Some aspects of the interplay between bipartite correlations and quantum channels

By

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PHYS10200605008

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A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

July, 2013

Homi Bhabha National Institute

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Krishna Kumar Sabapathy

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier in whole or in part for a degree / diploma at this or any other Institution / University.

Krishna Kumar Sabapathy

DEDICATIONS

To my parents and my well-wishers

ACKNOWLEDGEMENTS

This thesis has been written with an enormous amount of help from people around me. It is not without reason that Google scholar chose the tag line ‘Stand on the shoulders of giants’. This quote seems apt in the present context.

It has been a long journey and I am grateful first and foremost to my adviser **Prof. R. Simon**. Most of what I have learnt in my stay here is from various forms of interactions and discussions from him. I also wish to thank him for the interesting topics and subjects that I learnt and was exposed to in my study. It has been an honour to be able to get this opportunity and I am sure it has and continues to play an important role in my academic path.

I would like to thank **Prof. Mukunda** for the many inspiring lectures that he has delivered over the last few years that I had the privilege to attend and have greatly benefited from. I have learnt a lot from him right from my undergraduate days and I am greatly indebted to him for many things. There are also a lot of attributes of him that I aspire to be able to inculcate in my academic journey.

An important role in the completion of this thesis was played by **Dr. Solomon Ivan**. I have learnt a lot from him through many hours of lengthy discussions over the last few years. He has been very gracious in his time and helped me along through my stay here. I also wish to thank **Dr. Sibasish Ghosh** who has introduced me to many interesting topics and has been a constant source of inspiration in pursuing my academic interests. He has also provided valuable inputs that have helped me in my work at various junctures.

Finally, I would like to thank all my teachers at Matscience, my colleagues, and my batch mates who have helped me all along. I am also grateful to all the librarians who have been extremely helpful and patient with regards to many of my requests, and without whom this thesis would not be completed. I would also like to thank the administration and the persons incharge of the computing facilities, who have provided the support facilities and made my stay here as pleasant as possible.

List of Publications arising from the thesis

Journal

1. *Robustness of nonGaussian entanglement against noisy amplifier and attenuator environments*. **K. K. Sabapathy**, J. S. Ivan, and R. Simon.
Physical Review Letters, **107**, 130501 (2011).

2. *Nonclassicality breaking is the same as entanglement breaking for bosonic Gaussian channels*. J. S. Ivan, **K. K. Sabapathy**, and R. Simon.
Phys. Rev. A, **88**, 032302 (2013).

3. J. S. Ivan, **K. K. Sabapathy**, and R. Simon. *Operator-sum Representation for Bosonic Gaussian Channels*. Physical Review A, **84**, 042311 (2011).

Others (communicated)

1. *Quantum discord plays no distinguished role in characterization of complete positivity: Robustness of the traditional scheme*. **K. K. Sabapathy**, J. S. Ivan, S. Ghosh, and R. Simon. arXiv:1304.4857 [quant-ph].

2. *Quantum discord for two-qubit X-states: A comprehensive approach inspired by classical polarization optics* . **K. K. Sabapathy** and R. Simon.
arXiv:1311.0210 [quant-ph].

List of corrections and changes as suggested by the Thesis and Viva Voce Examiners

1. Added Refs. 35, 37, 38, 40, 41, 44, and 51 in the Introduction Chapter.
2. Added Refs. 147-150 on p.96 of Chapter 2.
3. The notions of critical noise and robustness of entanglement have been clarified in Definitions 1, 2, and 3 of Chapter 4.
4. Introduced Figs. 5.1, 5.2, and 5.4 in Chapter 5 to bring out in a transparent manner the similarities and the differences between nonclassicality breaking and entanglement breaking channels.
5. Corrected typos as suggested by the referees in all the Chapters.

The corrections and changes suggested by the Thesis and Viva Voce Examiners have been incorporated in the thesis.

Guide

Contents

Synopsis

This thesis explores ways in which quantum channels and correlations (of both classical and quantum types) manifest themselves, and also studies the interplay between these two aspects in various physical settings. Quantum channels represent all possible evolutions of states, including measurements, allowed by quantum mechanics, while correlations are intrinsic (nonlocal) properties of composite systems.

