# Part II

# Main Results

# Chapter 2

# CP maps and initial correlations

## 2.1 Introduction

Open quantum systems are systems that are in interaction with its environment. Therefore, open quantum systems play a very fundamental role in the study of every realistic or practical application of quantum systems. There has been a rapid growth in the understanding of various properties related to open quantum systems like its realization, control, and the role played by the noise in such dissipative systems, both in the theoretical and experimental domain.

Recent studies on various aspects of control of open quantum systems has appeared [97–110]. These studies have been motivated by applications to quantum computing [111–113], laser cooling [114, 115], quantum reservoir engineering [116, 117], managing decoherence [118–122], and also to other fields like chemical reactions and energy transfer in molecules [123–126]. There has also been a study of experimental aspects of environment induced decoherence in various physical scenarios including atomic systems [127–131], spin networks [132], and molecular physics [133, 134].

A related recent avenue has been to exploit the dissipation into the environment. Here, theoretical studies of basic tasks in quantum information theory like state preparation [135–139], distillation [140], storage [141], cooling [142], and including their experimental aspects [143, 144], are performed by engineering the system-environment coupling. Further, the issue of timing in such dissipative quantum information processing was addressed in [145].

In this chapter we study the induced dynamics of the system resulting from the dynamics of an open quantum system. In particular, we explore the role of the initial system-bath states, especially in respect of a possible connection to quantum discord, as brought out in recent literature. From the various manifestations of open quantum systems listed above, it is pertinent to understand this aspect of realization of an open quantum system.

Every physical system is in interaction with its environment, *the bath*, to a smaller or larger degree of strength. Therefore, the joint unitary dynamics or unitary Schrödinger evolutions of the system and bath induces a dissipative non-unitary dynamics for the system [146]. We now briefly recapitulate the folklore scheme or Stinepring dilation [69, 147–150]. The Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$ of the system and the bath are of dimensions $d_S$, $d_B$ respectively. The $(d_S^2 - 1)$-dimensional (convex) state space $\Lambda_S$ is a subset of $\mathcal{B}(\mathcal{H}_S)$. We also denote the collection of initial-system bath states by $\Omega^{SB} \subset \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$, the convex hull of $\Omega^{SB}$ being denoted $\overline{\Omega^{SB}}$. The definition of $\Omega^{SB}$ will become clear in a subsequent Section.

### 2.1.1 Folklore scheme

The folklore scheme (see Fig. 2.1) for realizing open system dynamics is to first elevate the system states $\rho_S$ to the (tensor) products $\rho_S \otimes \rho_B^{\rm fid}$, for a *fixed* fiducial bath state $\rho_B^{\rm fid}$. Then these composite uncorrelated system-bath states are evolved under a joint unitary $U_{SB}(t)$, and finally the bath degrees of freedom are traced out to obtain the evolved states

