Part I

Preliminaries

Chapter 1

Introduction

1.1 States, observables, and measurements

[1] Let us consider a quantum system A with Hilbert space $\mathcal{H}_A$ of dimension d_A. *Pure states* $|\psi\rangle$ are equivalence classes of unit vectors (unit rays) in this Hilbert space :

$$|\psi\rangle \in \mathcal{H}_A, \;\; \langle\psi|\psi\rangle = 1, \;\; e^{i\alpha}|\psi\rangle \sim |\psi\rangle, \; 0 \leq \alpha < 2\pi. \tag{1.1}$$

A *mixed state* $\hat{\rho}_A$ is a statistical ensemble of pure states :

$$\hat{\rho}_A = \sum_j p_j \, |\psi_j\rangle\langle\psi_j|, \;\; \sum_j p_j = 1, \;\; p_j > 0 \; \forall \; j. \tag{1.2}$$

Equivalently, one could view the state of a quantum system (1.2) as a linear operator acting on $\mathcal{H}_A$ that satisfies the following defining properties :

- Hermiticity : $\hat{\rho}_A = \hat{\rho}_A^\dagger$.
- Positivity : $\hat{\rho}_A \geq 0$.

[1] This introduction Chapter has been written by generously borrowing from Prof. Simon's lectures delivered over the course of my doctoral work.

- Unit trace condition : $\mathrm{Tr}[\hat{\rho}_A] = 1$.

For any state $\hat{\rho}_A$, there exists a special decomposition called the *spectral decomposition*. It states that any mixed state can be written as

$$\hat{\rho}_A = \sum_{j=1}^{r} \lambda_j \, |\psi_j\rangle\langle\psi_j|, \tag{1.3}$$

where λ_j are the eigenvalues (which are all positive) and $|\psi_j\rangle$ are the eigenvectors that satisfy $\langle\psi_j|\psi_k\rangle = \delta_{kj}$. Here, $r \leq d_A$ is the rank of $\hat{\rho}_A$. The trace condition implies that $\sum_j \lambda_j = 1$.

It is clear from the properties of a quantum state that the state space or collection of quantum states of a system form a convex set. We denote the state space of system A by $\Lambda(\mathcal{H}_A)$. The pure states are the extreme points of this convex set and mixed states are nonextremal.

A useful way to capture mixedness is through a quantity known as *purity*. Purity is defined as $\mathrm{Tr}(\hat{\rho}_A^2)$. We see that for pure states

$$\mathrm{Tr}\,\hat{\rho}_A^2 = \mathrm{Tr}\,\hat{\rho}_A = 1, \tag{1.4}$$

while for mixed states

$$\mathrm{Tr}\,\hat{\rho}_A^2 < \mathrm{Tr}\,\hat{\rho}_A \text{ and so } \mathrm{Tr}\,\hat{\rho}_A^2 < 1. \tag{1.5}$$

The least value of purity is assumed by the maximally mixed state, and evaluates to $1/d_A$. Let us further denote by $\mathcal{B}(\mathcal{H}_A)$ the complex linear space of bounded linear operators on $\mathcal{H}_A$. The state space $\Lambda(\mathcal{H}_A) \subset \mathcal{B}(\mathcal{H}_A)$.

For illustration we consider the simplest quantum system, a qubit [1–3]. For a qubit system $d_A = 2$ and the states of the system can be represented as

$$\hat{\rho}_A = \frac{1}{2}(\mathbb{1} + \mathbf{a}.\boldsymbol{\sigma}), \tag{1.6}$$

where $\mathbf{a} \in \mathcal{R}^3$, $|\mathbf{a}| \leq 1$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The state space here is a (solid) sphere of unit radius and is known as the Poincaré or Bloch sphere. In this case, and only in this case, are all boundary points extremals. One can write its spectral decomposition explicitly as

$$\hat{\rho}_A = \frac{1 + |\mathbf{a}|}{2}\left[\frac{1}{2}(\mathbb{1} + \hat{\mathbf{a}}.\boldsymbol{\sigma})\right] + \frac{1 - |\mathbf{a}|}{2}\left[\frac{1}{2}(\mathbb{1} - \hat{\mathbf{a}}.\boldsymbol{\sigma})\right], \tag{1.7}$$

where the eigenvalues are

$$\frac{1}{2}(1 \pm |\mathbf{a}|) \tag{1.8}$$

corresponding to eigenvectors

$$\frac{1}{2}(\mathbb{1} \pm \hat{\mathbf{a}}.\boldsymbol{\sigma}), \quad \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}. \tag{1.9}$$

It is easily seen that the eigenvectors (1.9) are rank-one orthogonal projectors that lie on the boundary of the Bloch sphere and for a general mixed state $|\mathbf{a}| < 1$. They are unique except for $\mathbf{a} = 0$ in which case all states are eigenstates.

Returning to the general case, a quantity of interest to us is the *von-Neumann* entropy of a state $\hat{\rho}_A$ and is given by

$$S(\hat{\rho}_A) = -\mathrm{Tr}[\hat{\rho}_A \log_2(\hat{\rho}_A)]. \tag{1.10}$$

From the spectral decomposition of $\hat{\rho}_A$ (1.3), it is easily seen that $S(\hat{\rho}_A)$ is just the *Shannon entropy* of the probability distribution comprising the eigenvalues of $\hat{\rho}_A$, i.e., $S(\hat{\rho}_A) = -\sum_j \lambda_j \log_2(\lambda_j)$.

Having introduced the notion of states, we next consider the important concept of observables of a quantum system.

Observables : Observables are physical variables of the system that are measurable. Observables $\hat{O}$ of a quantum system A are defined as hermitian operators acting on $\mathcal{H}_A$. Let the spectral resolution of a nondegenerate observable $\hat{O}$ be $\hat{O} = \sum_j \lambda_j P_j$, where λ_j's are the eigenvalues corresponding to the eigenvectors P_j. When $\hat{O}$ is measured in the state $\hat{\rho}_A$, the j^{th} outcome corresponding to measurement operator (one-dimensional projection) P_j occurs with probability $p_j = \text{Tr}(\hat{\rho}_A P_j)$.

One obtains a more general measurement scheme called POVM (positive operator-valued measure) when the projective measurement elements P_j are replaced by positive operators Π_j with $\sum_j \Pi_j = \mathbb{1}$, and the probabilities of outcomes are obtained in a similar manner : $p_j = \text{Tr}(\hat{\rho}_A \Pi_j)$.

1.1.1 Composite systems

We next consider the case of composite systems. A composite system is one that has two (bipartite) or more (multipartite) subsystems . For our purpose, it suffices to concern ourselves with bipartite systems.

The Hilbert space of a composite system is given by the tensor product of those of the individual subsystems. In other words $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The state space of the composite system is denoted by $\Lambda(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Let $\{|e_i\rangle\}_1^n$ and $\{|f_j\rangle\}_1^m$ be respective ONB in Hilbert spaces of subsystems A and B. Then the collection of mn vectors $\{|e_i\rangle \otimes |f_j\rangle\}$ forms a basis in $\mathcal{H}_{AB}$. A product operator $A \otimes B$, with the matrix elements of A given as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \tag{1.11}$$

can be written as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}. \tag{1.12}$$

A generic density operator of the composite system can be written as

$$\rho_{AB} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}. \tag{1.13}$$

Here each matrix block $((A_{ij}))$ is a $m \times m$ matrix. The density matrix of subsystem A is obtained through performing *partial trace* on subsystem B, i.e.,

$$\begin{aligned} \hat{\rho}_A &= \mathrm{Tr}_B\,(\hat{\rho}_{AB}) \\ &= \sum_j \langle f_j^B|\hat{\rho}_{AB}|f_j^B\rangle, \end{aligned} \tag{1.14}$$

where $|f_j^B\rangle,\ j = 1, 2, \cdots, m$ is any ONB in $\mathcal{H}_B$. In terms of the matrix entries in Eq. (1.13)

we have

$$\rho_A = \begin{bmatrix} \mathrm{Tr}\, A_{11} & \mathrm{Tr}\, A_{12} & \cdots & \mathrm{Tr}\, A_{1n} \\ \mathrm{Tr}\, A_{21} & \mathrm{Tr}\, A_{22} & \cdots & \mathrm{Tr}\, A_{2n} \\ \vdots & & & \\ \mathrm{Tr}\, A_{n1} & \mathrm{Tr}\, A_{n2} & \cdots & \mathrm{Tr}\, A_{nn} \end{bmatrix}. \tag{1.15}$$

If instead the partial trace was performed over subsystem A we have

$$\rho_B = \sum_i A_{ii}. \tag{1.16}$$

Another useful operation is the *partial transpose* operation. Performing the partial transpose on subsystem B on matrix ρ_{AB} in Eq. (1.13), we have

$$\rho_{AB}^{T_B} = \begin{bmatrix} A_{11}^T & A_{12}^T & \cdots & A_{1n}^T \\ A_{21}^T & A_{22}^T & \cdots & A_{2n}^T \\ \vdots & & & \\ A_{n1}^T & A_{n2}^T & \cdots & A_{nn}^T \end{bmatrix}. \tag{1.17}$$

We see that the transpose operation was performed on each of the sub-blocks of the composite state. If the transpose operation was performed on subsystem A we have by Eq. (1.13)

$$\rho_{AB}^{T_A} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}. \tag{1.18}$$

Having introduced the notion of composite systems, we briefly discuss the connection

between POVM's, projective measurements and composite systems.

POVM: We have seen earlier that a POVM is a measurement scheme where the measurement elements $\Pi = \{\Pi_j\}$ are positive operators rather than projections. We now give a simple example in which a POVM results from a projective measurement on a larger system.

Consider a state $\hat{\rho}_{AB}$ of composite system of the form

$$\hat{\rho}_{AB} = \hat{\rho}_A \otimes |\psi_B\rangle\langle\psi_B|. \tag{1.19}$$

Let $P = \{P^i_{AB}\}$ be a collection of (one-dimensional) projection operators on the composite system AB which is complete: $\sum_i P^i_{AB} = \mathbb{1}$. The probability of the result of the i^{th} measurement is given by

$$\begin{aligned} p_i &= \mathrm{Tr}(\hat{\rho}_{AB}\, P^i_{AB}) \\ &= \mathrm{Tr}(\hat{\rho}_A \otimes |\psi_B\rangle\langle\psi_B|\, P^i_{AB}) \\ &= \mathrm{Tr}_A(\hat{\rho}_A \langle\psi_B|\, P^i_{AB}\, |\psi_B\rangle). \end{aligned} \tag{1.20}$$

If we define $\Pi^A_i = \langle\psi_B|\, P^i_{AB}\, |\psi_B\rangle$, we can write the above equation in a more suggestive form

$$p_i = \mathrm{Tr}_A\,(\Pi^A_i\, \hat{\rho}_A). \tag{1.21}$$

The operators $\{\Pi^A_i\}$ are all positive and sum to identity on $\mathcal{H}_A$. Hence the set $\Pi = \{\Pi^A_i\}$ constitutes a POVM. We thus see how a POVM results from the projective measurement on a larger system.

The following theorem guarantees that there always exists a physical mechanism by which one can realise any given POVM [4].

Theorem 1 (Neumark) *: One can extend the Hilbert space $\mathcal{H}$ on which the POVM elements $\{\Pi_j\}$ act, in such a way that there exists in the extended space $\mathcal{K}$, a set of orthogonal projectors $\{P_j\}$ with $\sum_j P_j = \mathbb{1}_{\mathcal{K}}$, and such that Π_j is the result of projecting P_j from $\mathcal{K}$ to $\mathcal{H}$.*

Having collected some basic ideas relating to bipartite states, we next consider an important aspect of these bipartite states that we are interested in, which is correlation between the subsystems.

1.2 Correlations

Correlations are intrinsic (nonlocal) properties of composite systems. Of the various quantifiers of correlations, we first consider entanglement. A bipartite pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is not entangled if and only if it is of the product form

$$|\psi\rangle = |u\rangle \otimes |v\rangle. \tag{1.22}$$

A useful representation of pure bipartite states is the *Schmidt representation* [1]. The Schmidt form makes use of the *singular value decomposition* theorem [5]:

Theorem 2 *An arbitrary complex $m \times n$ matrix A of rank k can be written in the form $A = VDW^\dagger$, where $V_{m\times m}$ and $W_{n\times n}$ are unitary, and D diagonal (with k entries which are positive and the rest zero). The non-zero diagonal entries of D are the square roots of the eigenvalues of $AA^\dagger \sim A^\dagger A$.*

Let us write down a general pure state of system AB as

$$|\psi\rangle = \sum_{ij} c_{ij} |i\rangle \otimes |j\rangle, \tag{1.23}$$

where c_{ij} is a general complex (coefficient) matrix, where $\{|i\rangle\}$ and $\{|j\rangle\}$ are the computational basis of systems A and B respectively. By applying suitable local unitaries, one can diagonalize any coefficient matrix $((c_{ij}))$ to bring it to a diagonal form as guaranteed by the singular value decomposition. We then have

$$|\psi\rangle = \sum_{j=1}^{r} \lambda_j |e_j^A\rangle \otimes |f_j^B\rangle, \tag{1.24}$$

where $\{e_j\}$ and $\{f_j\}$ are orthonormal in $\mathcal{H}_A$, $\mathcal{H}_B$ respectively, and the coefficients $\{\lambda_j\}$ are positive and $\sum_j \lambda_j^2 = 1$, as follows from the normalization condition. The number of terms in the above decomposition is called the *Schmidt rank* and can utmost take the value $r = \min(d_A, d_B)$. Further, if the Schmidt rank r is one, then the pure state is a product state and therefore separable. If $r > 1$, then the state $|\psi\rangle$ is entangled.

A closely related and useful concept is *purification*. Purification is an association of a generic mixed state of a system A with a pure entangled state of a suitable composite system AR. To this end, let $\hat{\rho}_A = \sum_{j=1}^{r} \lambda_j |\psi_j\rangle\langle\psi_j|$ be the spectral decomposition of a mixed state $\hat{\rho}_A$. Let us append this system A with a system R with Hilbert space $\mathcal{H}_R$ of dimension equal to the rank r of $\hat{\rho}_A$. Let $\{|e\rangle_R^j\}_1^r$ be an ONB for system R. Then starting from a pure state written as

$$|\psi\rangle_{AR} = \sum_j \sqrt{\lambda_j} |\psi_j\rangle_R \otimes |e^j\rangle_R, \tag{1.25}$$

one obtains $\hat{\rho}_A = \mathrm{Tr}_R[|\psi\rangle\langle\psi|_{AR}]$, and $|\psi\rangle_{AR}$ is called a purification of $\hat{\rho}_A$. In other words, it is seen that one recovers $\hat{\rho}_A$ by taking partial trace on subsystem R of the pure state $|\psi\rangle_{AR}$. The purification $|\psi\rangle_{AR}$ has a local unitary freedom in system R in the following sense. Any other choice of an ONB of system R also returns the same state $\hat{\rho}_A$ under partial trace [1].

Having considered separable and entangled pure states, a natural question would be to *quantify* the amount of entanglement in a pure state. A simple measure for the quantifi-

cation of entanglement of bipartite pure states [6] is given by the entropy of the reduced state, i.e.

$$E(|\psi\rangle_{AB}) = S(\hat{\rho}_A), \tag{1.26}$$

where $\hat{\rho}_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ is obtained by partial trace over subsystem B. We see that for product pure states, the reduced state is a pure state and hence has zero entanglement. We note in passing that the entanglement measure in Eq. (1.26) is symmetric with respect to the two subsystems in the following sense. One could have taken partial trace of system A instead of B in Eq. (1.26). Since $S(\hat{\rho}_A) = S(\hat{\rho}_B)$, as can be easily seen from the Schmidt decomposition, the amount of entanglement is the same in either procedure.

We now wish to consider mixed states of the bipartite system AB. A *separable mixed* state is one which can be written as [7]

$$\hat{\rho}_{AB} = \sum_j p_j \hat{\rho}_A^j \otimes \hat{\rho}_B^j, \tag{1.27}$$

i.e., the bipartite density matrix can be written as a convex combination of product density matrices. If such a decomposition does not exist, then the state is said to be entangled.

1.2.1 Entanglement detection

The problem of studying entanglement for mixed states [8] turns out to be a non-trivial one. There have been many approaches to detect and quantify the entanglement [9, 10]. We now briefly review a few of the measures used often in the literature, and a few of these provide an operational method to detect entanglement [11].