Given a quantum system with Hilbert space $\mathcal{H}_S$, states of this system are operators ρ that satisfy $\rho = \rho^\dagger$, $\rho \geq 0$, and $\mathrm{Tr}\,\rho = 1$. The set of all such operators (density matrices) constitute the (convex) state space $\Lambda(\mathcal{H}_S)$. Observables $\hat{O}$ are hermitian operators acting on $\mathcal{H}_S$. Let the spectral resolution of $\hat{O}$ be $\hat{O} = \sum_j \lambda_j P_j$. When $\hat{O}$ is measured, the j^{th} outcome corresponding to measurement operator P_j (projection) occurs with probability $p_j = \mathrm{Tr}(\rho\, P_j)$. One obtains a more general measurement scheme called POVM (positive operator valued measurement) when the projective measurement elements P_j are replaced by positive operators Π_j with $\sum_j \Pi_j = \mathbb{1}$, and the probabilities of outcomes are obtained in a similar manner : $p_j = \mathrm{Tr}(\rho \Pi_j)$.

If a system is isolated, then its dynamics is governed by the unitary Schrödinger evolution. A unitary operator U effects the following transformation $\rho \rightarrow \rho' = U \rho\, U^\dagger$, ρ and $\rho' \in \Lambda(\mathcal{H}_S)$. But if the system is in interaction with its environment, then the evolutions of the system of interest resulting from unitary evolutions of the composite are more general, but nevertheless described by linear maps acting on the state space, directly rather than through $\mathcal{H}_S$.

Let Φ be a linear map that acts on states of the system. An obvious necessary requirement for Φ to be a valid evolution is that it takes states to states. We call a map that satisfies this condition as a *positive map*, i.e.,

$$\Phi \text{ is positive} \Leftrightarrow \Phi(\rho_S) = \rho_S^{'} \in \Lambda(\mathcal{H}_S), \text{ i.e., } \Phi\left(\Lambda(\mathcal{H}_S)\right) \subset \Lambda(\mathcal{H}_S). \tag{1}$$

It turns out that not all positive maps are physical evolutions. For positive maps to be physical evolutions, there is a further requirement to be met.

Let us consider a composite system in which the system is appended with an arbitrary ancilla or reservoir R. The Hilbert space of the composite system is $\mathcal{H}_S \otimes \mathcal{H}_R$, a tensor product of the individual subsystem Hilbert spaces. Let us denote the state space of this composite system by $\Lambda(\mathcal{H}_S \otimes \mathcal{H}_R)$.

It is both reasonable and necessary to require that local action of Φ takes states of the joint system also to states. In other words

$$\begin{aligned}(\Phi \otimes \mathbb{1})[\rho_{SR}] &= \rho_{SR}^{'} \in \Lambda(\mathcal{H}_S \otimes \mathcal{H}_R), \\ \text{i.e., } [\Phi \otimes \mathbb{1}]\left(\Lambda(\mathcal{H}_S \otimes \mathcal{H}_R)\right) &\subset \Lambda(\mathcal{H}_S \otimes \mathcal{H}_R).\end{aligned} \tag{2}$$

A positive map Φ that satisfies Eq. (2), is known as a *completely positive* (CP) trace-preserving (TP) map or a **quantum channel**.

It is known that every CP map can be realised in the following way. First, the systems states are elevated to product states on a larger Hilbert space (system + environment), with a fixed state of the environment: $\rho_S \to \rho_S \otimes \rho_R$, ρ_R fixed. Then the product states are evolved by a joint unitary evolution, and finally the environment degrees of freedom are traced out to give the evolved system states. It turns out that this provides a suitable framework for the description of open quantum systems.

An intrinsic property of composite systems that is of much importance is correlations between subsystems. One important aspect has been to segregate the classical and quantum contents of correlations. To this end, various measures and methods have been proposed. Entanglement has been the most popular of these correlations owing to its inherent advantages in performing quantum computation and communication tasks [1] and has been studied over the last few decades. But there are other correlations that are motivated from an information-theoretic or measurement perspective, which try to capture this classical-quantum boundary [2]. These include quantum discord, classical correlation, measurement induced disturbance, quantum deficit, and geometric variants of these measures. Of these, quantum discord and classical correlation have received enormous attention in recent years.