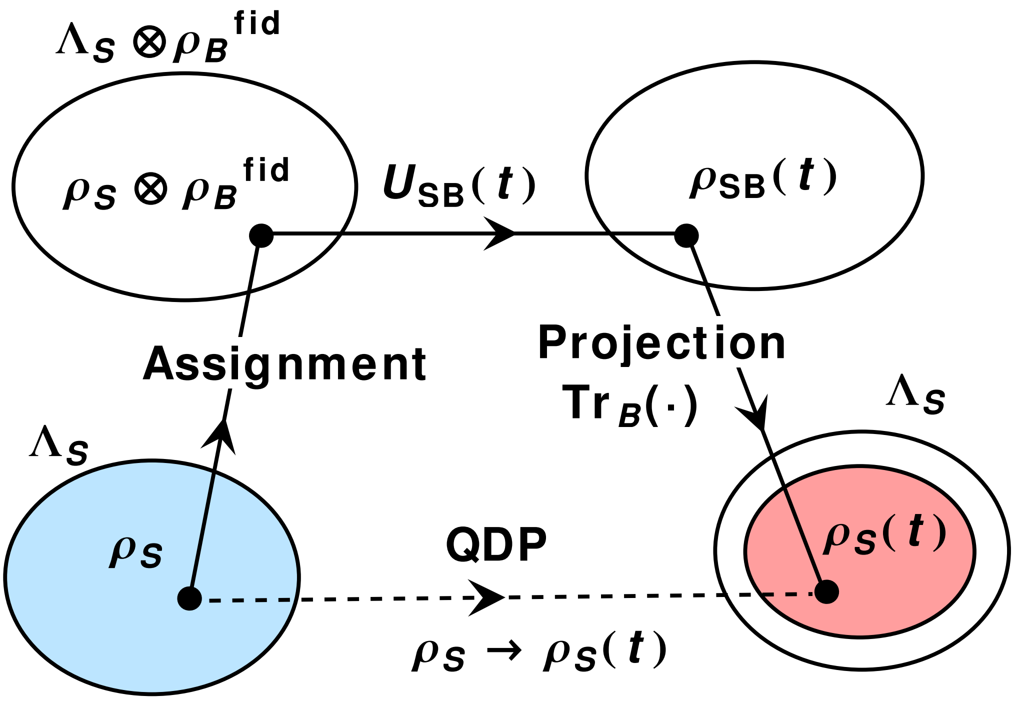


Figure 2.1: Showing the folklore scheme. In the folklore scheme, initial system states $\rho_S$ are elevated to product states of the composite, for a *fixed* fiducial bath state $\rho_B^{\mathrm{fid}}$, through the assignment map $\rho_S \to \rho_S \otimes \rho_B^{\mathrm{fid}}$. These uncorrelated system-bath states are evolved under a joint unitary $U_{SB}(t)$ to $U_{SB}(t)\rho_S \otimes \rho_B^{\mathrm{fid}} U_{SB}(t)^\dagger$ and, finally, the bath degrees of freedom are traced out to obtain the time-evolved states $\rho_S(t) = \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\rho_\mathrm{S} \otimes \rho_\mathrm{B}^{\mathrm{fid}}\, \mathrm{U_{SB}(t)}^\dagger\right]$ of the system. The resulting quantum dynamical process (QDP) $\rho_S \to \rho_S(t)$, parametrized by $\rho_B^{\mathrm{fid}}$ and $U_{SB}(t)$, is completely positive by construction. Initial system states are identified by the blue region and the final states by the red.

$\rho_S(t)$ of the system :

$$\rho_S \to \rho_S \otimes \rho_B^{\mathrm{fid}} \to U_{SB}(t)\,(\rho_S \otimes \rho_B^{\mathrm{fid}})\,U_{SB}(t)^\dagger$$
$$\to \rho_S(t) = \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\,(\rho_\mathrm{S} \otimes \rho_\mathrm{B}^{\mathrm{fid}})\,\mathrm{U_{SB}(t)}^\dagger\right]. \tag{2.1}$$

The resulting quantum dynamical process (QDP) $\rho_S \to \rho_S(t)$, parametrized by $\rho_B^{\mathrm{fid}}$ and $U_{SB}(t)$, is provably completely positive (CP) [67, 68, 70, 147, 148].

Indeed if $\rho_B^{\mathrm{fid}} = |\psi_B\rangle\langle\psi_B|$, and $\{|\nu_B\rangle\}$ is a complete basis for system $B$, then the operator-sum representation for the QDP can be written as

$$\rho_S(t) = \sum_k A_k(t)\,\rho_S\, A_k^\dagger(t), \tag{2.2}$$

where $A_k(t)$ are the sum-operators which are given by

$$A_k(t) = \langle v_B^k | U_{SB}(t) | \psi_B \rangle. \tag{2.3}$$

If instead we have a mixed state $\rho_B^{\text{fid}}$, then the operator-sum representation will just be a convex combination of maps resulting from each pure state in, say, the spectral resolution of $\rho_B^{\text{fid}}$.

While every CP map can be thus realized with uncorrelated initial states of the composite, there has been various studies in literature that explore more general realizations of CP maps [151–160]. Possible effects of system-bath initial correlations on the reduced dynamics for the system has been the subject of several recent studies [161–169]. Some of these works look at the connection between the concept of quantum discord and the complete positivity of the reduced dynamics [161,163,168]; these are of much interest to us.

### 2.1.2 SL scheme

A specific, carefully detailed, and precise formulation of the issue of initial system-bath correlations possibly influencing the reduced dynamics was presented not long ago by Shabani and Lidar [163]. In this formulation (see Fig. 2.2), the distinguished bath state $\rho_B^{\text{fid}}$ is replaced by a collection of (possibly correlated) system-bath initial states $\Omega^{SB} \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$. The dynamics gets defined through a joint unitary $U_{SB}(t)$:

$$\rho_{SB}(0) \rightarrow \rho_{SB}(t) = U_{SB}(t)\,\rho_{SB}(0)\,U_{SB}(t)^\dagger, \quad \forall\, \rho_{SB}(0) \in \Omega^{SB}. \tag{2.4}$$