Partial transpose test : Given a bipartite state $\hat{\rho}_{AB}$ whose matrix elements are written as

$$\hat{\rho}_{AB} = \sum \rho_{jk;mn} |j\rangle\langle m| \otimes |k\rangle\langle n|, \tag{1.28}$$

the partial transpose with respect to subsystem B leads to

$$\hat{\rho}_{AB}^{T_B} = \sum \rho_{jk;mn} |j\rangle\langle m| \otimes |n\rangle\langle k|, \tag{1.29}$$

and that on subsystem A gives

$$\hat{\rho}_{AB}^{T_A} = \sum \rho_{jk;mn} |m\rangle\langle j| \otimes |k\rangle\langle n|. \tag{1.30}$$

For 2×2 and 2×3 systems the lack of positivity of the state resulting from partial transpose provides a necessary and sufficient test to detect entanglement [12, 13]. If, say, $\hat{\rho}_{AB}^{T_B}$ is positive, then the state is separable; else it is entangled. The test however fails for higher dimensional systems as positivity under partial transpose (PPT) is not a sufficient condition for separability. In higher dimensions, if a state fails the partial transpose test, then it is entangled. But PPT is not a sufficient condition for separability, and there exist entangled states which are PPT [14, 15].

Positive maps : A positive map Γ is a linear map on the space of bounded linear operators on a given Hilbert space which takes positive operators to positive operators. i.e.,

$$\Gamma : \mathcal{B}(\mathcal{H}_S) \rightarrow \mathcal{B}(\mathcal{H}_S), \;\; \Gamma(A) = A^{'} \geq 0 \; \forall \; A \geq 0. \tag{1.31}$$

It is immediately seen that the one-sided action of a positive map on a separable state takes it to a density operator. The necessary and sufficient condition for a state $\hat{\rho}_{AB}$ to be separable is that $[\mathbb{1} \otimes \Gamma](\hat{\rho}_{AB}) \geq 0$ for all positive maps Γ [13]. It turns out that there is a very important subset of positive maps known as completely positive maps. The notion

of completely positive maps will be discussed in a later Section. For detecting entanglement, it is positive maps that are not completely positive that are useful. The transpose map (used in the partial transpose test) considered above is an example of a positive map that is not completely positive.

Entanglement witness : A self-adjoint bipartite operator W which has at least one negative eigenvalue and has nonnegative expectation on product states is called an entanglement witness [13, 16]. A state $\hat{\rho}_{AB}$ belongs to the set of separable states if it has a nonnegative mean value for all W, i.e.

$$\mathrm{Tr}\,(W\hat{\rho}_{AB}) \geq 0 \;\; \forall \;\; W, \tag{1.32}$$

where W is an entanglement witness. For every entangled state $\hat{\rho}_{AB}$, there exist an entanglement witness W such that $\mathrm{Tr}\,(W\hat{\rho}_{AB})) < 0$. We then say that the entanglement of $\hat{\rho}_{AB}$ is witnessed by W.

Reduction criterion : Consider the following map that is known as the reduction map :

$$\Gamma(\hat{\rho}_A) = \frac{(\mathbb{1}\mathrm{Tr}(\hat{\rho}_A) - \hat{\rho}_A)}{d_A - 1}. \tag{1.33}$$

The reduction separability criterion [17, 18] states that a necessary condition for a state to be separable is that it satisfies

$$\begin{aligned} & [\mathbb{1} \otimes \Gamma_B][\hat{\rho}_{AB}] \geq 0 \\ \Longrightarrow \quad & \hat{\rho}_A \otimes \mathbb{1} - \hat{\rho}_{AB} \geq 0. \end{aligned} \tag{1.34}$$

The reduction criterion is weaker than the partial transpose test [17].

Range criterion: The range criterion [19] states that if a state $\hat{\rho}_{AB}$ is separable then there exists a set of product vectors $\{|\psi_i^A\rangle \otimes |\phi_i^B\rangle\}$ such that it spans the range of $\hat{\rho}_{AB}$ and the set $\{|\psi_i^A\rangle \otimes |(\phi_i^B)^*\rangle\}$ spans that of $\hat{\rho}_{AB}^{T_B}$, where the complex conjugation is done in the same basis in which the partial transpose operation is performed.

Unextendable product basis: An unextendable product basis (UPB) [20] is a set S_u of orthonormal product vectors such that there is no product vector that is orthogonal to all of them. Let us denote by $S_u^\perp$ the subspace that is orthogonal to the subspace spanned by vectors in S_u. Therefore, any vector in $S_u^\perp$ is entangled. By the range criterion, we have that any mixed state with support on this orthogonal space $S_u^\perp$ is entangled. Using this concept of UPB, one can construct entangled states that are PPT.

1-Entropic type: There is an entropic way to quantify the statement that 'an entangled state gives more information about the total system than about the subsystems'. Indeed it was shown that the entropy of a subsystem can be greater that the entropy of the total system only when the state is entangled [21, 22]. In other words, for a separable state

$$S(\hat{\rho}_A) \leq S(\hat{\rho}_{AB}), \text{ and } S(\hat{\rho}_B) \leq S(\hat{\rho}_{AB}). \tag{1.35}$$

Majorization criterion: A vector $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ [23], denoted by $\mathbf{x} \prec \mathbf{y}$, both of dimension d, if

$$\begin{aligned} &\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j, \quad \text{for} \quad k = 1, \cdots, d-1; \\ &\sum_{j=1}^{d} x_j = \sum_{j=1}^{d} y_j, \end{aligned} \tag{1.36}$$

it being assumed that the components are arranged in decreasing order. The majorization

criterion states that if a state $\hat{\rho}_{AB}$ is separable [24], then

$$\boldsymbol{\lambda}_{AB} \prec \boldsymbol{\lambda}_A \text{ and } \boldsymbol{\lambda}_{AB} \prec \boldsymbol{\lambda}_B, \tag{1.37}$$

where $\boldsymbol{\lambda}_{(\cdot)}$ is the vector of eigenvalues of $\hat{\rho}_{(\cdot)}$ written in decreasing order. Therefore, we say that for a separable state the eigenvalues of the bipartite state is majorized by the ones of either reduced state.

Realignment criterion : The realignment map R is defined as

$$[R(\rho_{AB})]_{ij;kl} = [\rho_{AB}]_{ik;jl}. \tag{1.38}$$

The realignment criterion [25] states that if a state $\hat{\rho}_{AB}$ is separable, then $||R(\hat{\rho}_{AB})||_1 \leq 1$.

Bell-type : A Bell-type [26] inequality is one which tries to capture entanglement through probabilities of outcomes of suitably chosen observables. An example of one such inequality in the two-qubit setting is the CHSH inequality [27] :

$$\begin{aligned} &|\text{Tr}(\hat{O}_{CHSH}\,\hat{\rho}_{AB})| \leq 2, \\ &\hat{O}_{CHSH} = A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2). \end{aligned} \tag{1.39}$$

Here, A_1 and A_2 are respectively $\mathbf{a_1}.\boldsymbol{\sigma}$ and $\mathbf{a_2}.\boldsymbol{\sigma}$, $\mathbf{a}_1$, $\mathbf{a}_2 \in \mathcal{R}^3$. One similarly constructs operators B_1, B_2 with respect to two directions $\mathbf{b_1}$, $\mathbf{b_2}$. Any state $\hat{\rho}_{AB}$ that violates this inequality is an entangled state.

1.2.2 Entanglement quantification

In the previous Section we summarized a few ways to detect entanglement. We now briefly describe some measures of entanglement. We first begin by listing reasonable properties that any entanglement measure E would be expected to satisfy [28–31].

- $E(\hat{\rho}_{AB}) = 0$ for $\hat{\rho}_{AB}$ separable.
- E is invariant under local unitary transformations:

$$E(U_A \otimes U_B \hat{\rho}_{AB} U_A^\dagger \otimes U_B^\dagger) = E(\hat{\rho}_{AB}). \tag{1.40}$$

- E is non-increasing under local operations and classical communications (LOCC):

$$E(\Gamma(\hat{\rho}_{AB})) \leq E(\hat{\rho}_{AB}), \text{ for } \Gamma \in \text{LOCC}. \tag{1.41}$$

- E returns the value of von-Neumann entropy of the reduced states when evaluated on pure bipartite states, i.e.,

$$E(|\psi_{AB}\rangle) = S(\text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)). \tag{1.42}$$

- E is subadditive over a general product of bipartite entangled states, i.e.,

$$E(\hat{\rho}_{AB} \otimes \hat{\rho}_{A'B'}) \leq E(\hat{\rho}_{AB}) + E(\hat{\rho}_{A'B'}). \tag{1.43}$$

- *Normalization*: $E(\hat{\rho}_{\text{max}}) = \text{Log}_2 d$ for a maximally entangled state $\hat{\rho}_{\text{max}}$.

There are in addition some technical requirements that are also considered:

- E is a convex function on the state space.

- E is continuous on the state space [32].

It is known that the measures to be considered below do not necessarily satisfy all the above mentioned properties, but are nevertheless used depending on the context.

Entanglement of formation: The expression for entanglement of formation is given by [8]

$$E_F(\hat{\rho}_{AB}) = \min_{\{p_j, |\psi^j\rangle\}} \sum_j p_j S(\mathrm{Tr}_B(|\psi^j\rangle_{AB}\langle\psi^j|)), \tag{1.44}$$

where $\hat{\rho}_{AB} = \sum_j p_j |\psi^j\rangle\langle\psi^j|$ is a pure state ensemble, and the optimization is over all possible convex pure state decompositions of the original bipartite mixed state $\hat{\rho}_{AB}$.

This optimization has been solved analytically for very few examples. These include two-qubit states [33, 34], symmetric Gaussian states [35], general Gaussian states [36], Gaussian entanglement of formation [37], werner states and O-O states [38], isotropic states [39], some highly symmetric states [40], flower states [41], examples in 16×16 systems [42]; special classes of states using the Koashi-Winter relation in two-qubit states [43] and tripartite Gaussian states [44]; special examples using the Matsumoto-Shimono-Winter relation [45].

Entanglement cost: The entanglement cost [46–48] is defined as the asymptotic or regularized version of entanglement of formation [49]. In other words,

$$E_C(\hat{\rho}_{AB}) = \lim_{n\to\infty} \frac{E_F(\hat{\rho}_{AB}^{\otimes n})}{n}. \tag{1.45}$$

The entanglement cost has been evaluated for 3×3 anti-symmetric states [50], lower bounds for d-dimensional anti-symmetric states were obtained in [51], certain antisymmetric states with a non-identical bipartite separation [52], examples of highly symmetric

states [40], and flower states [41].

Distillable entanglement: The distillable entanglement [53–56] is a measure of how much entanglement can be extracted from an entangled state $\hat{\rho}_{AB}$ [57,58] in an asymptotic setting, i.e.,

$$E_D(\hat{\rho}_{AB}) = \sup\left\{r : \lim_{n\to\infty}\left(\inf_{\Gamma}\|\Gamma(\hat{\rho}^{\otimes n}) - \Phi^+_{2^{rn}}\|_1\right) = 0\right\}, \tag{1.46}$$

where Γ is an LOCC operation, and $\Phi^+_{2^x}$ stands for $(\Phi^+)^{\otimes x}_2$, Φ^+_2 being a Bell state. Here, $\|A\|_1$ stands for $\sum_j \lambda_j$, where λ_j's are the singular values of A. Further, it is known that $E_D(\hat{\rho}_{AB}) \leq E_F(\hat{\rho}_{AB}) \leq E_C(\hat{\rho}_{AB})$, i.e., distillable entanglement is a lower bound for entanglement of formation [10].

Relative entropy of entanglement: The relative entropy [59–61] is just the 'distance' of a given state $\hat{\rho}_{AB}$ to the closest separable state, i.e.

$$E_R(\hat{\rho}_{AB}) = \inf_{\sigma\in\mathcal{S}} S(\hat{\rho}_{AB}\|\hat{\sigma}), \tag{1.47}$$

where $\hat{\sigma}$ is an element of the set of separable states $\mathcal{S}$ and

$$S(\hat{a}\|\hat{b}) = \mathrm{Tr}[\hat{a}\,(\mathrm{Log}\hat{a} - \mathrm{Log}\hat{b})], \tag{1.48}$$

is known as the relative entropy between states $\hat{a}$, $\hat{b}$.

Squashed entanglement: The expression for squashed entanglement of $\hat{\rho}_{AB}$ is given by [62]

$$E_{\mathrm{sq}}(\hat{\rho}_{AB}) = \inf_{\hat{\rho}_{ABE}} \frac{1}{2} I(A:B|E), \tag{1.49}$$

where $I(A:B|E) = S_{AE} + S_{BE} - S_E - S_{ABE}$, S_X denotes the entropy of the state of system X and the infimum is taken over all density matrices $\hat{\rho}_{ABE}$ such that $\hat{\rho}_{AB} = \mathrm{Tr}_E(\hat{\rho}_{ABE})$. E_{sq} enjoys many interesting properties like additivity over tensor products and superadditivity in general. E_{sq} is a lower bound of entanglement of formation and an upper bound on distillable entanglement [62].

Logarithmic negativity: Logarithmic negativity is a straight-forward computable measure of entanglement [63] often used in the literature. It is defined as the logarithm of the sum of moduli of the eigenvalues of the partial transpose of a given bipartite state. The expression for logarithmic negativity is given by

$$E_N(\hat{\rho}_{AB}) = \mathrm{Log}_2\left[\|\hat{\rho}_{AB}^{\mathrm{Tr}_B}\|_1\right]. \tag{1.50}$$

1.2.3 Quantum discord, classical correlation and mutual information

We now move on to a different set of correlations that are motivated more directly from a measurement perspective and capture a different sort of classical-quantum boundary as opposed to the separable-entangled boundary. Among these measurement-based correlations, the ones of primary interest to us are three closely related quantities namely classical correlation, quantum discord and mutual information.

We may motivate the definition of quantum discord by first looking at the classical setting [64]. Given a probability distribution $p(x,y)$ in two variables, the mutual information $I(x,y)$ is defined as

$$I(x,y) = H(x) - H(x|y), \tag{1.51}$$

where $H(\cdot)$ stands for the Shannon entropy

$$H(x) = -\sum_x p(x)\,\mathrm{Log}[p(x)] \tag{1.52}$$

and $H(x|y)$ is the conditional entropy. Using Bayes rule we are lead to an equivalent expression for mutual information :

$$I(x,y) = H(x) + H(y) - H(x,y). \tag{1.53}$$

This second expression in (1.53) for mutual information naturally generalises to the quantum setting when the bipartite probability distribution is replaced by a bipartite state $\hat{\rho}_{AB}$ and the Shannon entropy $H(\cdot)$ by the von Neumann entropy $S(\cdot)$ of quantum states, and we have

$$I(\hat{\rho}_{AB}) = S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}_{AB}). \tag{1.54}$$

But the first expression (1.51) for classical mutual information does *not* possess a straightforward generalization to the quantum case. In the quantum case, the conditional entropy is defined with respect to a measurement, where the measurement is performed on one of the subsystems, say subsystem B. Let us consider a POVM $\Pi^B = \{\Pi_j^B\}$ where $\Pi_j^B \geq 0$ and $\sum_j \Pi_j^B = \mathbb{1}$. Then the (average) conditional entropy post measurement is given by

$$S^A = \sum_j p_j\, S(\hat{\rho}_j^A), \tag{1.55}$$

where the probabilities and states post measurement are given by

$$\begin{aligned} p_j &= \mathrm{Tr}_{AB}\,(\Pi_j^B\,\hat{\rho}_{AB}), \\ \hat{\rho}_j^A &= p_j^{-1}\,\mathrm{Tr}_B(\Pi_j^B\,\hat{\rho}_{AB}). \end{aligned} \tag{1.56}$$

Let us denote by $S^A_{\min}$ the minimum of S^A over all measurements or POVM's, i.e.,

$$S^A_{\min} = \min_{\Pi} \sum_j p_j S(\hat{\rho}^A_j). \tag{1.57}$$

The difference between these two classically equivalent expressions (optimized over all measurements) is called quantum discord $\mathcal{D}(\hat{\rho}_{AB})$ and is given by the expression :

$$\begin{aligned} \mathcal{D}(\hat{\rho}_{AB}) &= I(\hat{\rho}_{AB}) - \left[S(\hat{\rho}_A) - S^A_{\min}\right], \\ &= S(\hat{\rho}_B) - S(\hat{\rho}_{AB}) + S^A_{\min}. \end{aligned} \tag{1.58}$$