Let us now consider a bipartite system with Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_S$, where the two subsystem Hilbert spaces have been taken to be identical for simplicity. A pure bipartite state is said to be separable if it can be written as a (tensor) product of states of the individual subsystems. Else, the pure state is said to be entangled. While, a mixed state $\rho_{AB} \in \Lambda(\mathcal{H}_S \otimes \mathcal{H}_S)$ is said to separable if it can be written as a convex combination of product states, i.e.,

$$\rho_{AB} = \sum_j p_j \rho_j^A \otimes \rho_j^B. \tag{3}$$

A state that cannot be written in this form is called an entangled state. The set of separable states form a convex subset of the bipartite state space. The qualitative detection and quantitative estimation of entanglement have proved to be non-trivial. To this end, there have been many approaches that include Bell-type inequalities, entanglement witnesses, entropy based measures, distance (geometry) based measures, and criteria based on positive maps that are not completely positive.

Quantum discord is a ‘beyond-entanglement’ quantum correlation, since there exist separable states which return a non-zero value of quantum discord. A recent avenue has been to try and find advantages of these correlations, both in the theoretical and experimental domain, in respect of information precessing tasks. For example, some interesting applications of quantum discord in quantum computation, state merging, remote state preparation, and entanglement distillation have been reported.

We may motivate the definition of quantum discord by first looking at the classical setting. Given a probability distribution $p(x, y)$ in two variables, the mutual information $I(x, y)$ is defined as

$$I(x, y) = H(x) - H(x|y), \tag{4}$$

where $H(\cdot)$ stands for the Shannon entropy $H(x) = -\sum_x p(x)\,\mathrm{Log}[p(x)]$ and $H(x|y)$ is the conditional entropy. Using Bayes rule we are lead to an equivalent expression for mutual information :

$$I(x, y) = H(x) + H(y) - H(x, y). \tag{5}$$

The second expression (5) for mutual information naturally generalises to the quantum setting when the bipartite probability distribution is replaced by a bipartite state ρ_{AB} and the Shannon entropy $H(\cdot)$ by the von Neumann entropy $S(\cdot)$ of quantum states, and we have

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{6}$$

But the first expression (4) for classical mutual information does *not* possess a straightforward generalization to the quantum case. In the quantum case, the conditional entropy is defined with respect to a measurement, where the measurement is performed on one of the subsystems, say subsystem B. Let us consider a POVM $\Pi^B = \{\Pi_j^B\}$ where $\Pi_j^B \geq 0$ and

$\sum_j \Pi_j^B = \mathbb{1}$. Then the conditional entropy post measurement is given by

$$S^A = \sum_j p_j S(\rho_j^A), \tag{7}$$

where the probabilities and states post measurement are given by

$$\begin{aligned} p_j &= \mathrm{Tr}\,(\Pi_j^B \rho_{AB}), \\ \rho_j^A &= p_j^{-1}\,\mathrm{Tr}_B(\Pi_j^B \rho_{AB}). \end{aligned} \tag{8}$$

Let us denote by $S^A_{\min}$ the minimum of S^A over all measurements or POVM's. The difference between these two classically equivalent expressions (optimized over all measurements) is called quantum discord $\mathcal{D}(\rho_{AB})$:

$$\begin{aligned} \mathcal{D}(\rho_{AB}) &= I(\rho_{AB}) - \left[S(\rho_A) - S^A_{\min}\right], \\ &= S(\rho_B) - S(\rho_{AB}) + S^A_{\min}. \end{aligned} \tag{9}$$

The quantity $C(\rho_{AB}) = S(\rho_A) - S^A_{\min}$ is defined as the **classical correlation**. Thus, the mutual information which is supposed to capture the total correlation of a bipartite state is broken down into quantum discord, that captures the quantum correlations, and classical correlation $C(\rho_{AB})$.