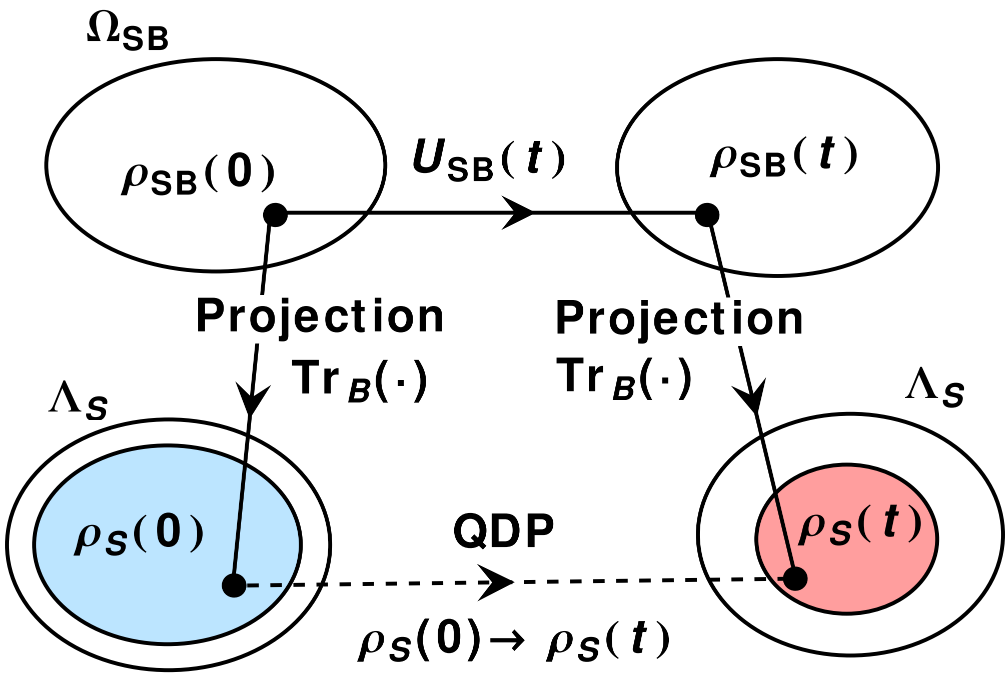


Figure 2.2: Showing the SL scheme. In sharp contrast to the folklore scheme, there is no assignment map in the SL scheme. The distinguished bath state $\rho_B^{\text{fid}}$ is replaced by a collection $\Omega^{SB}$ of (possibly correlated) system-bath initial states $\rho_{SB}(0)$. The dynamics gets defined through $\rho_{SB}(0) \rightarrow \rho_{SB}(t) = U_{SB}(t)\rho_{SB}(0)\,U_{SB}(t)^\dagger$ for all $\rho_{SB}(0) \in \Omega^{SB}$. With reduced system states $\rho_S(0)$ and $\rho_S(t)$ defined through the imaging or projection map $\rho_S(0) = \text{Tr}_\text{B}\,\rho_\text{SB}(0)$ and $\rho_S(t) = \text{Tr}_\text{B}\left[\text{U}_\text{SB}(\text{t})\rho_\text{SB}(0)\,\text{U}_\text{SB}(\text{t})^\dagger\right]$, this unitary dynamics of the composite induces on the system the QDP $\rho_S(0) \rightarrow \rho_S(t)$. As before, initial system states are identified by the blue region and the final states by the red.

This composite dynamics induces on the system the QDP

$$\rho_S(0) \rightarrow \rho_S(t), \tag{2.5}$$

with $\rho_S(0)$ and $\rho_S(t)$ defined through this natural imaging from $\Omega^{SB}$ to the system state space $\Lambda_S$ :

$$\rho_S(0) = \text{Tr}_\text{B}\,\rho_\text{SB}(0),\ \rho_\text{S}(\text{t}) = \text{Tr}_\text{B}\rho_\text{SB}(\text{t}). \tag{2.6}$$

It is clear the folklore scheme is a particular case of the SL scheme corresponding to $\Omega^{SB} = \{\rho_S \otimes \rho_B^{\text{fid}}\,|\,\rho_B^{\text{fid}} = \text{fixed}\}$.

This generalized formulation of QDP allows SL to transcribe the fundamental issue to this question: What are the necessary and sufficient conditions on the collection $\Omega^{SB}$ so that the induced QDP $\rho_S(0) \to \rho_S(t)$ in Eq. (2.5) is guaranteed to be CP *for all* joint unitaries $U_{SB}(t)$? Motivated by the work of Rodriguez-Rosario et al. [161], and indeed highlighting it as *'a recent breakthrough'*, SL advance the following resolution to this issue :

**Theorem 7 (Shabani-Lidar)** *: The QDP in Eq.* (2.5) *is CP for all joint unitaries* $U_{SB}(t)$ *if and only if the quantum discord vanishes for all* $\rho_{SB} \in \Omega_{SB}$*, i.e., if and only if the initial system-bath correlations are purely classical.*

Whether the Shabani-Lidar QDP so described is well-defined and completely positive is clearly an issue answered solely by the nature of the collection $\Omega^{SB}$.

