expression as the first canonical form, while the complete positivity condition has a more stringent form $ab \geq (1+\kappa^2)^2$. For the third and final canonical form, the nonclassicality breaking condition requires both a and b to be bounded from below by unity, whereas both the entanglement breaking and complete positivity conditions read $ab \geq 1$. Table 5.1 conveniently places these conditions side-by-side. In the case of first canonical form, (first row of Table 5.1), the complete positivity condition itself is vacuous for $\kappa = 1$, the classical noise channels.

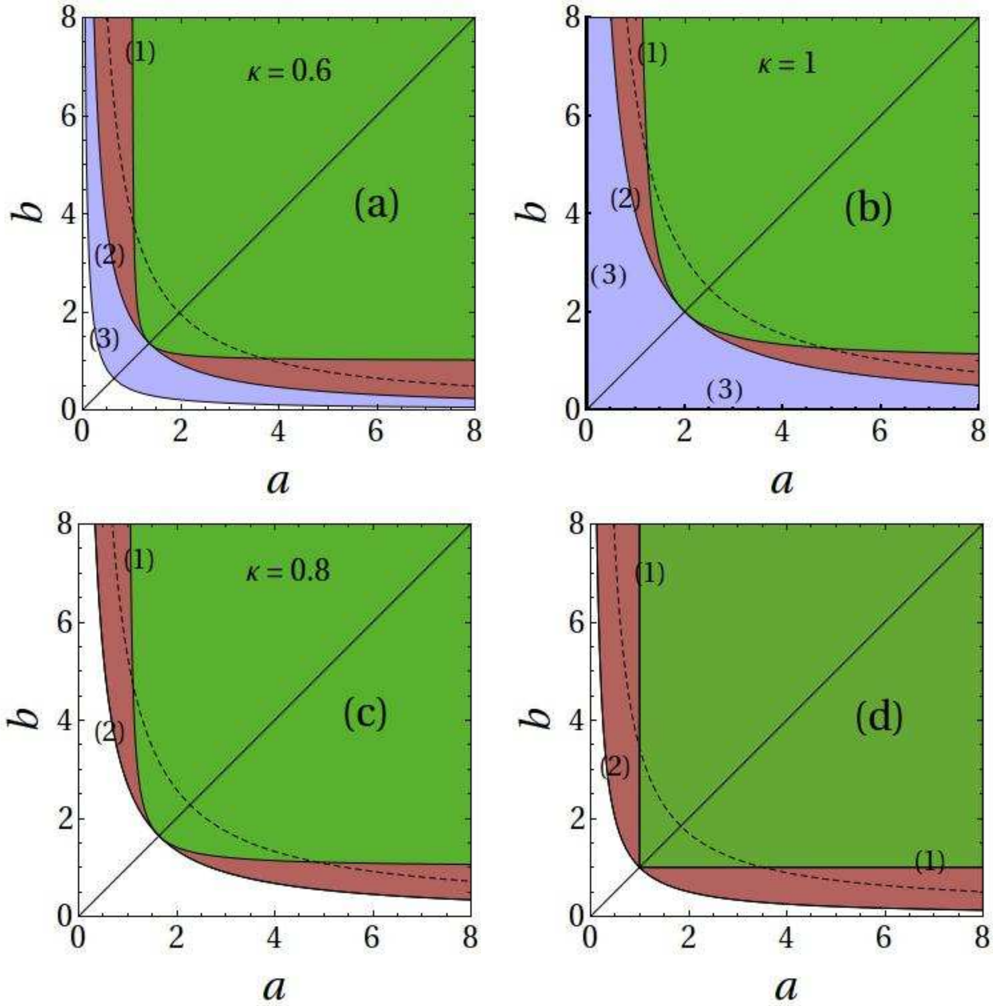

Figure 5.3: Showing a pictorial comparison of the nonclassicality breaking condition, the entanglement breaking condition, and the complete positivity condition in the channel parameter space (a, b), for fixed $\det X$. Curves (1), (2), and (3) correspond to saturation of these conditions in that order. Curve (3) thus corresponds to quantum-limited channels. Frame (a) refers to the first canonical form ($\kappa 1\!\!1$, diag(a, b)), frame (c) to the second canonical form ($\kappa\,\sigma_3$, diag(a, b)), and frame (d) to the third canonical form, singular X. Frame (b) refers to the limiting case $\kappa = 1$, classical noise channel. In all the four frames, the region to the right of (above) curve (1) corresponds to nonclassicality breaking channels; the region to the right of (above) curve (2) corresponds to entanglement breaking channels; curve (3) depicts the CP condition, so the region to the right of (above) it alone corresponds to physical channels. The region to the left (below) curve (3) is unphysical as channels. In frames (c) and (d), curves (2) and (3) coincide. In frame (b), curve (3) of (a) reduces to the a and b axis shown in bold. In frames (a) and (c), curves (1) and (2) meet at the point $(1 + \kappa^2, 1 + \kappa^2)$, in frame (b) they meet at $(2, 2)$, and in frame (d) at $(1, 1)$. The region between (2) and (3) corresponds to the set of channels which are not entanglement breaking. That in frame (c) and (d) the two curves coincide proves that this set is vacuous for the second and third canonical forms. That in every frame the nonclassicality breaking region is properly contained in the entanglement breaking region proves that a