It is the interplay between correlations of bipartite states and their evolution through quantum channels that is the unifying theme of this thesis. We explore some aspects of this interplay in the different chapters. There are four broad topics that are covered in this thesis :

- Initial bipartite correlations and induced subsystem dynamics : Does initial correlation of the system-bath states provide a generalization of the folklore product realization of CP maps?
- A geometric approach to computation of quantum discord for two-qubit X-states.

- Robustness of nonGaussian vs Gaussian entanglement against noise : We demonstrate simple examples of nonGaussian states whose entanglement survives longer that Gaussian entanglement under noisy channels
- Is nonclassicality breaking the same thing as entanglement breaking? The answer is shown to be in the affirmative for bosonic Gaussian channels.

In **Chapter 1**, we provide a basic introduction to the concepts that are used in the thesis. In addition to setting up the notations, this Chapter helps to render the thesis reasonably self-contained. We describe the properties of bipartite correlations of interest to us, namely, classical correlation, quantum discord, and entanglement.

We briefly describe the notion of quantum channels. We discuss in some detail the three well-established representations of CP maps [3]. These are the operator-sum representation, the unitary representation, and the Choi-Jamiolkowski isomorphism between bipartite states and channels. We indicate how one can go from one representation to another.

We also indicate an operational way to check as to when a positive map be can called a CP map. We list some properties of channels and indicate when a channel is unital, dual, extremal, entanglement breaking, bistochastic, and so on.

We then move on to a discussion of states and channels in the continuous variable setting, in particular the Gaussian case. Here, we begin by recapitulating the properties of Gaussian states. The phase space picture in terms of quasiprobability distributions is outlined and some basic aspects of the symplectic structure is recalled. We will be mainly concerned with the Wigner distribution and its associated characteristic function. Gaussian states are completely described in terms of the variance matrices and means. For these states, we describe the uncertainty principle, the canonical form of the variance matrix, Simon's criterion for detecting entanglement of two-mode Gaussian states, and a description of the more commonly used Gaussian states like the vacuum state, thermal state, squeezed state, and coherent state.

Then we proceed to a discussion of single-mode Bosonic Gaussian channels. These are trace-preserving completely positive maps that take input Gaussian states to Gaussian states at the output. These channels play a fundamental role in continuous variable quantum information theory. We discuss their phase space description, the CP condition, and enumeration of their canonical forms.

In the standard classification, Bosonic Gaussian channels group themselves into five broad classes. Namely, the attenuator, amplifier, phase conjugation, singular channels and, finally, the classical noise channels. We then briefly describe the operator-sum representation [4] for all single-mode Gaussian channels.

Of particular importance to us is the analysis of quantum-limited channels. Quantum-limited channels are channels that saturate the CP condition and hence do not contain extra additive classical noise over and above the minimum demanded by the uncertainty principle. We emphasise the fact that noisy channels can be factored as product of a pair of noiseless or quantum-limited channels. The action in the Fock basis is brought out.

The attenuator channel and the amplifier channel are of particular interest to us, and so we bring out some of its properties including the semigroup structure of the amplifier and the attenuator families of quantum-limited channels. The noisy versions of these channels can be easily obtained by composition of a pair of quantum-limited channels; this is explicitly shown for all channels and tabulated. In particular, we obtain a discrete operator-sum representation for the classical noise channel which may be contrasted with the familiar one in terms of a continuum of Weyl displacement operators. These representations lead to an interesting application which is pursued in Chapter 4. This Introductory chapter renders the passage to the main results of the thesis in subsequent chapters smooth.

In **Chapter 2** we consider the dynamics of a system that is in interaction with an environment, or in other words, the dynamics of an open quantum system.

Dynamics of open quantum systems is fundamental to the study of any realistic or practical application of quantum systems. Hence, there has been a rapidly growing interest in the understanding of various properties related to open quantum systems like its realization, control, and the role played by the noise in such dissipative systems, both in the theoretical and experimental domain. These studies have been motivated by applications to quantum computing, laser cooling, quantum reservoir engineering, managing decoherence, and also to other fields like chemical reactions and energy transfer in molecules.