The quantity in the square brackets above is defined as the classical correlation and is denoted by

$$C(\hat{\rho}_{AB}) = S(\hat{\rho}_A) - S^A_{\min}. \tag{1.59}$$

It is useful to keep in mind an alternate expression for mutual information which is given by

$$I(\hat{\rho}_{AB}) = S(\hat{\rho}_{AB} || \hat{\rho}_A \otimes \hat{\rho}_B), \tag{1.60}$$

where $S(\cdot||\cdot)$ is the relative entropy. We see that the mutual information is defined as the relative entropy between the given bipartite state and the (tensor) product of its reductions. Thus, the mutual information which is supposed to capture the total correlation of a bipartite state is broken down into quantum discord $\mathcal{D}(\hat{\rho}_{AB})$, that captures the quantum correlations, and classical correlation $C(\hat{\rho}_{AB})$:

$$I(\hat{\rho}_{AB}) = \mathcal{D}(\hat{\rho}_{AB}) + C(\hat{\rho}_{AB}) \tag{1.61}$$

There are many properties that are satisfied by quantum discord and classical correlations

and we list some of them below [65]:

- Quantum discord and classical correlation are dependent on the subsystem on which the measurement is performed. Hence, they are not symmetric under exchange of the subsystems in general.
- Both classical correlation and quantum discord are non-negative quantities.
- $\mathcal{D}(\hat{\rho}_{AB})$, $C(\hat{\rho}_{AB})$, and $I(\hat{\rho}_{AB})$ are invariant under local unitary transformations. This turns out to be useful for the computation of these quantities.
- $C(\hat{\rho}_{AB}) = 0$ only for a product state.
- $C(\hat{\rho}_{AB}) = E_F(\hat{\rho}_{AB})$ for any pure bipartite state $\hat{\rho}_{AB} = |\psi\rangle\langle\psi|$.
- $D(\hat{\rho}_{AB}) = E_F(\hat{\rho}_{AB})$ for any pure bipartite state $\hat{\rho}_{AB} = |\psi\rangle\langle\psi|$. In other words, both classical correlation and quantum discord reduce to the entanglement on pure bipartite states.
- A state has vanishing quantum discord when

$$\mathcal{D}(\hat{\rho}_{AB}) = 0 \quad \text{or} \quad I(\hat{\rho}_{AB}) = C(\hat{\rho}_{AB}). \tag{1.62}$$

  Further, any one-way zero discord state can be written as

$$\hat{\rho}_{AB} = \sum_i p_i \, |i\rangle\langle i| \otimes \hat{\rho}_b^i, \tag{1.63}$$

  where $\{p_j\}$'s form a probability distribution and $\{|i\rangle\}$ is an orthonormal basis in the subsystem where the measurement was performed. In other words, a one-way zero discord state is invariant under some von-Neumann measurement on the subsystem. Such states are also known as classical-quantum states.

As a simple example, we compute all the three correlations for a two-qubit Bell-state

$$|\psi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}. \tag{1.64}$$

For this pure state we have $I(|\psi\rangle) = 2\text{Log}_2 2 = 2$ bits. While $C(|\psi\rangle) = D(|\psi\rangle) = E(|\psi\rangle) = 1$ bit, illustrating $I(|\psi\rangle) = C(|\psi\rangle) + D(|\psi\rangle)$.

1.3 Positive maps and completely positive maps

Having briefly considered correlations, we now turn to another aspect of central importance to this thesis which is channels. We wish to know what are all the allowed physical evolutions of a given quantum system. If a system is isolated, then its dynamics is governed by the unitary Schrödinger evolution. A unitary evolution U effects the following transformation

$$\hat{\rho}_A \rightarrow \hat{\rho}_A^{'} = U\,\hat{\rho}_A\,U^{\dagger},\ \hat{\rho}_A \text{ and } \hat{\rho}_A^{'} \in \Lambda(\mathcal{H}_A). \tag{1.65}$$

But if the system is in interaction with its environment, then the evolutions of the system of interest resulting from unitary evolutions of the composite are more general, but nevertheless described by linear maps acting on the state space $\Lambda(\mathcal{H}_A)$ directly rather than through its action on $\mathcal{H}_A$.

Let Φ be a linear map that acts on states of the system, i.e., $\Phi : \Lambda(\mathcal{H}_A) \rightarrow \Lambda(\mathcal{H}_A)$. Writing this transformation in terms of the matrix elements, we have (dropping the system label A):

$$\rho \rightarrow \rho^{'} \ : \ \ \Phi_{ij;k\ell}\,\rho_{k\ell} = \rho_{ij}^{'}. \tag{1.66}$$

An obvious necessary requirement for Φ to be a valid evolution is that it takes states to

states. So we require that the action of Φ preserve hermiticity, trace and positivity of the states. For the hermiticity of the output states we have that

$$\rho^{'}_{ij} = (\rho^{'}_{ji})^{*},$$
$$\text{i.e.,}\ \ \Phi_{ij;\ell k} = \Phi^{*}_{ji;k\ell}. \tag{1.67}$$

The trace preserving condition manifests as

$$\sum_{i} \Phi_{ii;k\ell} = \delta_{k\ell}. \tag{1.68}$$

Finally, the positivity condition can be written as

$$v_i^{*}\, \rho^{'}_{ij}\, v_j \geq 0 \ \ \forall\, |v\rangle. \tag{1.69}$$

This property can be checked by assuming that $\rho = |u\rangle\langle u|$, a pure state as input. We have

$$v_i^{*}\, \Phi_{ij;k\ell}\, \rho_{k\ell}\, v_j \geq 0, \ \text{for every}\ |u\rangle,$$
$$\text{i.e.,}\ \ v_i^{*}\, \Phi_{ij;k\ell}\, u_k u_\ell^{*}\, v_j \geq 0, \ \text{for every}\ |u\rangle, |v\rangle. \tag{1.70}$$

It is instructive to write the matrix elements of Φ in terms of a new matrix we denote $\widetilde{\Phi}$,

$$\widetilde{\Phi}_{ij;k\ell} = \Phi_{ik;j\ell}. \tag{1.71}$$

The hermiticity preserving condition in Eq. (1.67) now reads

$$\widetilde{\Phi}_{i\ell;jk} = \widetilde{\Phi}^{*}_{jk;i\ell} \tag{1.72}$$

In other words, for the map Φ to be hermiticity preserving, we have that $\widetilde{\Phi}$ to be a hermi-

tian matrix :

$$\widetilde{\Phi} = \widetilde{\Phi}^{\dagger}. \tag{1.73}$$

The trace condition in Eq. (1.68) reads

$$\sum_i \widetilde{\Phi}_{ik;i\ell} = \delta_{k\ell}. \tag{1.74}$$

Finally, the positivity condition of Eq. (1.70) is then

$$\begin{aligned} & v_i^* \, \widetilde{\Phi}_{ik;j\ell} \, u_k u_\ell^* \, v_j \geq 0, \\ & \quad \Rightarrow \langle \psi | \widetilde{\Phi} | \psi \rangle \geq 0, \ \forall \ |\psi\rangle = |u\rangle \otimes |v\rangle. \end{aligned} \tag{1.75}$$

We note that the positivity condition states that $\widetilde{\Phi}$ is positive over all *product* vectors.

To summarize, we call a map that satisfies all the above conditions as a *positive map*, i.e.,

$$\begin{aligned} \Phi \text{ is positive } & \Leftrightarrow \Phi(\rho_S) = \rho_S^{'} \in \Lambda(\mathcal{H}_S), \\ & \text{i.e., } \Phi\left(\Lambda(\mathcal{H}_S)\right) \subset \Lambda(\mathcal{H}_S). \end{aligned} \tag{1.76}$$

It turns out that not all positive maps are physical evolutions. For positive maps to be physical evolutions, there is a further requirement to be met.

Let us consider a composite system which consists of the original system appended with an arbitrary ancilla or reservoir R. The Hilbert space of the composite system is $\mathcal{H}_S \otimes \mathcal{H}_R$, a tensor product of the individual subsystem Hilbert spaces. Let us denote the state space of this composite system by $\Lambda(\mathcal{H}_S \otimes \mathcal{H}_R)$.

To motivate the difference between positive and completely positive maps, we now give an example of a map that is positive but nevertheless nonpositive under local action, the

transpose map. The density operator for the Bell-state $|\psi^+\rangle$ in Eq. (1.64) is

$$|\psi^+\rangle\langle\psi^+| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{1.77}$$

If we now apply the transpose map locally on the B system, by Eq. (1.17), we get

$$(|\psi^+\rangle\langle\psi^+|)^{T_B} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.78}$$

We see that the above operator has negative eigenvalues. Therefore, the local action of the transpose map does not give a positive operator of the larger system, even though transpose map by itself is a positive map.

It is both reasonable and necessary to require that local action of Φ takes states of the joint system also to states for Φ to be a physical evolution. In other words

$$\begin{aligned} (\Phi \otimes \mathbb{1})[\rho_{SR}] &= \rho'_{SR} \in \Lambda(\mathcal{H}_S \otimes \mathcal{H}_R), \\ \text{i.e., } [\Phi \otimes \mathbb{1}]\,(\Lambda(\mathcal{H}_S \otimes \mathcal{H}_R)) &\subset \Lambda(\mathcal{H}_S \otimes \mathcal{H}_R). \end{aligned} \tag{1.79}$$

A positive map Φ that satisfies Eq. (1.79), is known as a *completely positive* (CP) trace-preserving (TP) map or a **quantum channel**.

An important class of positive maps from entanglement perspective is the so called *decomposable* maps [13]. A positive map Φ is said to be decomposable if

$$\Phi = \Phi_1 + \Phi_2 \circ T, \tag{1.80}$$

where Φ_1 and Φ_2 are both completely positive maps and T is the transpose map. One application of the decomposability notion is that positive maps that are decomposable are 'weaker' than the transpose map in the detection of entanglement. In other words, one would like to look for maps that are not decomposable to detect entangled states which are PPT [66].

We have seen above that a completely positive map is positive under local action on the system, appended with a system R of any dimension. It turns out that there is an operational criterion that captures this aspect.

Consider a composite system whose Hilbert space is given by $\mathcal{H}_S \otimes \mathcal{H}_S$, where $\mathcal{H}_S$ is the system Hilbert space. We denote the maximally entangled state of the composite system by :

$$|\psi\rangle_{\max} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle. \tag{1.81}$$

Let us denote $d\,|\psi\rangle\langle\psi|_{\max}$ by $\hat{\sigma}$, which is explicitly written down as

$$\hat{\sigma} = \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |i\rangle\langle j|. \tag{1.82}$$

We then have the following theorem regarding completely positive maps [67] :

Theorem 3 (Choi) *A positive map* Φ *is completely positive if and only if* $[\Phi \otimes \mathbb{1}](\hat{\sigma}) \geq 0$.

The matrix $D_\Phi = [\Phi \otimes \mathbb{1}](\hat{\sigma})$ is known as the dynamical matrix [68]. We now discuss the properties and representation of CP maps in some detail in the following Section.

1.4 Representations of CP maps

There are three well-known representations of completely positive maps. These are the unitary (Stinespring) dilation [69], the operator sum representation (OSR) [68, 70], and the Choi-Jamiolkowski isomohpishm (CJI) [67, 71].

1.4.1 Unitary representation

It is known that every CP map can be realised in the following way. Let us consider a composite system constructed from the system plus an environment R with Hilbert space $\mathcal{H}_R$. First, the systems states are elevated to product states on the larger Hilbert space (system + environment), with a fixed state of the environment :

$$\hat{\rho}_S \rightarrow \hat{\rho}_S \otimes |\psi\rangle_R\langle\psi|, \quad |\psi_R\rangle \text{ fixed.} \tag{1.83}$$

Then the product states are evolved by a joint unitary evolution U_{SR} :

$$\hat{\rho}_S \otimes |\psi\rangle_R\langle\psi| \rightarrow U_{SR}\,(\hat{\rho}_S \otimes |\psi\rangle_R\langle\psi|)\,U_{SR}^{\dagger}. \tag{1.84}$$

Finally, the environment degrees of freedom are traced out to give the evolved system states :

$$\Phi(\hat{\rho}_S) = \hat{\rho}_S^{'} = \mathrm{Tr}_R\left[U_{SR}\,(\hat{\rho}_S \otimes |\psi\rangle_R\langle\psi|)\,U_{SR}^{\dagger}\right]. \tag{1.85}$$

We see that the map $\Phi : \hat{\rho}_S \rightarrow \hat{\rho}_S^{'}$ is completely specified by the triplet $(\mathcal{H}_R,\, U_{SR},\, |\psi\rangle_R)$. A schematic diagram for the unitary representation is shown in Fig. 1.1. It is immediately clear that this representation is not unique as can be seen from the following example. Performing a unitary transformation on system R and appropriately changing the fixed

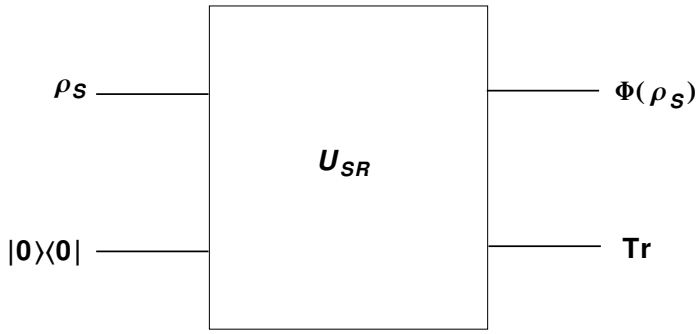

Figure 1.1: Showing the unitary realization of any quantum channel. The initial state is appended with a fixed environment state denoted $|0\rangle\langle 0|$ and the composite is evolved through a joint unitary U_{SR}. Then the environment degrees are ignored to obtain the evolved system state.

pure state of the environment, we obtain the same CP map Φ. In other words

$$U_{SR} \to U_{SR}(\mathbb{1}_S \otimes U_R), \quad |\psi_R\rangle \to U_R^\dagger |\psi_R\rangle, \tag{1.86}$$

will result in the same map. The map Φ is trace-preserving by construction. The unitary representation can also be written in the following form :

$$\Phi(\hat{\rho}_S) = \mathrm{Tr}_R\,(V_{SR}\,\hat{\rho}_S\,V_{SR}^\dagger), \tag{1.87}$$

where $V_{SR} = U_{SR}|\psi\rangle_R$ is an isometry from $\mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_R$. We recall that an isometry $V : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$ is a linear operator such that $V^\dagger V = \mathbb{1}_A$, which is satisfied by V_{SR} defined in Eq. (1.87).

1.4.2 Operator sum representation

An alternative and equivalent representation of a quantum channel is known as the operator sum representation (OSR). Every channel Φ can be expressed as

$$\Phi(\mathcal{H}_S) = \sum_k A_k \hat{\rho}_S A_k^\dagger, \tag{1.88}$$

where the operators A_k are called Kraus operators. The trace-preserving condition reads

$$\sum_k A_k^\dagger A_K = \mathbb{1}. \tag{1.89}$$

Let us now consider a new set of Kraus operators given by $\tilde{A}_k = V_{kj} A_j$. Let us impose the trace-preserving condition

$$\begin{aligned} &\sum_k \tilde{A}_k^\dagger \tilde{A}_K = \mathbb{1}, \\ \text{i.e.,}\quad &\sum_k \sum_{ij} A_i^\dagger V_{ki}^* V_{kj} A_j = \mathbb{1}. \end{aligned} \tag{1.90}$$

In other words, we require

$$\begin{aligned} &\sum_k V_{ki}^* V_{kj} = \delta_{ij}, \\ &\Rightarrow V^\dagger V = \mathbb{1}, \end{aligned} \tag{1.91}$$

i.e., V is required to be an isometry by the trace preserving condition in Eq. (1.89).

Let us denote by $\tilde{\Phi}$ the channel corresponding to the new set of Kraus operators $\{\tilde{A}_k\}$. The

operator sum representation is then given by

$$\begin{aligned}\tilde{\Phi}(\hat{\rho}_S) &= \sum_k \tilde{A}_k \hat{\rho}_S A_k^\dagger \\ &= \sum_k \sum_{ij} V_{kj} A_j \hat{\rho}_S A_i^\dagger V_{ki}^* \\ &= \sum_{ij} \sum_k V_{kj} V_{ki}^* A_j \hat{\rho}_S A_i^\dagger \\ &= \sum_{ij} \delta_{ij} A_j \hat{\rho}_S A_i^\dagger \\ &= \sum_j A_j \hat{\rho}_S A_j^\dagger. \end{aligned} \tag{1.92}$$

We see that $\tilde{\Phi} = \Phi$, i.e., there is a isometry freedom in the definition of the operator sum representation. In other words, if two sets of Kraus operators are related by an isometry, then the corresponding channels Φ, $\widetilde{\Phi}$ defined through OSR, will represent one and the same map.