## 2.2 Properties of SL $\Omega^{SB}$

In order that the QDP in Eq. (2.5) be *well defined* in the first place, the set $\Omega^{SB}$ should necessarily satisfy the following two properties; since our entire analysis rests critically on these properties, we begin by motivating them.

### 2.2.1 Property 1

No state $\rho_S(0)$ can have two (or more) pre-images in $\Omega^{SB}$. To see this fact unfold assume, to the contrary, that

$$\begin{aligned} \mathrm{Tr_B}\,\rho_{\mathrm{SB}}(0) = \mathrm{Tr_B}\,\rho'_{\mathrm{SB}}(0),\;\; \rho_{\mathrm{SB}}(0) \neq \rho'_{\mathrm{SB}}(0), \\ \text{for two states } \rho_{SB}(0),\;\; \rho'_{SB}(0) \in \Omega^{SB}. \end{aligned} \tag{2.7}$$

Clearly, the difference $\triangle\rho_{SB}(0) = \rho_{SB}(0) - \rho'_{SB}(0) \neq 0$ should necessarily meet the property $\mathrm{Tr_B}\triangle\rho_{\mathrm{SB}}(0) = 0$. Let $\{\lambda_u\}_{u=1}^{d_S^2-1}$ be a set of orthonormal hermitian traceless $d_S \times d_S$ matrices so that together with the unit matrix $\lambda_0 = \mathbb{1}_{d_S \times d_S}$ these matrices form a hermitian basis for $\mathcal{B}(\mathcal{H}_S)$, the set of all $d_S \times d_S$ (complex) matrices. Let $\{\gamma_v\}_{v=1}^{d_B^2-1}$, $\gamma_0 = \mathbb{1}_{d_B \times d_B}$ be a similar basis for $\mathcal{B}(\mathcal{H}_{\mathcal{B}})$. The $(d_S d_B)^2$ tensor products $\{\lambda_u \otimes \gamma_v\}$ form a basis for $\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$, and $\triangle\rho_{SB}(0)$ can be written in the form

$$\triangle\rho_{SB}(0) = \sum_{u=0}^{d_S^2-1} \sum_{v=0}^{d_B^2-1} C_{uv}\, \lambda_u \otimes \gamma_v,\; C_{uv} \text{ real.} \tag{2.8}$$

Now, the property $\mathrm{Tr_B}\, \triangle\rho_{\mathrm{SB}}(0) = 0$ is strictly equivalent to the requirement that the expansion coefficient $C_{u0} = 0$ for all $u = 0,\, 1,\, \cdots\, d_S^2 - 1$. Since the $[(d_S d_B)^2 - 1]$-parameter unitary group $SU(d_S d_B)$ acts *irreducibly* on the $[(d_S d_B)^2 - 1]$-dimensional subspace of $\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ consisting of all traceless $d_S d_B$-dimensional matrices [ this is the adjoint representation of $SU(d_S d_B)$ ], there exists an $U_{SB}(t) \in SU(d_S d_B)$ which takes $\triangle\rho_{SB}(0) \neq 0$ into a matrix whose expansion coefficient $C_{u0} \neq 0$ for some $u$. That is, if the initial $\triangle\rho_{SB}(0) \neq 0$ then one and the same system state $\rho_S(0)$ will evolve into two distinct

$$\begin{aligned} \rho_S(t) &= \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\rho_{\mathrm{SB}}(0)\mathrm{U_{SB}(t)}^\dagger\right], \\ \rho_S'(t) &= \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\rho_{\mathrm{SB}}'(0)\mathrm{U_{SB}(t)}^\dagger\right], \end{aligned} \tag{2.9}$$

for some $U_{SB}(t)$, rendering the QDP in equation (2.5) one-to-many, and hence ill-defined.