Here we study the induced dynamics of a system viewed as part of a larger composite system, when the system plus environment undergoes a unitary evolution. Specifically, we explore the effect of initial system-bath correlations on complete positivity of the reduced dynamics.

Traditional (Folklore) Scheme : In the folklore scheme, initial system states ρ_S are elevated to product states of the composite, for a *fixed* fiducial bath state ρ_B^{fid}, through the assignment map $\rho_S \to \rho_S \otimes \rho_B^{\mathrm{fid}}$. These uncorrelated system-bath states are evolved under a joint unitary $U_{SB}(t)$ to $U_{SB}(t)\rho_S \otimes \rho_B^{\mathrm{fid}}\, U_{SB}(t)^\dagger$ and, finally, the bath degrees of freedom are traced out to obtain the time-evolved states of the system of interest :

$$\rho_S \to \rho_S(t) = \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\rho_\mathrm{S} \otimes \rho_\mathrm{B}^{\mathrm{fid}}\, \mathrm{U_{SB}(t)}^\dagger\right]. \tag{10}$$

The resulting quantum dynamical process (QDP) $\rho_S \to \rho_S(t)$, parametrized by ρ_B^{fid} and $U_{SB}(t)$, is completely positive by construction.

Currently, however, the issue of system-bath initial correlations potentially affecting the reduced dynamics of the system has been attracting considerable interest. A specific, carefully detailed, and precise formulation of the issue of initial system-bath correlations possibly influencing the reduced dynamics was presented not long ago by Shabani and Lidar (SL) [5].

Shabani-Lidar scheme : In sharp contrast to the folklore scheme, there is no assignment map in the SL scheme. The distinguished bath state ρ_B^{fid} is replaced by a collection Ω^{SB} of (possibly correlated) system-bath initial states $\rho_{SB}(0)$. The dynamics gets defined through

$$\rho_{SB}(0) \rightarrow \rho_{SB}(t) = U_{SB}(t)\,\rho_{SB}(0)\,U_{SB}(t)^{\dagger}, \tag{11}$$

for all $\rho_{SB}(0) \in \Omega^{SB}$. With reduced system states $\rho_S(0)$ and $\rho_S(t)$ defined through the imaging or projection map $\rho_S(0) = \mathrm{Tr_B}\,\rho_{\mathrm{SB}}(0)$ and $\rho_S(t) = \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\,\rho_{\mathrm{SB}}(0)\,\mathrm{U_{SB}(t)}^{\dagger}\right]$, this unitary dynamics of the composite induces on the system the QDP $\rho_S(0) \rightarrow \rho_S(t)$.

Whether the SL QDP so described is well-defined and completely positive is clearly an issue answered solely by the nature of the collection Ω^{SB}. It is evident that the folklore scheme obtains as a special case of the SL scheme. This generalized formulation of QDP allows SL to transcribe the fundamental issue to this question: What are the necessary and sufficient conditions on the collection of initial states so that the induced QDP is guaranteed to be CP *for all* joint unitaries?

Motivated by the work of Rodriguez-Rosario et al. [6], SL advance the following resolution to this issue: *The QDP is CP for all joint unitaries if and only if the quantum discord vanishes for all initial system-bath states* $\in \Omega^{SB}$ *, i.e., if and only if the initial system-bath correlations are purely classical*. The SL theorem has come to be counted among the more important recent results of quantum information theory, and it is paraded by many authors as one of the major achievements of quantum discord. Meanwhile, the very recent work of Brodutch et al. [7], contests the claim of SL and asserts that vanishing quantum discord is sufficient but not necessary condition for complete positivity.

Our entire analysis in Chapter 2 rests on two, almost obvious, necessary properties of the set of initial system-bath states Ω^{SB} so that the resulting SL QDP would be well defined.
Property 1: No state $\rho_S(0)$ can have two (or more) pre-images in Ω^{SB}.
Property 2: While every system state need not have a pre-image *actually enumerated* in Ω^{SB}, the set of $\rho_S(0)$'s having pre-image in Ω^{SB} should be sufficiently large, such that the QDP can be extended by linearity to all states of the system, i.e., to the full state space of

the system.