1.4.3 Choi-Jamiolkowski representation

The third representation is the Choi-Jamiolkowski state corresponding to a given CP map Φ. Consider the composite system AB whose Hilbert space is given by $\mathcal{H}_S \otimes \mathcal{H}_S$. We will make use of the maximally entangled pure state given in Eq. (1.81). The Choi-Jamiolkowski state is obtained from the one-sided action of the CP map Φ. We have

$$\Gamma_\Phi = (\Phi \otimes \mathbb{1}) \frac{\hat{\sigma}}{d}. \tag{1.93}$$

The state Γ_Φ associated with Φ gives a complete description of the CP map.

Φ is trace-preserving only if $\mathrm{Tr}_A(\Gamma_\Phi) = \mathbb{1}/d$. We note that the dynamical matrix D_Φ is related to the Choi-Jamiolkowski state by: $D_\Phi = d\,\Gamma_\Phi$. The CJ-representation turns out to be useful in obtaining the operator sum representation of Φ, as will be detailed in the

next Section.

1.4.4 Connecting the three representations

We will now briefly describe how the three representations are interconnected, and how one can go from one representation to the another.

Unitary → OSR:

Let $\mathcal{E} = \{|e_R^j\rangle\}$ be an orthonormal basis for system R. We first begin with the unitary representation and perform the trace in basis $\mathcal{E}$. We have

$$\begin{aligned}\Phi(\hat{\rho}_S) &= \mathrm{Tr}_R\left[U_{SR}\ (\hat{\rho}_S \otimes |\psi\rangle_R\langle\psi|)\ U_{SR}^\dagger\right], \\ &= \sum_j \langle e_R^j|\left[U_{SR}\ (\hat{\rho}_S \otimes |\psi\rangle_R\langle\psi|)\ U_{SR}^\dagger\right]|e_R^j\rangle \\ &= \sum_j (\langle e_R^j|U_{SR}|\psi\rangle_R)\,\hat{\rho}_S\,({}_R\langle\psi|U_{SR}^\dagger|e_R^j\rangle)\end{aligned} \tag{1.94}$$

Let us now define operators $A_k : \mathcal{H}_S \to \mathcal{H}_S$ where

$$A_k = \langle e_R^k|U_{SR}|\psi_R\rangle. \tag{1.95}$$

Then the expression for $\Phi(\hat{\rho}_S)$ in Eq. (1.94) reduces to

$$\Phi(\hat{\rho}_S) = \sum_j A_k\,\hat{\rho}_S\,A_k^\dagger, \tag{1.96}$$

which is the operator sum representation. To check the trace condition we evaluate

$$\begin{aligned}\sum_j A_j^\dagger A_j &= \sum_j \langle\psi_R|U_{SR}^\dagger|e_R^j\rangle\langle e_R^j|U_{SR}|\psi_R\rangle \\ &= \langle\psi_R|U_{SR}^\dagger\left(\sum_j |e_R^j\rangle\langle e_R^j|\right)U_{SR}\,|\psi_R\rangle \\ &= \langle\psi_R|U_{SR}^\dagger U_{SR}|\psi_R\rangle \\ &= \langle\psi_R|\mathbb{1}_S\otimes\mathbb{1}_R|\psi_R\rangle \\ &= \mathbb{1}_S,\end{aligned} \tag{1.97}$$

as expected. Had we instead chosen some other complete basis to evaluate the partial trace in the unitary representation, we would have obtained another operator sum representation for the same map Φ, connected to the original one by an isometry as seen earlier in Eq. (1.92).

OSR $\rightarrow$ Unitary:

We will describe how to obtain the unitary representation starting from the operator sum representation. Let us begin with

$$\Phi(\hat{\rho}_S) = \sum_{k=1}^{r} A_k\,\hat{\rho}_S\,A_k^\dagger, \tag{1.98}$$

where r denotes the number of Kraus operators in the operators sum representation. Let us consider a composite system with Hilbert space $\mathcal{H}_R\otimes\mathcal{H}_S$, with system R of dimension r. Let us arrange the operators A_k in a suggestive form to obtain a bipartite operator

$V : \mathcal{H}_S \to \mathcal{H}_R \otimes \mathcal{H}_S$ defined as

$$\begin{aligned} V &= \sum_k |k\rangle \otimes A_k \\ &= \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{pmatrix}, \end{aligned} \tag{1.99}$$

and $\{|k\rangle\}_1^r$ is an ONB for system R. It is immediate that V is an isometry as can be seen from the fact that

$$\begin{aligned} V^\dagger V &= \sum_{j,k}^{r} \langle j|k\rangle \, A_j^\dagger A_k \\ &= \sum_j A_j^\dagger A_k \delta_{jk} \\ &= \mathbb{1}_S . \end{aligned} \tag{1.100}$$

We note that V is a $dr \times d$ matrix. We already see that the operator sum representation can be written as

$$\begin{aligned} \Phi(\hat{\rho}_S) &= \mathrm{Tr}_R V \hat{\rho}_S \, V^\dagger \\ &= \mathrm{Tr}_R \left(\sum_k |k\rangle \otimes A_k \right) \hat{\rho}_S \left(\sum_j \langle j| \otimes A_j^\dagger \right) \\ &= \sum_{jk} \mathrm{Tr}_R(|k\rangle\langle j|) \otimes A_k \hat{\rho}_S \, A_j^\dagger \\ &= \sum_{jk} A_k \hat{\rho}_S \, A_j^\dagger \, \delta_{jk} \\ &= \sum_k^r A_k \hat{\rho}_S \, A_k^\dagger . \end{aligned} \tag{1.101}$$

The isometry V which is $dr \times d$ matrix can be appropriately completed to a unitary $dr \times dr$ matrix. For convenience, we can make the choice $U_{SR}|1\rangle_R = V$, where $|1\rangle_R$ is the first

vector of the computational basis in system R. Then one recovers operator V as given in Eq. (1.99). In this way, we obtain the unitary representation of the CP map.

CJ $\rightarrow$ Φ :

The action of Φ on a state $\hat{\rho}_S$ can be written down from the dynamical matrix D_Φ associated with Φ. We have

$$\Phi(\hat{\rho}) = \mathrm{Tr}_R\,(D_\Phi\,\hat{\rho}^T). \tag{1.102}$$

This expression can be verified in a straight-forward manner :

$$\begin{aligned}
\mathrm{Tr}_R\,(D_\Phi\,[\mathbb{1}\otimes\hat{\rho}^T]) &= \sum_{ij}\,[\Phi\otimes\mathbb{1}_R]\mathrm{Tr}_R\,(|i\rangle\langle j|\otimes|i\rangle\langle j|\,\hat{\rho}^T) \\
&= \sum_{ij}\,\Big(\Phi[|i\rangle\langle j|]\,\langle j|\hat{\rho}^T|i\rangle\Big), \\
&= \sum_{ij}\,\Big(\Phi[|i\rangle\langle j|]\,\rho_{ij}\Big) \\
&= \Phi(\hat{\rho}).
\end{aligned} \tag{1.103}$$

CJ $\rightarrow$ **OSR** :

Here we outline a simple procedure to obtain the operator sum representation from the CJ state or dynamical matrix. Let us first write down a decomposition of the dynamical matrix D_Φ into pure states, the spectral resolution being a special choice with orthonormal vectors :

$$D_\Phi = \sum_j\,|\psi_j\rangle\langle\psi_j|. \tag{1.104}$$

Note that the vectors $\{|\psi_j\rangle\}$ are not normalized. Let us write down the vectors $\{|\psi_j\rangle\}$ as

$$|\psi_j\rangle = \sum_{mn} c^j_{mn} |m\rangle \otimes |n\rangle, \tag{1.105}$$

where c^j_{mn} is the coefficient matrix for every vector $|\psi_j\rangle$. To each of these vectors we associate an operator $\widetilde{K}^j$ using the Jamiolkowski isomorphism [71] which is defined as

$$\widetilde{K}^j = \sum_{mn} c^j_{mn} |m\rangle\langle n|. \tag{1.106}$$

In other words, we flip the second ket of the vector to a bra to obtain the associated operator. We see that the isomorphism associates a vector $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A$ to a linear operator $V : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$.

Let $\widetilde{\Phi}$ be the map whose Kraus operators are the $\widetilde{K}^j$'s. Consider the one-sided action of the CP map $\widetilde{\Phi}$:

$$\begin{aligned}
D_{\widetilde{\Phi}} = [\widetilde{\Phi} \otimes \mathbb{1}](\hat{\sigma}) &= \sum_{ij} [\widetilde{\Phi} \otimes \mathbb{1}][|i\rangle\langle j| \otimes |i\rangle\langle j|] \\
&= \sum_{ij} \sum_{k} \widetilde{K}^k |i\rangle\langle j| (\widetilde{K}^k)^\dagger \otimes |i\rangle\langle j| \\
&= \sum_{ij} \sum_{k} \sum_{m} c^k_{mi} |m\rangle\langle n| (c^k_{nj})^* \otimes |i\rangle\langle j| \\
&= \sum_{k} \left(\sum_{mi} c^k_{mi} |mi\rangle \right) \left(\sum_{nj} (c^k_{nj})^* \langle nj| \right) \\
&= \sum_{k} |\psi_k\rangle\langle\psi_k| = D_\phi,
\end{aligned} \tag{1.107}$$

proving the assertion. By Eq. (1.107), the association from the vector to the operator is made transparent by the following identity:

$$\sum_{ij} \left[\widetilde{K}^k \otimes \mathbb{1} \right] (|i\rangle\langle j| \otimes |i\rangle\langle j|) \left[(\widetilde{K}^k)^\dagger \otimes \mathbb{1} \right] = |\psi_k\rangle\langle\psi_k|, \tag{1.108}$$

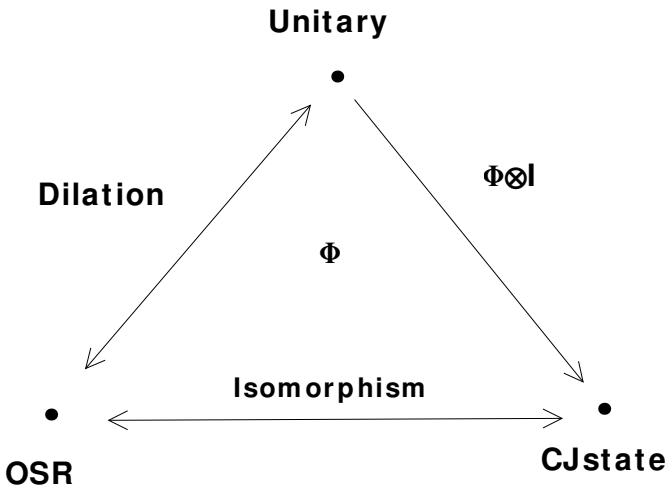

Figure 1.2: Showing a schematic diagram for the various ways in which the three representations of CPTP maps, namely, the unitary representation, the operator sum representation and the Choi-Jamiolkowski representation are related.

A schematic diagram with the connections between the various representations is shown in Fig. 1.2. A few remarks are in order with respect to obtaining the operator sum representation from the CJ state. Note that we first began with the dynamical matrix for which the trace is not unity, and in fact $\mathrm{Tr}(\Gamma_\Phi) = d_A$. This facilitated the obtaining of the Kraus operators directly. That the resulting CP map is trace-preserving is a consequence of the fact that $\mathrm{Tr}_A \Gamma_\Phi = \mathbb{1}/d$, i.e., $\mathrm{Tr}_A D_\phi = \mathbb{1}$.

We see that the rank of the CJ state Γ_Φ or dynamical matrix D_Φ corresponding to a channel Φ gives the minimum number of Kraus operators in the operator sum representation. We will call the operator sum representation of a channel *minimal* when the number of Kraus operators is the minimum number possible. Let us denote the rank of Γ_Φ as r. So we have that

$$r \leq d^2. \tag{1.109}$$

In other words, the maximum number of Kraus operators in the minimal representation is d^2. One way to obtain this minimal representation is to consider the spectral decomposition of the Γ_Φ. We have seen earlier that any isometry on the set of operators also gives rise to equivalent operator sum representations. These are precisely the various rank-one decompositions of Γ_Φ. Finally, we see that the unitary representation requires an ancilla system R of dimension $r \leq d^2$ to realize any channel acting on a system with Hilbert space dimension d.

1.4.5 Properties of CP maps

We now provide a useful guide to the various properties of CPTP maps in a suitable representation.

Dual : Given a channel Φ with Kraus operators $\{A_k\}$, the *dual* Φ' is defined as the CP map that has an operator sum representation with Kraus operators $\{\tilde{A}_k = A_k^\dagger\}$, i.e.,

$$\Phi'(\hat{\rho}_S) = \sum_k \tilde{A}_k \hat{\rho}_S \tilde{A}_k^\dagger. \tag{1.110}$$

Since Φ is trace-preserving, it implies that

$$\sum_k A_k^\dagger A_k = \sum_k \tilde{A}_k \tilde{A}_k^\dagger = \mathbb{1}. \tag{1.111}$$

Unital : A unital CP map is one that takes the identity operator to itself, i.e.,

$$\Phi(\mathbb{1}) = \mathbb{1}. \tag{1.112}$$

From the operator sum representation of Φ, we have the following condition on the Kraus

operators :

$$\sum_k A_k A_k^\dagger = \mathbb{1}. \tag{1.113}$$

Therefore, the dual of a quantum channel is a unital CP map as can be seen by Eq. (1.110). An alternative way to see this fact is by considering the Choi-Jamiolkoski state Γ_Φ. We have that a CP map Φ is unital if and only if

$$\mathrm{Tr}_B(\Gamma_\Phi) = \frac{\mathbb{1}}{d_A}, \quad \mathrm{Tr}_B(D_\Phi) = \mathbb{1}. \tag{1.114}$$

Bistochastic : A channel Φ that is also unital is called a bistochastic map. So the conditions in terms of the Kraus operators are given by :

$$\begin{aligned} \sum_k A_k^\dagger A_k &= \mathbb{1}, \\ \text{and} \quad \sum_k A_k A_k^\dagger &= \mathbb{1}. \end{aligned} \tag{1.115}$$

Alternately, a channel Φ is bistochastic if and only if

$$\mathrm{Tr}_A(\Gamma_\Phi) = \frac{\mathbb{1}}{d}, \text{ and } \quad \mathrm{Tr}_B(\Gamma_\Phi) = \frac{\mathbb{1}}{d}. \tag{1.116}$$

Random unitary : A random unitary channel [72] is a channel which is a convex combination of unitary channels. In other words, the operator sum representation of a random unitary channel can be given in the form :

$$\Phi(\hat{\rho}_S) = \sum_k p_k U_k \hat{\rho}_S U_k^\dagger. \tag{1.117}$$

By definition, a random unitary channel is bistochastic. But not every bistochastic channel

is random unitary. Examples of channels which are bistochastic but not random unitary were provided in the finite-dimensional setting in [73–75] and for continuous variable systems in [76].

Extremal: The set of quantum channels acting on a given Hilbert space forms a convex set, i.e.,

$$\Phi = p_1\Phi_1 + (1 - p_1)\Phi_2, \tag{1.118}$$

is also a CPTP map when Φ_1, Φ_2 are channels. An extremal channel is one that cannot be written as a convex combination of other quantum channels. A simple example of an extremal map is a unitary channel, i.e.,

$$\Phi(\hat{\rho}_S) = U\,\hat{\rho}_S\,U^\dagger. \tag{1.119}$$

By definition, a random unitary channel is not extremal. A theorem by Choi [67] gives a way to check if a channel is extremal or not.

Theorem 4 (Choi) *A CPTP map Φ with minimal operator sum representation $\Phi(\hat{\rho}_S) = \sum_k A_k\,\hat{\rho}_S\,A_K^\dagger$ is extremal if and only if the operators $\{\mathcal{A}_{kj} = A_k^\dagger A_j\}$ are linearly independent.*

If the number of Kraus operators is r, then the number of operators $\{\mathcal{A}_{kj}\}$ is r^2. Since we require linear independence of $\{\mathcal{A}_{kj}\}$ for an extremal channel, we have $r^2 \leq d^2$. In other words, the operator sum representation of an extremal channel can have utmost d Kraus operators in the minimal representation.

Entanglement-breaking: A channel $\Phi : \Lambda(\mathcal{H}_S) \to \Lambda(\mathcal{H}_S)$ is said to be entanglement-breaking [77] if its one-sided action takes every bipartite state $\hat{\rho}_{SR} \in \Lambda(\mathcal{H}_S \otimes \mathcal{H}_R)$ to a separable state for an arbitrary system R. Much like the CP condition, there is an opera-

tional way to check whether a channel Φ is entanglement-breaking or not. We have the following theorem [77]:

Theorem 5 (Horodecki-Shor-Ruskai) *A channel is entanglement-breaking iff*

$$\Gamma_\Phi = \frac{1}{d}[\Phi \otimes \mathbb{1}](\hat{\sigma}). \tag{1.120}$$

is separable. Further, every entanglement-breaking channel has an operator sum representation in which every Kraus operator is rank-one.