### 2.2.2 Property 2

While every system state $\rho_S(0)$ need not have a pre-image *actually enumerated* in $\Omega^{SB}$, the set of $\rho_S(0)$'s having pre-image should be sufficiently large. Indeed, Rodriguez-Rosario et al. [161] have rightly emphasised that it should be *'a large enough set of states such that the QDP in Eq.* (2.5) *can be extended by linearity to all states of the system'*. It is easy to see that if $\Omega^{SB}$ fails this property, then the very issue of CP would make no sense. For, in carrying out verification of CP property, the QDP would be required to act, as is well known [67], on $\{|j\rangle\langle k|\}$ for $j,\, k = 1, 2, \cdots d_S$; i.e., on generic complex $d_S$-dimensional square matrices, and not just on positive or hermitian matrices alone. Since the basic issue on hand is to check if the QDP as a map on $\mathcal{B}(\mathcal{H}_S)$ is CP or not, it is essential that it be well defined (at least by linear extension) on the entire *complex* linear space $\mathcal{B}(\mathcal{H}_S)$.

## 2.3 Main Result

With the two properties of $\Omega^{SB}$ thus motivated, we proceed to prove our main result. We 'assume', *for the time being*, that every pure state $|\psi\rangle$ of the system has a pre-image in $\Omega^{SB}$. This assumption may appear, at first sight, to be a drastic one. But we show later that it entails indeed *no loss of generality*.

It is evident that, for every pure state $|\psi\rangle$, the pre-image in $\Omega^{SB}$ has to necessarily assume the (uncorrelated) product form $|\psi\rangle\langle\psi| \otimes \rho_B$ , $\rho_B$ being a state of the bath which could possibly depend on the system state $|\psi\rangle$.

Now, let $\{|\psi_k\rangle\}_{k=1}^{d_S}$ be an orthonormal basis in $\mathcal{H}_S$ and let $\{|\phi_\alpha\rangle\}_{\alpha=1}^{d_S}$ be another orthonormal basis related to the former through a complex Hadamard unitary matrix $U$. Recall that a unitary U is Hadamard if $|U_{k\alpha}| = 1/\sqrt{d_S}$, independent of $k, \alpha$. For instance, the characters of the cyclic group of order $d_S$ written out as a $d_S \times d_S$ matrix is Hadamard. The fact that the $\{|\psi_k\rangle\}$ basis and the $\{|\phi_\alpha\rangle\}$ basis are related by a Hadamard means that $|\langle\psi_k|\phi_\alpha\rangle|$ is independent of both $k$ and $\alpha$, and hence equals $1/\sqrt{d_S}$ uniformly. We may refer to such a pair as *relatively unbiased bases*.

Let $|\psi_k\rangle\langle\psi_k| \otimes O_k$ be the pre-image of $|\psi_k\rangle\langle\psi_k|$ in $\Omega^{SB}$ and $|\phi_\alpha\rangle\langle\phi_\alpha| \otimes \widetilde{O}_\alpha$ that of $|\phi_\alpha\rangle\langle\phi_\alpha|$, $k, \alpha = 1, 2, \cdots, d_S$. Possible dependence of the bath states $O_k$ on $|\psi_k\rangle$ and $\widetilde{O}_\alpha$ on $|\phi_\alpha\rangle$ has not been ruled out as yet. Since the maximally mixed state of the system can be expressed in two equivalent ways as $d_S^{-1} \sum_k |\psi_k\rangle\langle\psi_k| = d_S^{-1} \sum_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$, *uniqueness* of its pre-image in $\Omega^{SB}$ (Property 1) demands

$$\sum_{k=1}^{d_S} |\psi_k\rangle\langle\psi_k| \otimes O_k = \sum_{\alpha=1}^{d_S} |\phi_\alpha\rangle\langle\phi_\alpha| \otimes \widetilde{O}_\alpha. \tag{2.10}$$

Taking projection of both sides on $|\psi_j\rangle\langle\psi_j|$, and using the Hadamard property $|\langle\psi_j|\phi_\alpha\rangle|^2 = d_S^{-1}$, we have

$$O_j = \frac{1}{d_S}\sum_{\alpha=1}^{d_S} \widetilde{O}_\alpha, \quad j = 1, 2, \cdots, d_S, \tag{2.11}$$

while projection on $|\phi_\beta\rangle\langle\phi_\beta|$ leads to

$$\widetilde{O}_\beta = \frac{1}{d_S}\sum_{k=1}^{d_S} O_k, \;\; \beta = 1, 2, \cdots, d_S. \tag{2.12}$$

The $2d_S$ constraints of Eqs. (2.11), (2.12) together imply that $O_j = \widetilde{O}_\beta$ uniformly for all $j, \beta$. Thus the pre-image of $|\psi_k\rangle\langle\psi_k|$ is $|\psi_k\rangle\langle\psi_k| \otimes \rho_B^{\rm fid}$ and that of $|\phi_\alpha\rangle\langle\phi_\alpha|$ is $|\phi_\alpha\rangle\langle\phi_\alpha| \otimes \rho_B^{\rm fid}$, for all $k, \alpha$, for some *fixed fiducial bath state* $\rho_B^{\rm fid}$. And, perhaps more importantly, the pre-image of the maximally mixed state $d_S^{-1}\mathbb{1}$ necessarily equals $d_S^{-1}\mathbb{1} \otimes \rho_B^{\rm fid}$ as well.