Using these two requirements, we prove that *both the SL theorem and the assertion of Brodutch et al. are too strong to be tenable*. We labour to point out that rather than viewing this result as a negative verdict of the SL theorem, it is more constructive to view our result as demonstrating a kind of robustness of the traditional scheme.

In **Chapter 3** we undertake a comprehensive analysis of the problem of computation of correlations in two-qubit systems, especially the so-called X-states which have come to be accorded a distinguished status in this regard. Our approach exploits the very geometric nature of the problem, and clarifies some issues regarding computation of correlations in X-states. It may be emphasised that the geometric methods used here have been the basic tools of (classical) polarization optics for a very long time, and involve constructs like Stokes vectors, Poincaré sphere, and Mueller matrix [8].

As noted earlier, the expressions for quantum discord and classical correlation are

$$\begin{aligned} \mathcal{D}(\rho_{AB}) &= S(\rho_B) - S(\rho_{AB}) + S^A_{\min}, \\ C(\rho_{AB}) &= S(\rho_A) - S^A_{\min}. \end{aligned} \tag{12}$$

It is seen that the only term that requires an optimization is the conditional entropy post measurement, $S^A_{\min}$. Given a composite state ρ_{AB}, the other entropic quantities are immediately evaluated. Central to the simplicity and comprehensiveness of our analysis is the recognition that computation of $S^A_{\min}$ for two-qubit X-states is a *one-parameter optimization problem*, much against the impression given by a large section of the literature.

Our analysis begins by placing in context the use of the Mueller-Stokes formalism for estimating $S^A_{\min}$. Given a two-qubit state ρ_{AB}, it can always be written as

$$\rho_{AB} = \frac{1}{4} \sum_{a,b=0}^{3} M_{ab}\, \sigma_a \otimes \sigma_b^*, \tag{13}$$

the associated 4×4 matrix M being real; σ_1, σ_2, σ_3 are the Pauli matrices and σ_0 equals the unit matrix. Writing a POVM element on the B side as $K = \frac{1}{2}\sum_a S_a \sigma_a$, the output state of A post measurement is obtained as the action of M on the input Stokes vector S^{in} corresponding to the POVM element. This may be compactly expressed as

$$S^{\text{out}} = MS^{\text{in}} \tag{14}$$

and has a form analogous to the input-output relation in polarization optics. In view of this analogy, we may call M the Mueller matrix associated with $\hat{\rho}_{AB}$. In general M need not correspond to a trace-preserving map, since the conditional output states need not be normalized. So they need to be normalised for calculating the conditional entropy. The manifold of these normalized conditional states is an ellipsoid, a convex subset of the Poincaré sphere, completely parametrised by the local unitarily invariant part of the M matrix and, thereby, the local unitarily invariant part of the bipartite state ρ_{AB}. The boundary of this output ellipsoid corresponds to the images of all possible rank-one POVM's or light-like S^{in}.

While this geometric picture a two-qubit state being fully captured by its Mueller matrix—or equivalently by this output ellipsoid—applies to every two-qubit state, X-states are distinguished by the fact that the centre C of the output ellipsoid, the origin O of the Poincaré sphere and I, the image of maximally mixed input $S^{\text{in}} = (1, 0, 0, 0)^T$ are all collinear and lie on one and the same principal axis of the ellipsoid.

One realizes that the Mueller matrix of any X-state can, by local unitaries, be brought to a canonical form wherein the only nonvanishing off-diagonal elements are m_{03} and m_{30}, and thus X-states form, in the canonical form, a five parameter family with m_{11}, m_{22}, m_{33}, m_{30}, and m_{03} as the canonical parameters ($m_{00} = \text{Tr}\,\rho_{AB} = 1$ identically). With this realization our entire analysis in Chapter 3 is geometric in flavour and content. The principle results of the Chapter may be summarized as follows :

- All X-states of vanishing discord are fully enumerated and contrasted with earlier results.
- Computation of quantum discord of X-states is proved to be an optimization problem in one real variable.
- It is shown that the optimal POVM never requires more that three elements.
- In the manifold of X-states, the boundary between states requiring three elements for optimal POVM and those requiring just two is fully detailed.