That there exists an operator sum representation having rank-one elements for every entanglement-breaking channel is a consequence of the fact that every separable state has a decomposition in terms of products of projectors.

Having assembled the basic notions of correlations and channels of interest to us, we next consider some preliminaries regarding continuous variable systems.

1.5 Single mode of radiation

Let us consider as our quantum system a single-mode of a radiation field (a harmonic oscillator) [78]. The Hilbert space is the space of all (complex) square integrable functions ψ over one real variable, the configuration space, and is denoted by $\mathcal{L}^2(\mathcal{R})$:

$$\psi \in \mathcal{L}^2(\mathcal{R}) \longleftrightarrow \int dx\, |\psi(x)|^2 < \infty. \tag{1.121}$$

The creation and annihilation operators, denoted by $\hat{a}$, $\hat{a}^\dagger$ of the quantum system satisfy the standard bosonic commutation relation:

$$[\hat{a},\, \hat{a}^\dagger] = 1. \tag{1.122}$$

In terms of the hermitian position and momentum variables, these ladder operators have the expression

$$\hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}, \quad \hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}}, \tag{1.123}$$

and the equivalent commutation relation reads

$$[\hat{q}, \hat{p}] = i, \tag{1.124}$$

where we have set $\hbar = 1$. Let us arrange the operators $\hat{q},\ \hat{p}$ as a column vector :

$$\hat{\xi} = \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}. \tag{1.125}$$

Then the commutation relations, using Eq. (1.124), can be compactly written as

$$[\hat{\xi}_i, \hat{\xi}_j] = i\beta_{ij}, \tag{1.126}$$

where

$$\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.127}$$

Consider a linear transformation on $\hat{q}$ and $\hat{p}$ specified by a 2×2 real matrix S :

$$\hat{\xi} \to \hat{\xi}' = S\hat{\xi}. \tag{1.128}$$

Since the new variables also need to satisfy the canonical commutation relations of (1.126), we have that

$$S\,\beta\,S^T = \beta. \tag{1.129}$$

In other words, S is an element of the symplectic group $Sp(2,\mathcal{R})$.

An important aspect to note is that these linear transformations are induced by unitary evolutions generated by Hamiltonians that are quadratic in the mode operators [78]. In other words any $\hat{H} = \sum_{ij} h_{ij}\hat{\xi}_i\hat{\xi}_j$, $((h_{ij}))$ being real symmetric, the corresponding unitary transformation $\hat{U} = e^{-i\hat{H}}$ induces

$$\hat{U}^\dagger \hat{\xi}\, \hat{U} = S(h)\hat{\xi}, \quad S(h) \in Sp(2,\mathcal{R}). \tag{1.130}$$

Passive transformations :

A subset of transformations of particular interest to us are the what are known as passive transformations [78]. Passive transformations are those symplectic transformations that are phase-space rotations as well. We only consider the single-mode case for simplicity. We denote the collection of passive transformations on single-mode systems by $K(1)$. We have

$$K(1) = \{S \,|\, S \in Sp(2,\mathcal{R}) \cap SO(2,\mathcal{R})\}. \tag{1.131}$$

It turns out that $K(1) = SO(2,\mathcal{R})$ is isomorphic to $U(1)$. All the above properties suitably generalise to the multi-mode case. Passive transformations conserve photon number and play an important role in the definition of squeezing [78].

1.6 Phase space distributions

The study of phase space distributions can be motivated from the possibility of using these functions as a 'weight' functions in an integral representation of a given operator [79,80]. Before we describe the notion of phase space distributions, we now briefly discuss an important class of operators known as the Weyl displacement operators.

For each complex number $\alpha \in C$, there is an associated operator $\mathcal{D}(\alpha)$, which is defined as

$$\mathcal{D}(\alpha) = \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}]. \tag{1.132}$$

The operators $\{\mathcal{D}(\alpha)\}$ are known as the displacement operators. From the definition we see that $\mathcal{D}(\alpha)$ is unitary and $\mathcal{D}^\dagger(\alpha) = \mathcal{D}(-\alpha) = \mathcal{D}(\alpha)^{-1}$. The operators $\mathcal{D}(\alpha)$ are called displacement operators for the following reason :

$$\begin{aligned} \mathcal{D}(\alpha)^\dagger \, \hat{a} \, \mathcal{D}(\alpha) &= \hat{a} + \alpha \\ \mathcal{D}(\alpha)^\dagger \, \hat{a}^\dagger \, \mathcal{D}(\alpha) &= \hat{a}^\dagger + \alpha^* \end{aligned} \tag{1.133}$$

The composition of two displacement operators with independent arguments gives :

$$\mathcal{D}(\alpha)\mathcal{D}(\beta) = \exp\left[\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)\right] \mathcal{D}(\alpha + \beta). \tag{1.134}$$

We finally mention the orthogonality property :

$$\mathrm{Tr}\,[\mathcal{D}(\alpha)\,\mathcal{D}^{-1}(\beta)] = \pi\delta^{(2)}(\alpha - \beta). \tag{1.135}$$

It may be 'visually' seen from the definition (1.132) that $\mathcal{D}(\alpha)$ is simply the 'quantized' version of the plane wave $\exp[\alpha z^* - \alpha^* z]$ over the classical $q - p$ phase-space, with $z = (q + ip)/\sqrt{2}$. It should thus come as no surprise that the collection $\{\mathcal{D}(\alpha), \alpha \in C\}$ satisfy a completeness relation corresponding to the completeness of the plane waves (Fourier integral theorem). Consequently, the displacement operators $\mathcal{D}(\alpha)$ form a basis for expansion of generic operators acting on $\mathcal{H} = \mathcal{L}^2(\mathcal{R})$ [79].

The displacement operators can be expressed in various ways to correspond to various ordering schemes. Ordering refers to the order in which the ladder operators are written in the polynomial expansion of the displacement operators. Two particular choices of

ordering are the normal ordering and the anti-normal ordering. The normal ordering of the displacement operator $\mathcal{D}(\alpha)$ is given by the expression

$$\mathcal{D}(\alpha) = \exp[-|\alpha|^2/2]\, \exp[\alpha \hat{a}^\dagger]\, \exp[-\alpha^* \hat{a}], \tag{1.136}$$

and the anti-normal ordering by

$$\mathcal{D}(\alpha) = \exp[|\alpha|^2/2]\, \exp[-\alpha^* \hat{a}]\, \exp[\alpha \hat{a}^\dagger]. \tag{1.137}$$

The expression in (1.132) corresponds to Weyl or symmetric ordering. The s-ordered displacement operator, $s \in [-1, 1]$, denoted by $\mathcal{D}(\alpha; s)$ is defined as

$$\mathcal{D}(\alpha; s) = \exp[s|\alpha|^2/2]\, \mathcal{D}(\alpha). \tag{1.138}$$

So normal ordering corresponds to $s = 1$, anti-normal ordering to $s = -1$, and the symmetric or Weyl ordering corresponds to the case $s = 0$. The s-ordered monomial $\{(\hat{a}^\dagger)^n \hat{a}^m\}_s$ is defined as

$$\mathcal{D}(\alpha; s) = \sum_{n,m=0}^{\infty} \{(\hat{a}^\dagger)^n \hat{a}^m\}_s \, \frac{\alpha^n (-\alpha^*)^m}{n!m!}, \tag{1.139}$$

or equivalently we have

$$\{(\hat{a}^\dagger)^n \hat{a}^m\}_s = \left. \frac{\partial^{n+m} \mathcal{D}(\alpha; s)}{\partial \alpha^n \partial (-\alpha^*)^m} \right|_{\alpha=0}. \tag{1.140}$$

The s-ordered displacement operators facilitate the definition of the s-ordered characteristic function. The s-ordered characteristic function associated with a given density operator $\hat{\rho}$ is defined as

$$\chi_s(\hat{\rho}, \alpha) = \mathrm{Tr}[\mathcal{D}(\alpha; s)\, \hat{\rho}]. \tag{1.141}$$

From the completeness of the displacement operators, we have the following representation or inverse relation for any operator $\hat{\rho}$ [79]:

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} \chi_s(\hat{\rho}, \alpha)\, \mathcal{D}^{-1}(\alpha; s). \tag{1.142}$$

The s-ordered quasiprobability associated with a state $\hat{\rho}$ is defined as the two-dimensional Fourier transform of the corresponding s-ordered characteristic function. We have

$$W_s(\hat{\rho}, \xi) = \int \frac{d^2\alpha}{\pi}\, e^{\xi\alpha^* - \xi^*\alpha}\, \chi_s(\hat{\rho}, \alpha). \tag{1.143}$$

We now briefly detail three frequently used quasiprobabilities from among the one-parameter family of quasiprobabilities, namely, the Wigner, the Q and the ϕ distributions [80].

Wigner function :

The Wigner function $W_0(\hat{\rho}, \alpha)$ associated with a given state $\hat{\rho}$ results as the symmetric-ordered ($s = 0$) quasiprobability :

$$W(\hat{\rho}, \alpha) \equiv W_0(\hat{\rho}, \alpha) = \int \frac{d^2\xi}{\pi}\, e^{\alpha\xi^* - \xi\alpha^*}\, \chi_0(\hat{\rho}, \xi). \tag{1.144}$$

The Wigner function (and indeed every s-ordered quasiprobability) is real and normalized in accordance with the hermiticity and trace condition of a density operator $\hat{\rho}$:

$$\begin{aligned} W(\hat{\rho}, \alpha) &= W(\hat{\rho}, \alpha)^*, \\ \int \frac{d^2\alpha}{\pi}\, W(\hat{\rho}, \alpha) &= 1. \end{aligned} \tag{1.145}$$

The Wigner representation is particularly useful for evaluating expectation values of operators written in the symmetric ordered form :

$$\mathrm{Tr}\left[\hat{\rho}\, \{(\hat{a}^\dagger)^n \hat{a}^m\}_0\right] = \int \frac{d^2\alpha}{\pi}\, W(\hat{\rho}, \alpha)\, (\alpha^*)^n \alpha^m. \tag{1.146}$$

We see that the symmetric-ordered operators are just replaced by the c-number equivalents and the density operator is replaced by the associated Wigner function.

A useful property of the Wigner function is the ease with which symplectic transformations reflect in the Wigner description. We had seen earlier that unitary transformations generated by Hamiltonians quadratic in mode operators, lead to a symplectic transformation of the mode operators. We have in the Wigner picture :

$$\begin{aligned}
&\hat{\rho} \rightarrow \hat{\rho}^{'} = U(S)\,\hat{\rho}\,U(S)^{\dagger} \\
&\quad \Longleftrightarrow \hat{\xi} \rightarrow \hat{\xi}^{'} = S\hat{\xi} \\
&\quad \Longleftrightarrow W(\hat{\rho},\xi) \rightarrow W(\hat{\rho}^{'},\xi) = W(\hat{\rho}, S^{-1}\xi) \\
&\quad \Longleftrightarrow \chi(\hat{\rho},\xi) \rightarrow \chi(\hat{\rho}^{'},\xi) = \chi(\hat{\rho}, S^{-1}\xi) \\
&\quad \Longleftrightarrow V \rightarrow V^{'} = S\,V\,S^{T}, \qquad (1.147)
\end{aligned}$$

where in the last line of Eq. (1.147), V stands for the variance matrix associated with $\hat{\rho}$. We will consider the notion of variance matrix in the next Section. We will repeatedly appeal to the above transformations in phase space as well as the corresponding transformations at the level of the variance matrix in the following Sections.

Husimi function :

The Husimi Q-function is the anti-normal ordered quasiprobability and is defined as

$$Q(\hat{\rho},\alpha) \equiv W_{-1}(\hat{\rho},\alpha) = \int \frac{d^2\xi}{\pi}\, e^{\alpha\xi^{*}-\xi\alpha^{*}}\, \chi_{-1}(\hat{\rho},\xi). \qquad (1.148)$$

It can be shown that the Q-function can alternately be written as [80] :

$$Q(\hat{\rho},\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle. \qquad (1.149)$$

We see that the Q-function is always pointwise positive irrespective of the state $\hat{\rho}$ and its

numerical value is bounded from above by 1, i.e., $Q(\hat{\rho}, \alpha) \leq 1$. Further, Q is normalised :

$$\int \frac{d^2\alpha}{\pi} Q(\hat{\rho}, \alpha) = 1. \tag{1.150}$$

We see that the Q-function is a probability distribution over the complex plane. We mention in passing that though every Q-function is a probability distribution, the converse however is not true. The Q-function facilitates the computation of the ensemble averages of anti-normally ordered operators analogous to how the Wigner function was useful for computing ensemble averages of symmetric-ordered operators.

Diagonal 'weight' function :

The third important quasiprobability we shall be interested in is the normal-ordered distribution corresponding to $s = 1$. The quasiprobability corresponding to $s = 1$ is called the Sudarshan-Glauber diagonal weight function denoted by ϕ [81, 82]. The diagonal weight $\phi(\hat{\rho}, \alpha)$ associated with a density matrix $\hat{\rho}$ is defined as

$$\phi(\hat{\rho}, \alpha) \equiv W_1(\hat{\rho}, \alpha) = \int \frac{d^2\xi}{\pi} e^{\alpha\xi^* - \xi\alpha^*} \chi_1(\hat{\rho}, \xi). \tag{1.151}$$

Every density matrix $\hat{\rho}$ can be expressed in the 'diagonal' form in the (over-complete) coherent state basis as

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} \phi(\alpha) |\alpha\rangle\langle\alpha|. \tag{1.152}$$

We note that coherent states form a complete non-orthogonal set. From the trace condition of $\hat{\rho}$, we have that

$$\int \frac{d^2\alpha}{\pi} \phi(\alpha) = 1. \tag{1.153}$$

Unlike the Q-function and the Wigner function which are well-behaved on the complex

plane, the diagonal weight function can be highly singular. Finally, we note that the diagonal function ϕ helps to easily evaluate the ensemble averages of normally-ordered operators which is of much interest from an experimental perspective [83].

1.7 Gaussian states

We now begin with a brief discussion on the notion of a variance matrix associated with a state $\hat{\rho}$ [78, 84, 85]. Consider the 2×2 matrix of operators $\hat{\xi}\,\hat{\xi}^T$. The following identity associated with this operator matrix is obtained by using the commutation and anti-commutation relations of the mode operators :

$$\begin{aligned} 2\,(\hat{\xi}\hat{\xi}^T)_{ij} &= 2\,\hat{\xi}_i\,\hat{\xi}_j \\ &= \{\hat{\xi}_i, \hat{\xi}_j\} + [\hat{\xi}_i, \hat{\xi}_j] \\ &= \{\hat{\xi}_i, \hat{\xi}_j\} + i\beta_{ij}. \end{aligned} \tag{1.154}$$

Taking the expectation value in the state $\hat{\rho}$, we have

$$2\langle\hat{\xi}\hat{\xi}^T\rangle_{ij} = \mathrm{Tr}\,(\{\hat{\xi}_i, \hat{\xi}_j\}\,\hat{\rho}) + i\beta_{ij}. \tag{1.155}$$

Let us assume without loss of generality that state is one for which the means are zero. We now define the variance matrix V of a given state $\hat{\rho}$ as

$$V_{ij} = \mathrm{Tr}\,(\{\hat{\xi}_i, \hat{\xi}_j\}\,\hat{\rho}). \tag{1.156}$$

The matrix V is real, symmetric and positive definite. These properties of the variance matrix would also hold for a classical probability distribution. However, for the quantum case there is an additional condition that V has to satisfy for it to be a valid variance

matrix. This additional constraint is the uncertainty principle [78]:

$$V + i\beta \geq 0. \tag{1.157}$$

It is known that every variance matrix can be diagonalised by a symplectic transformation [78, 86]. For the single mode case, by choosing a suitable symplectic transformation S, the variance matrix can be diagonalised, i.e.,

$$\begin{aligned} V \rightarrow V_{\text{can}} &= S\, V\, S^T \\ &= \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix}, \end{aligned} \tag{1.158}$$

where κ is called the symplectic eigenvalue of V. In this canonical form, the uncertainty principle of Eq. (1.157) reads

$$\kappa \geq 1. \tag{1.159}$$

We now describe a particularly important class of states known as Gaussian states [78, 85, 87, 88]. We assume that the state has zero first moments [this can be achieved by a rigid phase space translation that is effected by the action of a suitable (unitary) displacement operator]. A Gaussian state is one whose Wigner function is a Gaussian function:

$$W(\hat{\rho}, \alpha) = \frac{1}{2\sqrt{\text{Det}V}} \exp[-\frac{1}{2}\alpha^T\, V^{-1}\, \alpha], \tag{1.160}$$

where the complex number $\alpha = x + iy$ can also be viewed as the vector $(x, y)^T$. Equivalently, Gaussian states are states whose corresponding (symmetric-ordered) characteristic function is a Gaussian function. In other words we have

$$\chi(\hat{\rho}, \xi) = \exp[-\frac{1}{2}\xi^T \beta V \beta^T \xi], \tag{1.161}$$

where V is the variance matrix associated with the state $\hat{\rho}$. We wish to emphasis that a Gaussian state is completely specified by its first (means) and second moments (variances).