nonclassicality breaking channel is certainly an entanglement breaking channel. The dotted curve in each frame indicates the orbit of a generic entanglement breaking Gaussian channel under the action of a local unitary squeezing after the channel action. That the orbit of every entanglement breaking channel passes through the nonclassicality breaking region, proves that the nonclassicality in all the output states of an entanglement breaking channel can be removed by a fixed unitary squeezing, thus showing that every entanglement breaking channel is 'essentially' a nonclassicality breaking channel.

Canonical form	NB	EB	CP
$(\kappa\,\mathbb{1},\ \mathrm{diag}(a,b))$	$(a-1)(b-1)\geq\kappa^4$	$ab\geq(1+\kappa^2)^2$	$ab\geq(1-\kappa^2)^2$
$(\kappa\,\sigma_3,\ \mathrm{diag}(a,b))$	$(a-1)(b-1)\geq\kappa^4$	$ab\geq(1+\kappa^2)^2$	$ab\geq(1+\kappa^2)^2$
$(\mathrm{diag}(1,0),\ Y)$,	$a,\ b\geq 1$, a, b being eigenvalues of Y	$ab\geq 1$	$ab\geq 1$
$(\mathrm{diag}(0,0),\ \mathrm{diag}(a,b))$	$a,\ b\geq 1$	$ab\geq 1$	$ab\geq 1$

Table 5.1: A comparison of the nonclassicality breaking (NB) condition, the entanglement breaking (EB) condition, and the complete positivity (CP) condition for the three canonical classes of channels.

This comparison is rendered pictorial in Fig. 5.3, in the channel parameter plane (a,b), for fixed values of $\det X$. Saturation of the nonclassicality breaking condition, the entanglement breaking condition, and the complete positivity condition are marked (1), (2), and (3) respectively in all the four frames. Frame (a) depicts the first canonical form for $\kappa = 0.6$ (attenuator channel). The case of the amplifier channel takes a qualitatively similar form in this pictorial representation. As $\kappa \to 1$, from below ($\kappa < 1$) or above ($\kappa > 1$), curve (3) approaches the straight lines $a = 0$, $b = 0$ shown as solid lines in Frame (b) which depicts this limiting $\kappa = 1$ case (the classical noise channel). Frame (c) corresponds to the second canonical form (phase conjugation channel) for $\kappa = 0.8$ and Frame (d) to the third canonical form. It may be noticed that in Frames (c) and (d) the curves (2) and (3) merge, indicating and consistent with that fact that channels of the second and third canonical forms are aways entanglement breaking.