Taking another pair of relatively unbiased bases $\{|\psi_k^{'}\rangle\}$, $\{|\phi_\alpha^{'}\rangle\}$ one similarly concludes that the pure states $|\psi_k^{'}\rangle\langle\psi_k^{'}|$, $|\phi_\alpha^{'}\rangle\langle\phi_\alpha^{'}|$ too have pre-images $|\psi_k^{'}\rangle\langle\psi_k^{'}| \otimes \rho_B^{\rm fid}$, $|\phi_\alpha^{'}\rangle\langle\phi_\alpha^{'}| \otimes \rho_B^{\rm fid}$ respectively, with the same fixed fiducial bath state $\rho_B^{\rm fid}$ as before. This is so, since the maximally mixed state is *common* to both sets.

Considering in this manner enough number of pure states or projections $|\psi\rangle\langle\psi|$ sufficient to span—by linearity—the entire system state space $\Lambda_S$, and hence $\mathcal{B}(\mathcal{H}_S)$, and using the fact that convex sums goes to corresponding convex sums under pre-imaging, one readily concludes that *every element* $\rho_{SB}(0)$ of $\Omega^{SB}$ (irrespective of whether ${\rm Tr}_B\,\rho_{SB}(0)$ is pure or mixed) *necessarily* needs to be of the product form $\rho_S(0) \otimes \rho_B^{\rm fid}$, for some *fixed* bath state $\rho_B^{\rm fid}$. But this is exactly the folklore realization of non-unitary dissipative dynamics, to surpass which was the primary goal of the SL scheme. We have thus proved our principal result:

> No initial correlations—*even classical ones*—are permissible in the SL scheme. That is, quantum discord is no less destructive as far as CP property of QDP is concerned.

It is true that we have proved this result under an assumption but, as we show below, this assumption entails no loss of generality at all.

As we have noted, if at all a pure state $\rho_S(0) = |\psi\rangle\langle\psi|$ has a pre-image in $\Omega^{SB}$ it would necessarily be of the product form $|\psi\rangle\langle\psi| \otimes \rho_B$, for some (possibly $|\psi\rangle$-dependent) bath state $\rho_B$. While this is self-evident and is independent of SL, it is instructive to view it as a consequence of the *necessary condition part* of SL theorem. Then our principal conclusion above can be rephrased to say that validity of SL theorem for pure states of the system readily leads to the folklore product-scheme as the *only solution* within the SL framework. This interesting aspect comes through in an even more striking manner in our proof below that our earlier 'assumption' is one without loss of generality.

### 2.3.1 Assumption entails no loss of generality

Let us focus, to begin with, on the convex hull $\overline{\Omega^{SB}}$ of $\Omega^{SB}$ rather than the full (complex) linear span of $\Omega^{SB}$ to which we are entitled. Let us further allow for the possibility that the image of $\overline{\Omega^{SB}}$ under the convexity-preserving linear map $\rho_{SB}(0) \rightarrow \text{Tr}_B\rho_{SB}(0)$ fills not the entire (convex) state space $\Lambda_S$—the $(d_S^2 - 1)$-dimensional generalized Bloch sphere—of the system, but only a portion thereof, possibly a very small part. Even so, in order that our QDP in equation (2.5) be well-defined, this portion would *occupy a non-zero volume* of the $(d_S^2 - 1)$-dimensional state space of the system (Property 2).

Let us consider one set of all mutually commuting elements of the system state space $\Lambda_S$. If the full state space were available under the imaging $\rho_{SB}(0) \rightarrow \text{Tr}_B\rho_{SB}(0)$ of