Simple examples of pure Gaussian states include the vacuum state or the ground state of the harmonic oscillator $|0\rangle$, coherent states $|\alpha\rangle = D(\alpha)|0\rangle$, squeezed state $S(\eta)|0\rangle$, which is obtained by the action of the squeeze transformation $S(\eta)$ on the vacuum state. The thermal state is an example of a mixed Gaussian state.

From the canonical form of the variance matrix in Eq. (1.158), we infer that by applying a suitable symplectic transformation, the variance matrix of any pure Gaussian state can be brought to the identity matrix, while any mixed Gaussian state can be brought to the form $\kappa \mathbb{1}_{2\times 2}$, where $\kappa > 1$ is the symplectic eigenvalue.

1.7.1 Two-mode systems

The Hilbert space of the two-mode system is $\mathcal{L}^2(\mathcal{R}) \otimes \mathcal{L}^2(\mathcal{R}) = \mathcal{L}^2(\mathcal{R}^2)$ and consists of vectors that are square integrable over a two-plane. As in the single mode case, we arrange the quadrature operators $\hat{q}_1,\ \hat{p}_1$, and $\hat{q}_2,\ \hat{p}_2$ associated with the modes as a column vector

$$\hat{\xi} = (\hat{q}_1,\ \hat{p}_1,\ \hat{q}_2,\ \hat{p}_2)^T. \tag{1.162}$$

Then the canonical commutation relations read

$$\begin{aligned} [\hat{\xi}_i,\ \hat{\xi}_j] &= i\,\Omega_{ij}, \\ \text{where,} \quad \Omega &= \beta \oplus \beta. \end{aligned} \tag{1.163}$$

The mode operators $\hat{a}_1,\ \hat{a}_2$ are defined for each mode in the standard way.

A Gaussian state of a two-mode system (with zero mean) is completely described by a

4×4 variance matrix which satisfies the uncertainty relation :

$$V + i\,\Omega \geq 0. \tag{1.164}$$

Of importance to us is the detection of entanglement of two-mode Gaussian states. It turns out that there is necessary and sufficient criteria for detecting entanglement of two-mode Gaussian states [89]. For this we require the use of the transpose map Λ. The transpose map transcribes on the Wigner function faithfully into a mirror reflection of the underlying phase space. In other words, we have :

$$\hat{\xi} \rightarrow \hat{\xi}^{'} = \Lambda\hat{\xi} = (\hat{q},\, -\hat{p}). \tag{1.165}$$

We now state the following necessary and sufficient condition for detecting entanglement of two-mode Gaussian states [89].

Theorem 6 (Simon) *A two-mode Gaussian state with variance matrix V is separable if and only if the local application of the transpose map by one and only one of the parties leads to a valid variance matrix. The state is entangled otherwise.*

The mirror reflection corresponding to partial transpose $\widetilde{\Lambda}$, with the transpose performed on the second mode, can be written as $\widetilde{\Lambda} = \mathrm{diag}(1, 1, 1, -1)$. The separability criterion can then be written as the additional requirement

$$V + i\,\widetilde{\Omega} \geq 0, \text{ where } \widetilde{\Omega} = \widetilde{\Lambda}\,\Omega\,\widetilde{\Lambda}, \tag{1.166}$$

over and above the uncertainty principle (1.164).

1.8 Gaussian channels

Having outlined the basic and fundamental properties of Gaussian states, we now consider the notion of Gaussian channels. Before we begin with the description of Gaussian channels, we wish to motivate the notion of Gaussian channels from the analogous classical setting.

It is well known in classical probability theory that a Gaussian probability distribution denoted by

$$\mathcal{P}_{\rm G}(\boldsymbol{\xi}) = \frac{1}{\sqrt{(2\pi)^n {\rm DetV}}}\, e^{-\frac{1}{2}\boldsymbol{\xi}^{\rm T} V^{-1}\boldsymbol{\xi}} \tag{1.167}$$

remains Gaussian under all affine transformations of the form $\boldsymbol{\xi} \rightarrow A\boldsymbol{\xi} + \boldsymbol{b}$ and convolutions with Gaussian distributions. The affine transformation $\boldsymbol{\xi} \rightarrow A\boldsymbol{\xi}$, induces the following transformation on the characteristic function, i.e.,

$$\chi_{\rm G}(\boldsymbol{x}) \rightarrow \chi_G(B\boldsymbol{x}), \quad B = (A^{-1})^T. \tag{1.168}$$

So we see that the translation by the vector $\boldsymbol{b}$ reflects as a linear phase factor in the characteristic function, and the homogeneous transformation A reflects as a corresponding homogeneous transformation $B = (A^{-1})^T$ on $\chi_{\rm G}(\boldsymbol{x})$. There are no restrictions on A and $\boldsymbol{b}$ for a Gaussian probability to be taken to a Gaussian probability under such a transformation.

The analogue of Gaussian probability distributions in quantum mechanics are Gaussian Wigner distributions. It is true that a Gaussian Wigner function is taken to a Gaussian probability under all affine transformations. But to remain a valid Wigner distribution, additional constraints have to be satisfied in the form of the uncertainty principle in Eq. (1.157) which we detail below.

The action of any Gaussian channel on system A may be realized through the action of a

Gaussianity preserving unitary on a suitably enlarged system :

$$\rho_A \rightarrow \rho_A^{'} = \mathrm{Tr}_B \left[U_{AB} \left(\rho_A \otimes \rho_B \right) U_{AB}^{\dagger} \right]. \tag{1.169}$$

Here ρ_B is a Gaussian state of the ancilla B, and U_{AB} is a linear canonical transformation on the enlarged composite system consisting of the system of interest A and the ancilla B. That all Gaussian channels can indeed be realized in this manner has been shown by the work of Holevo and coauthors [90–94].

For arbitrary input state with symmetric-ordered characteristic function $\chi_W(\xi;\rho)$, we have resulting from (1.169)

$$\chi^{\mathrm{in}}(\xi;\rho) \rightarrow \chi^{\mathrm{out}}(\xi;\rho) = \chi(X\xi;\rho) \exp\left[-\frac{\xi^T Y \xi}{2} \right], \tag{1.170}$$

where X and Y are real matrices, and Y being positive definite. The pair (X, Y) are completely specified by the unitary representation (1.169).

So the action of a Gaussian channel thus manifests simply as a linear transformation on the variance matrix V. Under the action of a Gaussian channel described by (X, Y) [90] :

$$V \rightarrow V' = X^T V X + Y. \tag{1.171}$$

Suppose we are instead given a general (X, Y) which effect the transformation in (1.171). For $V^{'}$ to be a valid variance matrix for arbitrary input, (X, Y) have to satisfy a constraint that is a consequence of the uncertainty principle, which we detail below.

Let us consider the one-sided action of a Gaussian map described by (X, Y) on a two-mode squeezed vacuum state with squeeze parameter r. The two-mode squeezed vacuum state

is represented in the Fock basis as

$$|\psi_r\rangle = \operatorname{sech} r \sum_{k=0}^{\infty} (\tanh r)^k |k,\ k\rangle, \tag{1.172}$$

and its variance matrix is given by

$$V_{\text{out}}(r) = \begin{pmatrix} c_{2r} & 0 & s_{2r} & 0 \\ 0 & c_{2r} & 0 & -s_{2r} \\ s_{2r} & 0 & c_{2r} & 0 \\ 0 & -s_{2r} & 0 & c_{2r} \end{pmatrix}, \tag{1.173}$$

where $c_{2r} = \cosh\ 2r$, $s_{2r} = \sinh\ 2r$.

The result of this one-sided action by the map (X, Y) is a two-mode mixed Gaussian state specified by variance matrix

$$V_{\text{out}}(r) = \begin{pmatrix} c_{2r}(X^T X) + Y & s_{2r}(X^T \sigma_3) \\ s_{2r}(\sigma_3 X) & c_{2r}(\mathbb{1}_2) \end{pmatrix}, \tag{1.174}$$

σ_3 being the standard Pauli matrix. It is clear that $V_{\text{out}}(r)$ should obey the mandatory uncertainty principle

$$V_{\text{out}}(r) + i\,\Omega \geq 0, \tag{1.175}$$

for all values of squeezing. In fact, this requirement in terms of the uncertainty principle is both a necessary and sufficient condition on (X, Y) to correspond to a Gaussian channel, and it may be restated in the form [91, 95]

$$Y + i\,\Omega \geq i\,X^T\,\Omega\,X. \tag{1.176}$$

1.8.1 Canonical forms for quantum-limited and noisy channels

Given a Gaussian channel Γ we can construct, 'quite cheaply', an entire family of Gaussian channels by simply preceding and following Γ with unitary (symplectic) Gaussian channels $U(S_1), U(S_2)$ corresponding respectively to symplectic matrices S_1, S_2. Therefore in classifying Gaussian channels it is sufficient to classify these orbits or double cosets and, further, we may identify each orbit with the 'simplest' looking representative element of that orbit (the canonical form). Since

$$U(S_1)\,\Gamma\,U(S_2)\ :\ \chi(\xi) \rightarrow \chi(S_2\,X\,S_1\,\xi)\exp[-\frac{1}{2}\xi^T\,S_1^T\,Y\,S_1\,\xi], \tag{1.177}$$

the task actually reduces to enumeration of the orbits of (X, Y) under the transformation $(X, Y) \rightarrow (X^{'}, Y^{'}) = (S_2\,X\,S_1,\ S_1^T\,Y\,S_1)$.

We wish to make one important remark regarding Gaussian channels. The injection of an arbitrary amount of classical (Gaussian) noise into the state is obviously a Gaussian channel : $\chi(\xi) \rightarrow \chi(\xi)\ \exp[-a\,|\xi|^2/2]$, $a > 0$. It is called the classical noise channel. Now, given a Gaussian channel we may follow it up with a classical noise channel to obtain another Gaussian channel. A Gaussian channel will be said to be *quantum-limited* if it cannot be realized as another Gaussian channel followed by a classical noise channel. Conversely, the most general Gaussian channel is a quantum-limited Gaussian channel followed by a classical noise channel, and it follows that quantum-limited channels are the primary objects which need to be classified into orbits.

In other words, for a given X, the minimal Y, say Y_0, that saturates the inequality in (1.176) represents the threshold Gaussian noise that needs to be added to $\chi(X\xi)$ to make atonement for the failure of X to be a symplectic matrix, and thus rendering the map completely positive; if X happens to be a symplectic matrix, then the corresponding minimal $Y_0 = 0$. And $Y \neq 0$ whenever X is not a symplectic matrix.

In the single-mode case where (X, Y) are 2×2 matrices, $S_1, S_2 \in Sp(2, R)$ can be so chosen

Quantum-limited Channel (X, Y_0)	X	Y_0	Noisy Channel $Y = Y_0 + a\mathbb{1}$
$\mathcal{D}(\kappa; 0)$	$-\kappa\sigma_3$	$(1+\kappa^2)\mathbb{1} \quad \kappa > 0$	$\mathcal{D}(\kappa; a)$
$C_1(\kappa; 0)$	$\kappa\mathbb{1}$	$(1-\kappa^2)\mathbb{1} \quad 0 \leq \kappa < 1$	$C_1(\kappa; a)$
$C_2(\kappa; 0)$	$\kappa\mathbb{1}$	$(\kappa^2-1)\mathbb{1} \quad \kappa > 1$	$C_2(\kappa; a)$
$\mathcal{A}_1(0)$	0	$\mathbb{1}$	$\mathcal{A}_1(a)$
$\mathcal{A}_2(0)$	$(\mathbb{1}+\sigma_3)/2$	$\mathbb{1}$	$\mathcal{A}_2(a)$
$\mathcal{B}_2(0)$	$\mathbb{1}$	0	$\mathcal{B}_2(a)$
$\mathcal{B}_1(0)$	$\mathbb{1}$	0	$\mathcal{B}_1(a)$

Table 1.1: Showing the quantum-limited bosonic Gaussian channels. The noisy versions of these channels are obtained by replacing Y_0 by $Y = Y_0 + a\mathbb{1}$ and so $Y > Y_0$.

that X' equals a multiple of identity, a multiple of σ_3, or $(\mathbb{1} + \sigma_3)/2$ while Y' equals a multiple of identity or $(\mathbb{1} + \sigma_3)/2$. Thus the canonical form of a Gaussian channel X, Y is fully determined by the rank and determinant of (X, Y), and classification of *quantum-limited bosonic Gaussian channels* [91, 92] is shown in Table 1.1

By following the above listed quantum-limited channels by injection of classical noise of magnitude a we get respectively $\mathcal{D}(\kappa; a)$, $C_1(\kappa; a)$, $C_2(\kappa; a)$, $\mathcal{A}_1(a)$, $\mathcal{A}_2(a)$, and $\mathcal{B}_2(a)$; the last case $\mathcal{B}_1(a)$ is special in that it is obtained from $\mathcal{B}_1(0)$ by injection of noise into *just one quadrature*: $\chi(\xi) \to \chi(\xi)\ \exp[-a\,\xi^T(\mathbb{1} + \sigma_3)\xi/4]$.

It is clear in the case of $\mathcal{D}(\kappa; 0)$ that $X = -\kappa\sigma_3$ corresponds to (scaled) phase conjugation or matrix transposition of the density operator. And the phase conjugation is the most famous among positive maps which are not CP [12, 13, 89]; it is the injection of additional classical noise of magnitude (not less than) $1 + \kappa^2$, represented by Y_0, that mends it into a CP map. It may be noted that the quantum-limited end of both the $\mathcal{B}_1$ and $\mathcal{B}_2$ families is the trivial identity channel.

The reason for the special emphasis on quantum-limited channels in our enumeration of the Holevo classification is this: every noisy Gaussian channel [except $\mathcal{B}_1(a)$] can be realized, as we shall see later, as the composite of a pair of quantum-limited channels. This fact will be exploited to study an application in Chapter 4.

Channel	Kraus operators	OSR
$\mathcal{D}(\kappa;0)$	$T_\ell(\kappa)=\sum_{n=0}^{\ell}(\sqrt{1+\kappa^2})^{-(n+1)}(\sqrt{1+\kappa^{-2}})^{-(\ell-n)}\sqrt{{}^{\ell}C_n}\lvert\ell-n\rangle\langle n\rvert$	$\sum_{\ell=0}^{\infty}T_\ell(\kappa)(\cdot)T_\ell(\kappa)^\dagger$
$\mathcal{C}_1(\kappa;0)$	$B_\ell(\kappa)=\sum_{m=0}^{\infty}\sqrt{{}^{m+\ell}C_\ell}(\sqrt{1-\kappa^2})^\ell\kappa^m\lvert m\rangle\langle m+\ell\rvert$	$\sum_{\ell=0}^{\infty}B_\ell(\kappa)(\cdot)B_\ell(\kappa)^\dagger$
$\mathcal{C}_2(\kappa;0)$	$A_\ell(\kappa)=\kappa^{-1}\sum_{m=0}^{\infty}\sqrt{{}^{m+\ell}C_\ell}\left(\sqrt{1-\kappa^{-2}}\right)^\ell(\kappa^{-1})^m\lvert m+\ell\rangle\langle m\rvert$	$\sum_{\ell=0}^{\infty}A_\ell(\kappa)(\cdot)A_\ell(\kappa)^\dagger$
$\mathcal{A}_1(0)$	$B_k=\lvert 0\rangle\langle k\rvert$	$\sum_k B_k(\cdot)B_k^\dagger$
$\mathcal{A}_2(0)$	$V_q=\lvert q/\sqrt{2}\rangle\langle q\rvert$	$\int dq\,V_q(\cdot)V_q^\dagger$
$\mathcal{B}_2(a)$	$D_\alpha=(\pi a)^{-1/2}\exp[-\lvert\alpha\rvert^2/2a]\,\mathcal{D}(\alpha)$	$\int d^2\alpha\,D_\alpha(\cdot)D_\alpha^\dagger$
$\mathcal{B}_1(a)$	$Z_q\equiv(\pi a)^{-1/4}\exp[-q^2/2a]\,\mathcal{D}(q/\sqrt{2})$	$\int dq\,Z_q(\cdot)Z_q^\dagger$

Table 1.2: Showing the OSR of the quantum-limited bosonic Gaussian channels and the classical noise channels.