It is clear that the nonclassicality breaking condition is stronger than the entanglement breaking condition. Thus, a nonclassicality breaking channel is necessarily entanglement breaking : But there are channel parameter ranges wherein the channel is entanglement breaking, though not nonclassicality breaking. The dotted curves in Fig. 5.3 represent orbits of a generic entanglement breaking channel Γ, fixed by the product ab (κ having been already fixed), when Γ is followed up by a variable local unitary squeezing $\mathcal{U}(r)$. To see that the orbit of every entanglement breaking channel passes through the nonclassicality breaking region, it suffices to note from Table 5.1 that the nonclassicality breaking boundary has $a = 1$, $b = 1$ as asymptotes whereas the entanglement breaking boundary

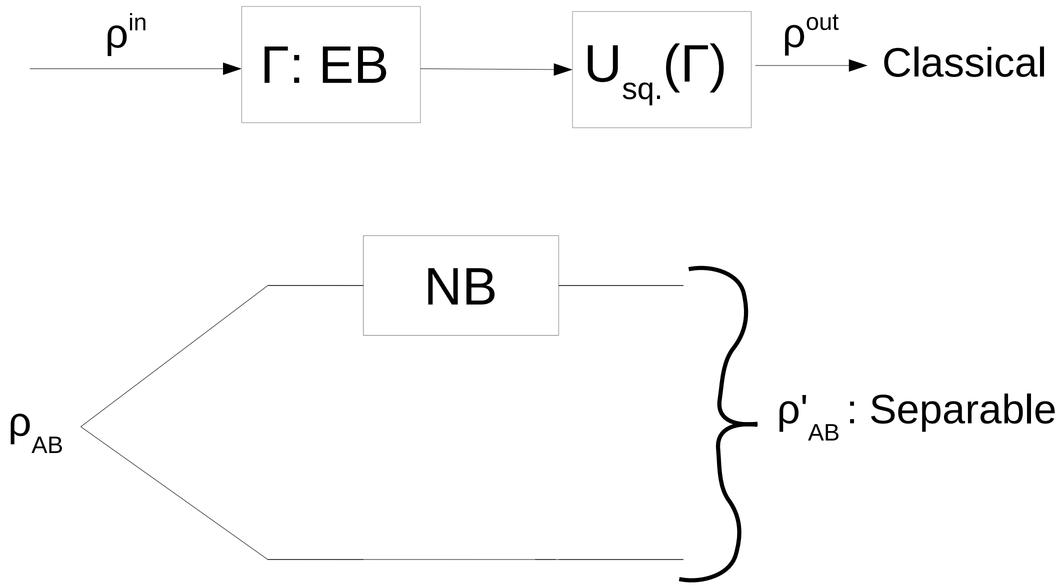

Figure 5.4: Showing the relationship between nonclassicality breaking and entanglement breaking channels established in the present Chapter. The output state corresponding to any input to an entanglement breaking channel is rendered classical by a single squeeze transformation that depends only on the channel parameters and independent of the input states. In other words, an entanglement breaking channel followed by a given squeeze transformation renders the original channel nonclassicality breaking. In contrast, every nonclassicality breaking channel is also entanglement breaking.

has $a = 0$, $b = 0$ as the asymptotes. That is, for every entanglement breaking channel there exists a particular value of squeeze-parameter r_0, depending only on the channel parameters and not on the input state, so that the entanglement breaking channel Γ followed by unitary squeezing of extent r_0 always results in a nonclassicality breaking channel $\mathcal{U}(r_0)\,\Gamma$. It is in this precise sense that nonclassicality breaking channels and entanglement breaking channels are essentially one and the same.

Stated somewhat differently, if at all the output of an entanglement breaking channel is nonclassical, the nonclassicality is of a ‘weak’ kind in the following sense. Squeezing is not the only form of nonclassicality. Our result not only says that the output of an entanglement breaking channel could at the most have a squeezing-type nonclassicality, it further says that the nonclassicality of *all* output states can be removed by a *fixed* unitary squeezing transformation. This is depicted schematically in Fig. 5.4.

5.6 Conclusions

We have explored the notion of nonclassicality breaking and its relation to entanglement breaking. We have shown that the two notions are effectively equivalent in the context of bosonic Gaussian channels, even though at the level of definition the two notions are quite different, the latter requiring reference to a bipartite system. Our analysis shows that some nonclassicality could survive an entanglement breaking channel, but this residual nonclassicality would be of a particular weaker kind.

The close relationship between entanglement and nonclassicality has been studied by several authors in the past [89, 316, 317, 337–340, 354–356]. It would seem that our result brings this relationship another step closer.

Finally, we have presented details of the analysis only in the case of single-mode bosonic Gaussian channels. We believe the analysis is likely to generalize to the case of n-mode channels in a reasonably straight forward manner.