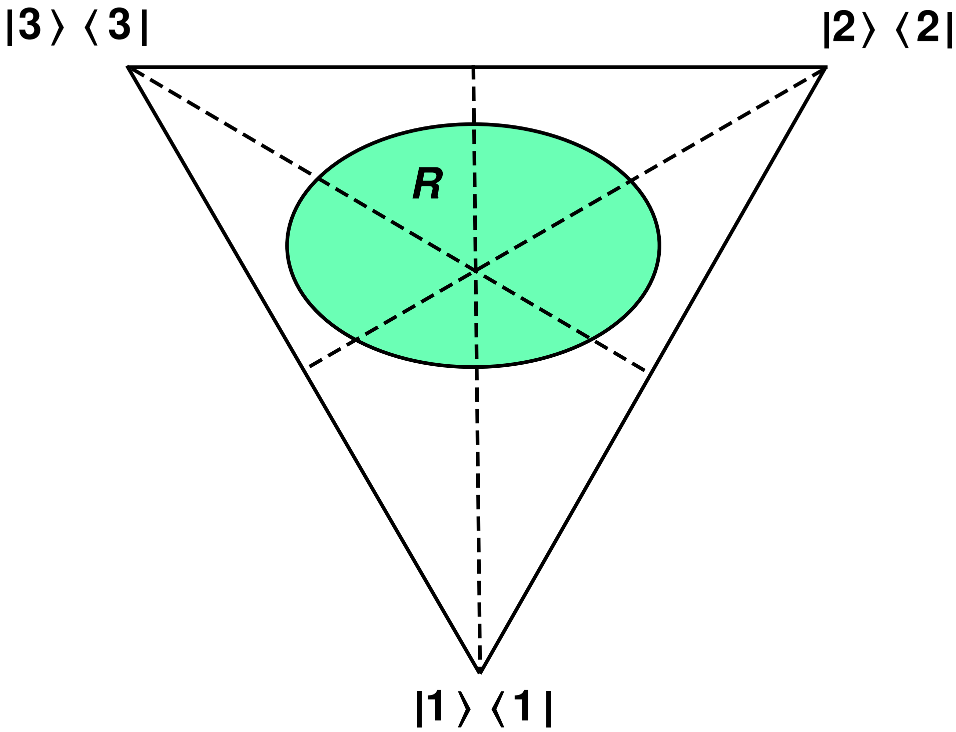


Figure 2.3: Depicting, for the case $d_S = 3$ (qutrit), the image of $\overline{\Omega^{SB}}$ under $\mathrm{Tr}_B(\cdot)$ in the plane spanned by the commuting (diagonal) $\lambda$-matrices $(\lambda_3, \lambda_8)$.

$\overline{\Omega^{SB}}$, then the resulting mutually commuting images would have filled the entire $(d_S - 1)$-simplex, the classical state space of a $d_S$-level system, this being respectively the triangle and the tetrahedron when $d_S = 3, 4$ [170, 171]. Since the full state space is not assumed to be available, these commuting elements possibly fill only a, perhaps very small but nevertheless of nontrivial measure, proper convex subset of the $(d_S - 1)$-simplex, depicted in Fig. 2.3 as region $R$ for the case $d_S = 3$, (qutrit).

Elements of these simultaneously diagonal density matrices of the system can be expressed as convex sums of orthogonal pure states or one-dimensional projections. For a generic element in this region, the spectrum is non-degenerate, and hence the projections are unique and commuting, being the eigenstates of $\rho_S(0)$, and correspond to the $d_S$ vertices of the $(d_S - 1)$-simplex. In the case of qutrit, it is pictorially seen in Fig. 2.3 that only the points on the three dotted lines correspond to doubly degenerate density matrices and the centre alone is triply degenerate, rendering transparent the fact that being nondegenerate is a generic attribute of region $R$.

Now consider the pre-image $\rho_{SB}(0)$ in $\overline{\Omega^{SB}}$ of such a generic non-degenerate $\rho_S(0) \in R$. Application of the SL requirement of vanishing discord (again, only the necessity part of the SL theorem) to this $\rho_{SB}(0)$ implies, by definition [64, 172], that the pre-image has the form

$$\rho_{SB}(0) = \sum_{j=1}^{d_S} p_j |j\rangle\langle j| \otimes \rho_{Bj}(0), \tag{2.13}$$

where the probabilities $p_j$ and the pure states $|j\rangle\langle j|$ are uniquely determined (in view of nondegeneracy) by the spectral resolution

$$\rho_S(0) = \mathrm{Tr}_B \rho_{SB}(0) = \sum_{j=1}^{d_S} p_j |j\rangle\langle j|. \tag{2.14}$$

And $\rho_{Bj}(0)$'s are bath states, possibly dependent on $|j\rangle\langle j|$ as indicated by the label $j$ in $\rho_{Bj}(0)$. These considerations hold for every nondegenerate element of region $R$ of probabilities $\{p_j\}$. In view of generic nondegeneracy, the requirement (2.13) implies that each of the $d_S$ pure states $|j\rangle\langle j|$ has pre-image of the form $|j\rangle\langle j| \otimes \rho_{Bj}(0)$ in the *linear span* of the pre-image of R—at least as seen by the QDP (2.5). That is, $\rho_{Bj}(0)$'s can have no dependence on the probabilities $\{p_j\}$.