1.8.2 Operator sum representation

We now briefly touch upon the operator sum representation of single-mode bosonic Gaussian channels [76]. The operator sum representation was obtained by considering the unitary representation of Gaussian channels. The system is first appended with a fixed Gaussian environment state (vacuum state for example), then the joint system is evolved through a two-mode Gaussian unitary transformation, and finally the environment mode is traced out in a suitable basis (Fock states for example) to obtain the resulting Kraus operators.

The Kraus operators thus constructed in [76] for all the quantum-limited channels and the classical noise channels is presented in Table 1.2. We see that in this representation, the beamsplitter, amplifier and phase conjugation channels have a discrete index Kraus representation whereas the classical noise channels and the singular channels have a continuous index Kraus representation.

Of particular interest to us is the action of the beamsplitter and amplifier channels on the Fock basis. We first consider the quantum-limited beamsplitter channel. We wish to consider the action of the channel on the operator basis consisting of the Fock operators $\{\lvert m\rangle\langle n\rvert\}$. From Table 1.1, we see that the action of the quantum-limited beamsplitter

channel on a general operator $|m\rangle\langle n|$ is given by :

$$|m\rangle\langle n| \rightarrow \sum_{\ell=0}^{\infty} B_\ell(\kappa)|m\rangle\langle n|B_\ell^\dagger(\kappa)$$
$$= \sum_{\ell=0}^{\min\{m,n\}} \sqrt{{}^mC_\ell\, {}^nC_\ell}\,(1-\kappa^2)^\ell \kappa^{m+n-2\ell}|m-\ell\rangle\langle n-\ell|. \qquad (1.178)$$

We see that the resulting operator is of finite rank. Further, for an input Fock state $|n\rangle\langle n|$, the output consists of all Fock state projectors up to the value n.

A similar analysis of the action of the quantum-limited amplifier channel on the operator $|m\rangle\langle n|$ leads to :

$$|m\rangle\langle n| \rightarrow \sum_{\ell=0}^{\infty} A_\ell(\kappa)|m\rangle\langle n|A_\ell^\dagger(\kappa)$$
$$= \kappa^{-2}\kappa^{-(n+m)} \sum_{\ell=0}^{\infty} \sqrt{{}^{n+\ell}C_\ell\, {}^{m+\ell}C_\ell}\,(1-\kappa^{-2})^\ell\, |m+\ell\rangle\langle n+\ell|. \qquad (1.179)$$

In contrast to the quantum-limited beamsplitter case, we see that the output operator is of infinite rank.

1.8.3 Semigroup property

It is clear from Table 1.1 (action in phase space) that successive actions of two quantum-limited beamsplitter channels with parameter values κ_1, κ_2 is a quantum-limited beamsplitter channel whose parameter κ equals the product $\kappa_1\kappa_2$ of the individual channel parameters :

$$C_1(\kappa_1) : \ \chi_W(\xi) \rightarrow \chi'_W(\xi) = \chi_W(\kappa_1\,\xi)\ \exp\,[-(1-\kappa_1^2)|\xi|^2/2],$$
$$C_1(\kappa_2) : \ \chi'_W(\xi) \rightarrow \chi''_W(\xi) = \chi'_W(\kappa_2\,\xi)\ \exp\,[-(1-\kappa_2^2)|\xi|^2/2]$$
$$= \chi_W(\kappa_1\kappa_2\,\xi)\ \exp\,[-(1-\kappa_1^2\kappa_2^2)|\xi|^2/2]. \qquad (1.180)$$

It is instructive to see how this semigroup property emerges in the Kraus representation. Let $\{B_{\ell_1}(\kappa_1)\}$ and $\{B_{\ell_2}(\kappa_2)\}$ be the Kraus operators of the two channels. The product of two Kraus operators $B_{\ell_1}(\kappa_1)$, $B_{\ell_2}(\kappa_2)$, one from each set, is

$$B_{\ell_1}(\kappa_1)B_{\ell_2}(\kappa_2) = \sum_{m=0}^{\infty} \sqrt{{}^{\ell_1+\ell_2}C_{\ell_1}} \left(\sqrt{1-\kappa_1^2}\right)^{\ell_1} \left(\sqrt{1-\kappa_2^2}\right)^{\ell_2} \times \sqrt{{}^{m+\ell_1+\ell_2}C_{\ell_1+\ell_2}}\, (\kappa_1\kappa_2)^m\, \kappa_2^{\ell_1} |m\rangle\langle m+\ell_1+\ell_2|. \tag{1.181}$$

The action of the product channel on the input operator $|r\rangle\langle r+\delta|$ is

$$\begin{aligned}
&\sum_{\ell_1,\ell_2} B_{\ell_1}(\kappa_1)B_{\ell_2}(\kappa_2)|r\rangle\langle r+\delta|B_{\ell_2}(\kappa_2)^\dagger\, B_{\ell_1}(\kappa_1)^\dagger \\
&= \sum_{\ell_1,\ell_2,m,n} \sqrt{{}^{\ell_1+\ell_2}C_{\ell_1}} \left(\sqrt{1-\kappa_1^2}\right)^{\ell_1} \left(\sqrt{1-\kappa_2^2}\right)^{\ell_2} \sqrt{{}^{m+\ell_1+\ell_2}C_{\ell_1+\ell_2}}\, (\kappa_1\kappa_2)^m \kappa_2^{\ell_1} \\
&\quad \times \sqrt{{}^{\ell_1+\ell_2}C_{\ell_1}} \left(\sqrt{1-\kappa_1^2}\right)^{\ell_1} \left(\sqrt{1-\kappa_2^2}\right)^{\ell_2} \sqrt{{}^{n+\ell_1+\ell_2}C_{\ell_1+\ell_2}}\, (\kappa_1\kappa_2)^n \kappa_2^{\ell_1} \\
&\quad \times |m\rangle\langle m+\ell_1+\ell_2|r\rangle\langle r+\delta|n+\ell_1+\ell_2\rangle\langle n|.
\end{aligned} \tag{1.182}$$

Denoting $\ell_1+\ell_2=\ell$, the expression on the RHS of Eq. (1.182) becomes

$$\text{RHS} = \sum_{\ell=0}^{r} \sum_{\ell_1=0}^{\ell} \sum_{m,n=0}^{\infty} {}^{\ell}C_{\ell_1} \kappa_2^{2\ell_1} (1-\kappa_1^2)^{\ell_1} (1-\kappa_2^2)^{(\ell-\ell_1)} (\kappa_1\kappa_2)^{m+n} \times \sqrt{{}^{\ell+m}C_{\ell}\, {}^{\ell+n}C_{\ell}}\, \delta_{r,m+\ell}\, \delta_{r+\delta,n+\ell}\, |m\rangle\langle n|. \tag{1.183}$$

The sum over ℓ_1 is the binomial expansion of $[(1-\kappa_1^2)\kappa_2^2 + (1-\kappa_2^2)]^\ell = (1-\kappa_1^2\kappa_2^2)^\ell$ and, in addition, we have the constraints $m+\ell=r$ and $n+\ell=r+\delta$. With this the expression (1.183) reduces to

$$\text{RHS} = \sum_{\ell=0}^{r} (1-\kappa_1^2\kappa_2^2)^\ell \sqrt{{}^{r}C_{\ell}\, {}^{r+\delta}C_{\ell}}\, (\kappa_1\kappa_2)^{2r-2\ell+\delta} |r-\ell\rangle\langle r-\ell+\delta|. \tag{1.184}$$

Comparing Eqs. (1.184) and (1.178) we find that the expression in (1.184) is precisely the action of a quantum-limited attenuator channel with parameter $\kappa_1\kappa_2$. In other words, we

have that

$$\sum_{\ell_1,\ell_2=0}^{\infty} B_{\ell_1}(\kappa_1)B_{\ell_2}(\kappa_2)|r\rangle\langle r+\delta|B^{\dagger}_{\ell_2}(\kappa_2)B^{\dagger}_{\ell_1}(\kappa_1) = \sum_{\ell=0}^{\infty} B_{\ell}(\kappa_1\kappa_2)|r\rangle\langle r+\delta|B^{\dagger}_{\ell}(\kappa_1\kappa_2). \quad (1.185)$$

An identical result can be similarly obtained for the behaviour of $|r+\delta\rangle\langle r|$, and thus we have proved the semigroup property

$$C_1(\kappa_1)\circ C_1(\kappa_2) = C_1(\kappa_1\kappa_2). \quad (1.186)$$

We now analyze the composition of two quantum-limited amplifier channels, as in the beamsplitter channel case. It follows from the very definition of the amplifier channel that the composition of two quantum-limited amplifier channels with parameters κ_1 and κ_2 is also a quantum-limited amplifier channel with parameter $\kappa = \kappa_1\kappa_2 > 1$:

$$C_2(\kappa_2)\circ C_2(\kappa_1) : \chi_W(\xi) \rightarrow \chi_W^{'}(\xi) = \chi_W(\kappa_1\kappa_2\,\xi)\ \exp\left[-(\kappa_1^2\kappa_2^2-1)|\xi|^2/2\right]. \quad (1.187)$$

That is,

$$C_2(\kappa_2)\circ C_2(\kappa_1) = C_2(\kappa_1\kappa_2) = C_2(\kappa_1)\circ C_2(\kappa_2). \quad (1.188)$$

It will be instructive to examine how this fact emerges from the structure of the Kraus operators. Let the set $\{A_{\ell_1}(\kappa_1)\}$ be the Kraus operators of the first amplifier and let $\{A_{\ell_2}(\kappa_2)\}$ be that of the second. Then the product of a pair of Kraus operators, one from each set, is

$$A_{\ell_1}(\kappa_1)A_{\ell_2}(\kappa_2) = (\kappa_1\kappa_2)^{-1}\sqrt{\ell_1+\ell_2 C_{\ell_1}}\sum_{n=0}^{\infty}\sqrt{n+\ell_1+\ell_2 C_{\ell_1+\ell_2}}\left(\sqrt{1-\kappa_1^{-2}}\right)^{\ell_1}$$
$$\times\left(\sqrt{1-\kappa_2^{-2}}\right)^{\ell_2}(\kappa_1\kappa_2)^{-n}\kappa_1^{-\ell_2}|n+\ell_1+\ell_2\rangle\langle n|. \quad (1.189)$$

Thus, under the successive action of these two amplifier channels the operator $|j\rangle\langle j+\delta|$

goes to

$$\sum_{\ell_1,\ell_2} A_{\ell_1}(\kappa_1)A_{\ell_2}(\kappa_2)|j\rangle\langle j+\delta|A_{\ell_2}(\kappa_2)^\dagger A_{\ell_1}(\kappa_1)^\dagger$$
$$= (\kappa_1\kappa_2)^{-2} \sum_{\ell_1,\ell_2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}^{\ell_1+\ell_2}C_{\ell_1} \left(1-\kappa_1^{-2}\right)^{\ell_1} \left(1-\kappa_2^{-2}\right)^{\ell_2} (\kappa_1\kappa_2)^{-(n+m)}\kappa_1^{-2\ell_2}$$
$$\times \sqrt{{}^{n+\ell_1+\ell_2}C_{\ell_1+\ell_2}\ {}^{m+\ell_1+\ell_2}C_{\ell_1+\ell_2}}|n+\ell_1+\ell_2\rangle\langle n|j\rangle\langle j+\delta|m\rangle\langle m+\ell_1+\ell_2|. \quad (1.190)$$

Denoting $\ell_1+\ell_2=\ell$, the RHS of the expression in Eq. (1.190) reduces to

$$(\kappa_1\kappa_2)^{-2} \sum_{\ell=0}^{\infty}\sum_{\ell_1=0}^{\ell}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}^{\ell}C_{\ell_1} (1-\kappa_1^{-2})^{\ell_1} (\kappa_1^{-2}(1-\kappa_2^{-2}))^{(\ell-\ell_1)} (\kappa_1\kappa_2)^{-(n+m)}$$
$$\times \sqrt{{}^{n+\ell}C_{\ell}\ {}^{m+\ell}C_{\ell}}\ \delta_{m,j+\delta}\,\delta_{n,j}\,|n+\ell\rangle\langle n+\delta+\ell|. \quad (1.191)$$

As in the beamsplitter case, the summation over the index ℓ_1 is a binomial expansion, and the expression in Eq. (1.191) reduces to

$$(\kappa_1\kappa_2)^{-2} \sum_{\ell=0}^{\infty} (1-\kappa_1^{-2}\kappa_2^{-2})^{\ell} (\kappa_1\kappa_2)^{-(j+j+\delta)} \sqrt{{}^{j+\ell}C_{\ell}\ {}^{j+\ell+\delta}C_{\ell}}\,|j+\ell\rangle\langle j+\ell+\delta|. \quad (1.192)$$

Comparing Eqs. (1.192) and (1.179), we see that the latter is the Kraus representation for a single quantum-limited amplifier channel. That is,

$$\sum_{\ell_1,\ell_2} A_{\ell_1}(\kappa_1)A_{\ell_2}(\kappa_2)|j\rangle\langle j+\delta|A_{\ell_2}(\kappa_2)^\dagger A_{\ell_1}(\kappa_1)^\dagger = \sum_{\ell} A_{\ell}(\kappa_1\kappa_2)|j\rangle\langle j+\delta|A_{\ell}(\kappa_1\kappa_2)^\dagger. \quad (1.193)$$

A similar behaviour holds for $|j+\delta\rangle\langle j|$ as well. And this is what we set out to demonstrate.

1.8.4 Noisy channels from quantum-limited ones

Our considerations so far have been in respect of quantum-limited channels. We turn our attention now to the case of noisy channels. It turns out that every noisy channel, except

$\mathcal{B}_1(a)$ which corresponds to injection of classical noise in just one quadrature, can be realised (in a non-unique way) as composition of two quantum-limited channels, so that the Kraus operators are products of those of the constituent quantum-limited channels.

We have noted already in the previous subsection that the composition of two quantum-limited attenuator (or amplifier) channels is again a quantum-limited attenuator (or amplifier) channel. This special semigroup property however does not obtain under composition for other quantum-limited channels. In general, composition of two quantum-limited channels results in a channel with additional classical noise. We will now consider pairs of quantum-limited channels from Table 1.1 and construct the Kraus operators of the resulting noisy channel.

The composite $C_2(\kappa_2;0)\circ C_1(\kappa_1;0),\ \kappa_2>1,\ \kappa_1<1$

It is clear from the very definition of these channels through their action on the characteristic function that the composite $C_2(\kappa_2;0)\circ C_1(\kappa_1;0)$ is a noisy amplifier, a classical noise channel, or a noisy attenuator depending on the numerical value of $\kappa_2\kappa_1$: it equals $C_1(\kappa_2\kappa_1;2(\kappa_2^2-1))$ for $\kappa_2\kappa_1<1$, $\mathcal{B}_2(2(\kappa_2^2-1))$ for $\kappa_1\kappa_2=1$, and $C_2(\kappa_2\kappa_1;2\kappa_2^2(1-\kappa_1^2))$ for $\kappa_2\kappa_1>1$, as may be readily read off from Table 1.3.

The Kraus operators for the composite is given by the set $\{A_m(\kappa_2)B_n(\kappa_1)\}$ with (m,n) running independently over the range $0\leq m,n<\infty$. By computing the products $A_m(\kappa_2)B_n(\kappa_1)$,

$X, Y \rightarrow$ $\downarrow$	$\mathcal{D}(\kappa_1; 0)$	$\mathcal{C}_1(\kappa_1; 0)$	$\mathcal{C}_2(\kappa_1; 0)$	$\mathcal{A}_2(0)$
$\mathcal{D}(\kappa_2; 0)$	$\mathcal{C}_1(\kappa_2\kappa_1; 2\kappa_2^2(1+\kappa_1^2))$, for $\kappa_2\kappa_1 < 1$. $\mathcal{C}_2(\kappa_2\kappa_1; 2(1+\kappa_2^2))$, for $\kappa_2\kappa_1 > 1$. $\mathcal{B}_2(2(1+\kappa_2^2))$, for $\kappa_2\kappa_1 = 1$.	$\mathcal{D}(\kappa_2\kappa_1; 2\kappa_2^2(1-\kappa_1^2)$	$\mathcal{D}(\kappa_2\kappa_1; 0)$	$\mathcal{A}_2(2\kappa_2^2)$
$\mathcal{C}_1(\kappa_2; 0)$	$\mathcal{D}(\kappa_2\kappa_1; 0)$	$\mathcal{C}_1(\kappa_2\kappa_1; 0)$	$\mathcal{C}_1(\kappa_2\kappa_1; 2\kappa_2^2(\kappa_1^2-1))$, for $\kappa_2\kappa_1 < 1$. $\mathcal{C}_2(\kappa_2\kappa_1; 2(1-\kappa_2^2))$, for $\kappa_2\kappa_1 > 1$. $\mathcal{B}_2(2(1-\kappa_2^2))$ for $\kappa_2\kappa_1 = 1$	$\mathcal{A}_2(0)$
$\mathcal{C}_2(\kappa_2; 0)$	$\mathcal{D}(\kappa_2\kappa_1; 2(\kappa_2^2-1))$	$\mathcal{C}_1(\kappa_2\kappa_1; 2(\kappa_2^2-1))$, for $\kappa_2\kappa_1 < 1$. $\mathcal{C}_2(\kappa_2\kappa_1; 2\kappa_2^2(1-\kappa_1^2))$, for $\kappa_2\kappa_1 > 1$. $\mathcal{B}_2(2(\kappa_2^2-1))$. for $\kappa_2\kappa_1 = 1$.	$\mathcal{C}_2(\kappa_2\kappa_1; 0)$	$\mathcal{A}_2(2(\kappa_2^2-1))$
$\mathcal{A}_2(0)$	$\mathcal{A}_2(\sqrt{\kappa_1^2+2}-1)$	$\mathcal{A}_2(\sqrt{2-\kappa_1^2}-1)$	$\mathcal{A}_2(\kappa_1-1)$	$\mathcal{A}_2(\sqrt{2}-1)$

Table 1.3: Showing the composition $X \circ Y$ of quantum-limited channels X, Y *assumed to be in their respective canonical forms simultaneously*. The composition results, in several cases, in noisy channels thereby enabling description of noisy Gaussian channels, including the classical noise channel $\mathcal{B}_2(a)$, in terms of *discrete sets* of linearly independent Kraus operators.

we have,

$$
\begin{aligned}
A_{\ell+\delta}(\kappa_2)B_{\ell}(\kappa_1) &= \sum_{j=0}^{\infty} g_1(\delta)_{\ell j}|j+\delta\rangle\langle j|,\\
A_{\ell}(\kappa_2)B_{\ell+\delta}(\kappa_1) &= \sum_{j=0}^{\infty} \tilde{g}_1(\delta)_{\ell j}|j\rangle\langle j+\delta|,\\
g_1(\delta)_{\ell j} &= \kappa_2^{-1}\left(\kappa_2^{-1}\kappa_1\right)^{j-\ell}\sqrt{{}^{j+\delta}C_{\ell+\delta}\,{}^{j}C_{\ell}}\\
&\quad\times\left(\sqrt{1-\kappa_1^2}\right)^{\ell}\left(\sqrt{1-\kappa_2^{-2}}\right)^{\ell+\delta},\ \text{for } j\geq\ell,\\
&= 0,\ \text{for } j<\ell;\\
\tilde{g}_1(\delta)_{\ell j} &= \kappa_2^{-1}\left(\kappa_2^{-1}\kappa_1\right)^{j-\ell}\sqrt{{}^{j+\delta}C_{\ell+\delta}\,{}^{j}C_{\ell}}\\
&\quad\times\left(\sqrt{1-\kappa_2^{-2}}\right)^{\ell}\left(\sqrt{1-\kappa_1^2}\right)^{\ell+\delta},\ \text{for } j\geq\ell,\\
&= 0,\ \text{for } j<\ell.
\end{aligned}
\tag{1.194}
$$

The composite $C_1(\kappa_2;0)\circ C_2(\kappa_1;0),\ \kappa_2<1,\ \kappa_1>1$

Again the composite $C_1(\kappa_2;0)\circ C_2(\kappa_1;0)$ is a noisy amplifier, a classical noise channel, or a noisy attenuator depending on the numerical value of $\kappa_2\kappa_1$ and the details may be read off from Table 1.3. The Kraus operators for the composite $C_1(\kappa_2;0)\circ C_2(\kappa_1;0)$ are given by $\{B_m(\kappa_2)A_n(\kappa_1)\}$, $0\leq m,n<\infty$. We have

$$
\begin{aligned}
B_{\ell+\delta}(\kappa_2)A_{\ell}(\kappa_1) &= \sum_{j=0}^{\infty} g_2(\delta)_{\ell j}|j\rangle\langle j+\delta|,\\
B_{\ell}(\kappa_2)A_{\ell+\delta}(\kappa_1) &= \sum_{j=0}^{\infty} \tilde{g}_2(\delta)_{\ell j}|j+\delta\rangle\langle j|,\\
g_2(\delta)_{\ell j} &= \kappa_1^{-1}\sqrt{{}^{j+\ell+\delta}C_{\ell+\delta}\,{}^{j+\delta+\ell}C_{\ell}}\,\kappa_1^{-(j+\delta)}\left(\sqrt{1-\kappa_1^{-2}}\right)^{\ell}\kappa_2^{j}\left(\sqrt{1-\kappa_2^{2}}\right)^{\ell+\delta},\\
\tilde{g}_2(\delta)_{\ell j} &= \kappa_1^{-1}\sqrt{{}^{j+\ell+\delta}C_{\ell+\delta}\,{}^{j+\delta+\ell}C_{\ell}}\,\kappa_1^{-j}\left(\sqrt{1-\kappa_1^{-2}}\right)^{\ell+\delta}\kappa_2^{j+\delta}\left(\sqrt{1-\kappa_2^{2}}\right)^{\ell}.
\end{aligned}
\tag{1.195}
$$

The composite $\mathcal{D}(\kappa_2) \circ \mathcal{D}(\kappa_1),\ \kappa_2 > 0,\ \kappa_1 > 0$

Similar to the earlier two cases, the composite $\mathcal{D}(\kappa_2; 0) \circ \mathcal{D}(\kappa_1; 0)$ is a noisy amplifier, a classical noise channel, or a noisy attenuator depending on the numerical value of $\kappa_2\kappa_1$, as in the earlier two cases, and the details can be read off from Table 1.3. It may be noted, again from Table 1.3, that this case tends to be more noisy than the earlier two cases.

The Kraus operators for this composite are given by $\{T_m(\kappa_2)T_n(\kappa_1)\},\ 0 \leq\ m, n < \infty$. The products $T_m(\kappa_2)T_n(\kappa_1)$ have the form

$$
\begin{aligned}
T_{\ell+\delta}(\kappa_2)T_\ell(\kappa_1) &= \sum_{j=0}^{\infty} g_3(\delta)_{\ell j}|j+\delta\rangle\langle j|, \\
T_\ell(\kappa_2)T_{\ell+\delta}(\kappa_1) &= \sum_{j=0}^{\infty} \tilde{g}_3(\delta)_{\ell j}|j+\delta\rangle\langle j|, \\
g_3(\delta)_{\ell j} &= \left(\sqrt{1+\kappa_1^2}\right)^{-1}\left(\sqrt{1+\kappa_2^2}\right)^{-1}\ \sqrt{{}^{\ell}C_j\ {}^{\ell+\delta}C_j}\left[\sqrt{(1+\kappa_2^2)(1+\kappa_1^{-2})}\right]^{-(\ell-j)} \\
&\quad \times\left[\sqrt{(1+\kappa_1^2)(1+\kappa_2^{-2})}\right]^{-j}\left(\sqrt{1+\kappa_2^{-2}}\right)^{-\delta},\ \text{ for } j \leq \ell, \\
&= 0,\ \text{ for } j > \ell, \\
\tilde{g}_3(\delta)_{\ell j} &= \left(\sqrt{1+\kappa_1^2}\right)^{-1}\left(\sqrt{1+\kappa_2^2}\right)^{-1}\ \sqrt{{}^{\ell}C_j\ {}^{\ell+\delta}C_j}\left[\sqrt{(1+\kappa_2^2)(1+\kappa_1^{-2})}\right]^{-(\ell-j)} \\
&\quad \times\left[\sqrt{(1+\kappa_1^2)(1+\kappa_2^{-2})}\right]^{-j}\left(\sqrt{1+\kappa_1^{2}}\right)^{-\delta},\ \text{ for } j \leq \ell, \\
&= 0,\ \text{ for } j > \ell.
\end{aligned}
\tag{1.196}
$$

The composite $\mathcal{D}(\kappa_2;0)\circ C_1(\kappa_1;0),\ \kappa_2>0,\ \kappa_1<1$

Kraus operators of this composite, which always corresponds to a noisy transpose channel (see Table 1.3), are $\{T_m(\kappa_2)B_n(\kappa_1)\},\ 0\le m,n<\infty$. We have

$$\begin{aligned} T_m(\kappa_2)B_n(\kappa_1) &= \sum_{j=0}^{\infty}\xi_{mn}^{j}|m-j\rangle\langle n+j| \\ \xi_{mn}^{j} &= \left(\sqrt{1+\kappa_2^2}\right)^{-1}\sqrt{{}^{m}C_{j}\,{}^{n+j}C_{j}}\left(\sqrt{1+\kappa_2^2}\right)^{-j}\left(\sqrt{1+\kappa_2^{-2}}\right)^{-(m-j)} \\ &\quad\times\kappa_1^{j}\left(\sqrt{1-\kappa_1^2}\right)^{n},\ \text{for } j\le m; \\ &= 0,\ \text{for } j>m. \end{aligned} \tag{1.197}$$

The composite $C_1(\kappa_2;0)\circ\mathcal{D}(\kappa_1;0),\ \kappa_1>0,\ \kappa_2<1$

This composite channel corresponds to a *quantum-limited* transpose channel (see Table 1.3). The Kraus operators $\{B_m(\kappa_2)T_n(\kappa_1)\},\ 0\le m,n<\infty$ (which as a set should be equivalent to $\{T_\ell(\kappa_2\kappa_1)\},\ 0\le\ell<\infty$), are

$$\begin{aligned} B_m(\kappa_2)T_n(\kappa_1) &= \sum_{j=m}^{n}\xi_{mn}^{j}|j-m\rangle\langle n-j|, \\ \xi_{mn}^{j} &= \sqrt{{}^{j}C_{m}\,{}^{n}C_{j}}\left(\sqrt{1-\kappa_2^2}\right)^{m}\kappa_2^{j-m}\left(\sqrt{1+\kappa_1^2}\right)^{-(n-j+1)} \\ &\quad\times\left(\sqrt{1+\kappa_1^{-2}}\right)^{-j},\ \text{for } n\ge m; \\ &= 0,\ \text{for } n<m. \end{aligned} \tag{1.198}$$

The composite $C_2(\kappa_2;0)\circ\mathcal{D}(\kappa_1;0),\ \ \kappa_2>1,\ \kappa_1>0$

This composite channel corresponds, for all κ_1,κ_2, to a noisy transpose channel, similar to the case of $\mathcal{D}(\kappa_2;0)\circ C_1(\kappa_1;0)$ considered earlier. The Kraus operators $\{A_m(\kappa_2)T_n(\kappa_1)\},\ 0\le$

$m, n < \infty$ have the form

$$
\begin{aligned}
A_m(\kappa_2)T_n(\kappa_1) &= \sum_{j=0}^{n} \xi_{mn}^{j} |j+m\rangle\langle n-j|, \\
\xi_{mn}^{j} &= \kappa_2^{-1}\left(\sqrt{1+\kappa_1^2}\right)^{-1} \sqrt{{}^{m+j}C_j \, {}^{n}C_j}\left(\sqrt{1-\kappa_2^{-2}}\right)^{m} \kappa_2^{-j} \\
&\quad \times \left(\sqrt{1+\kappa_1^2}\right)^{-(n-j)} \left(\sqrt{1+\kappa_1^{-2}}\right)^{-j}, \text{ for } j \leq n; \\
&= 0, \text{ for } j > n.
\end{aligned} \tag{1.199}
$$

The composite $\mathcal{D}(\kappa_2; 0) \circ C_2(\kappa_1; 0),\ \kappa_2 > 0,\ \kappa_1 > 1$

This composite is a *quantum-limited* transpose channel (see Table 1.3), with Kraus operators $\{T_m(\kappa_2)A_n(\kappa_1)\},\ 0 \leq m, n < \infty$. The product Kraus operators are computed as

$$
\begin{aligned}
T_m(\kappa_2)A_n(\kappa_1) &= \sum_{k=0}^{m} \xi_{mn}^{k} |m-k\rangle\langle k-n|, \\
\xi_{mn}^{k} &= \left(\sqrt{1+\kappa_2^2}\right)^{-(k+1)} \left(\sqrt{1+\kappa_2^{-2}}\right)^{-(\ell-k)} \sqrt{{}^{\ell}C_k} \\
&\quad \times \kappa_1^{-1} \sqrt{{}^{k}C_n}\left(\sqrt{1-\kappa_1^{-2}}\right)^{n} (\kappa_1^{-1})^{k-n}, \text{ for } k > n, \\
&= 0, \text{ for } k < n.
\end{aligned} \tag{1.200}
$$

Remark : We wish to make a final remark regarding the Kraus operators for the composite channels obtained as the product of the Kraus operators of the quantum-limited channels as detailed above. The Kraus operators for the composites in Eqs. (1.194), (1.195), (1.196), (1.197), (1.199) were shown to be linearly independent in Ref. [76]. However, the Kraus operators in (1.198) and (1.200), are linearly dependent. Nevertheless, they still give rise to a valid operator sum representation for the corresponding composite channels.

Channel	EB region	Y
$C_1(\kappa;\alpha)$	$\alpha \geq 2\kappa^2$	$Y \geq (\kappa^2+1)\,\mathbb{1}$
$C_2(\kappa;\alpha)$	$\alpha \geq 2$	$Y \geq (\kappa^2+1)\,\mathbb{1}$
$\mathcal{D}(\kappa;\alpha)$	$\alpha \geq 0$	$Y \geq (\kappa^2+1)\,\mathbb{1}$
$\mathcal{A}_2((1+\sigma_2)/2;\alpha)$	$\alpha \geq 0$	$Y \geq \mathbb{1}$

Table 1.4: Showing the EB bosonic Gaussian channels.

1.9 Entanglement-breaking bosonic Gaussian channels

We now consider the important notion of entanglement-breaking bosonic Gaussian channels. We recall (1.120) that a channel Γ *acting on system* S is entanglement-breaking if the bipartite output state $(\Gamma \otimes \mathbb{1}_E)(\hat{\rho}_{SE})$ is separable for every input state $\hat{\rho}_{SE}$, the ancilla system E being arbitrary [77].

A bosonic Gaussian channel is said to be entanglement-breaking if its one-sided action on a two-mode state is separable for all input bipartite states. It turns out that for single-mode bosonic Gaussian channels, the entanglement-breaking condition can be written down compactly by resorting to Simon's criterion [89].

A single-mode bosonic Gaussian channel Φ is said to EB if and only if $T \circ \Phi$ is also a channel, where T stands for the transpose operation. By Eq. (1.176) we have that

$$\begin{aligned} &\Lambda\, Y\, \Lambda + i\beta \geq i\,\Lambda\, X^T \beta\, X\, \Lambda \\ \implies\ &Y - i\beta \geq i\, X^T \beta\, X, \end{aligned} \tag{1.201}$$

where, as noted in Eq. (1.165), Λ transcribes for the transpose map. We add that this requirement is in addition to the constraint satisfied by (X, Y) for Φ to be a channel in Eq. (1.176). Further, we note that if a given channel (X, Y) is EB, then adding additional classical noise will also result in an EB channel.

Using the criterion provided in Eq.(1.201), we classify the EB Gaussian channels [96] for each of the canonical forms and tabulate then in Table 1.4. We see that the quantum-

limited phase conjugation channels $\mathcal{D}(\kappa;0)$ and singular channels $\mathcal{A}_2(0)$ are already entanglement breaking. Hence, these classes of channels are always entanglement breaking irrespective of the noise. One other quantum-limited channel that is EB is the $\kappa = 0$ end of the attenuator channel $C_1(\kappa;0)$, i.e. $C_1(0,0)$. The noisy channels $C_1(\kappa;\alpha)$ and $C_2(\kappa;\alpha)$ are EB for $\alpha \geq 2\kappa^2$ and 2 respectively. We will explore more properties of EB Gaussian channels in Chapter 5.