Since every pure state of the system constitutes one of the vertices of some $(d_S - 1)$-simplex in $\Lambda_S$ comprising one set of all mutually commuting density operators $\rho_S(0)$, the conclusion that a pure state effectively has in the linear span of $\Omega^{SB}$ a pre-image, and one necessarily of the product form, *applies to every pure state*, showing that the 'assumption' in our earlier analysis indeed *entails no loss of generality*.

## 2.4 Conclusion

To summarize, it is clear that the dynamics described by $\rho_{SB}(0) \to \rho_{SB}(t) = U_{SB}(t)\,\rho_{SB}(0)\,U_{SB}(t)^{\dagger}$, $\rho_{SB}(0) \in \Omega^{SB}$ would *'see'* only the full (complex) linear span of $\Omega^{SB}$, and *not so much the actual enumeration* of $\Omega^{SB}$ as such. But as indicated by the imaging (projection) map $\rho_{SB}(0) \to \rho_S(0) = \mathrm{Tr}_{\mathrm{B}}\,\rho_{\mathrm{SB}}(0)$, the only elements of this linear span which are immediately relevant for the QDP are those which are hermitian, positive semidefinite, and have unit trace; these are precisely the elements of $\overline{\Omega^{SB}}$, the convex hull of $\Omega^{SB}$. Since no system state can have two or more pre-images (see Property 1), in order to render the QDP in (2.5) well defined these relevant elements are forced to constitute a *faithful linear embedding*, in $\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$, of (a nontrivial convex subset of) the system's state space. In the SL scheme of things, this leaves us with just the folklore embedding $\rho_S(0) \to \rho_{SB}(0) = \rho_S(0) \otimes \rho_B^{\mathrm{fid}}$. This is the primary conclusion that emerges.

*Remark on convexity :*

Let us view this from a slightly different position. Since there is no conceivable manner in which a linear map $U_{SB}(t)$ acting on elements of $\Omega^{SB}$ could be prevented from acting on convex sums (indeed, on the linear span) of such elements, we may assume—without loss of of generality—$\Omega^{SB}$ to be convex and ask, consistent with the SL theorem: What are the possible choices for the collection $\Omega^{SB}$ to be *convex and at the same time consist entirely of states of vanishing quantum discord*. One possibility comprises elements of the form $\rho_{SB}(0) = \rho_S(0) \otimes \rho_B^{\mathrm{fid}}$ for a fixed bath state $\rho_B^{\mathrm{fid}}$ and arbitrary system state $\rho_S(0)$. This case of $\overline{\Omega^{SB}} = \Lambda_S \otimes \rho_B^{\mathrm{fid}}$ is recognized to be simply the folklore case. The second one consists of elements of the form $\rho_{SB}(0) = \sum_j p_j |j\rangle\langle j| \otimes \rho_{Bj}(0)$, for a *fixed (complete) set of orthonormal pure states* $\{|j\rangle\langle j|\}$. This case restricted to *mutually commuting density operators* of the system seems to be the one studied by Rodriguez-Rosario et al. [161], but the very notion of CP itself is unlikely to make much sense in this non-quantum case of *classical state space* (of dimension $d_S - 1$ rather than $d_S^2 - 1$), the honorific 'a recent breakthrough' notwithstanding.

The stated goal of SL was to give a *complete characterization* of possible initial correlations that lead to CP maps. It is possibly in view of the (erroneous) belief that there was a large class of permissible initial correlations out there within the SL framework, and that that class now stands fully characterized by the SL theorem, that a large number of recent papers tend to list complete characterization of CP maps among the principal achievements of quantum discord [173–180]. Our result implies, with no irreverence whatsoever to quantum discord, that characterization of CP maps may not yet be rightfully paraded as one of the principal achievements of quantum discord.

The SL theorem has influenced an enormous number of authors, and it is inevitable that those results of these authors which make essential use of the sufficiency part of the SL theorem need recalibration in the light of our result.

There are other, potentially much deeper, implications of our finding. Our analysis—strictly within the SL framework—has shown that this framework brings one exactly back to the folklore scheme itself, as if it were a *fixed point*. This is not at all a negative result for two reasons. First, it shows that quantum discord is no 'cheaper' than entanglement as far as complete positivity of QDP is concerned. Second, and more importantly, the fact that the folklore product-scheme survives attack under this powerful, well-defined, and fairly general SL framework demonstrates its, perhaps unsuspected, *robustness*. In view of the fact that this scheme has been at the heart of most applications of quantum theory to real situations, virtually in every area of physical science, and even beyond, its robustness the SL framework has helped to establish is likely to prove to be of far-reaching significance.