It may be stressed that our analysis in this Chapter is from first principles. It is comprehensive and geometric in nature. Our approach not only reproduces and unifies all known results in respect of X-states, but also brings out entirely new insights.

In **Chapter 4** we explore the connection between bipartite entanglement and local action of noisy channels in the context of continuous variable systems. Quantum entanglement in continuous variable systems has proved to be a valuable resource in quantum information processes like teleportation, cloning, dense coding, quantum cryptography, and quantum computation.

These early developments in quantum information technology involving continuous variable (CV) systems largely concentrated on Gaussian states and Gaussian operations, mainly due to their experimental viability within the current optical technology. The symplectic group of linear canonical transformations is available as a handy and powerful tool in this Gaussian scenario, leading to an elegant classification of permissible Gaussian processes or channels.

However, the fact that states in the nonGaussian sector could offer advantage for several quantum information tasks has resulted more recently in considerable interest in non-Gaussian states, both experimental and theoretical. The use of nonGaussian resources for teleportation, entanglement distillation, and its use in quantum networks have been studied. So there has been interest to explore the essential differences between Gaussian states and nonGaussian states as resources for performing quantum information tasks.

Allegra et al. [9] have studied the evolution of what they call *photon number entangled states* (PNES),

$$|\psi\rangle_{\text{PNES}} = \sum_{n} c_n \, |n,\, n\rangle, \tag{15}$$

in a *noisy* attenuator environment. They conjectured based on numerical evidence that, for a given energy, Gaussian entanglement is more robust than nonGaussian ones. Earlier Agarwal et al. [10] had shown that entanglement of the NOON state,

$$|\psi\rangle_{\text{NOON}} = \frac{1}{\sqrt{2}}(|n, 0\rangle + |0, n\rangle), \tag{16}$$

is more robust than Gaussian entanglement in the *quantum limited* amplifier environment. Subsequently, Nha et al. [11] showed that nonclassical features, including entanglement, of several nonGaussian states survive a *quantum limited* amplifier environment much longer than Gaussian entanglement. Since the conjecture of [9] refers to noisy environment, while the analysis of [10, 11] to the noiseless or quantum-limited case, the conclusions of the latter amount to neither confirmation nor refutation of the conjecture of [9]. In the meantime, Adesso argued [12] that the well known extremality [13] of Gaussian states implies 'proof and rigorous validation' of the conjecture of [9].

In the work described in Chapter 4 we employ the recently developed Kraus representation of bosonic Gaussian channels [4] to study analytically the behaviour of nonGaussian states in *noisy* attenuator or and amplifier environments. Both NOON states and a simple form of PNES are considered. Our results show conclusively that the conjecture of [9] is too strong to be maintainable, the 'proof and rigorous validation' of [12] notwithstanding.

An important point that emerges from this study is the fact that Gaussian entanglement resides entirely 'in' the variance matrix or second moments, and hence disappears when environmental noise raises the variance matrix above the vacuum or quantum noise limit. That our chosen nonGaussian states survive these environments shows that their entanglement resides in the higher moments, in turn demonstrating that their entanglement is genuine nonGaussian. Indeed, the variance matrix of our PNES and NOON states for $N = 5$ is six times 'more noisy' than that of the vacuum state.

We study in **Chapter 5** an interesting relationship between nonclassicality and entanglement in the context of bosonic Gaussian channels. We motivate and resolve the following issue : *which Gaussian channels have the property that their output is guaranteed to be classical independent of the input state*?

We recall that the density operator $\hat{\rho}$ representing any state of radiation field is '*diagonal*' in the coherent state '*basis*' [14], and this happens because of the over-completeness property of the coherent state basis. An important notion that arises from the diagonal representation is the *classicality-nonclassicality divide*. Since coherent states are the most elementary of all quantum mechanical states exhibiting classical behaviour, any state that can be written as a convex sum of these elementary classical states is deemed classical. Any state which cannot be so written as a convex sum of coherent states is deemed nonclassical.

This classicality-nonclassicality divide leads to the following natural definition, inspired by the notion of entanglement breaking channels : we define a channel Γ to be *nonclassicality breaking* if and only if the output state $\hat{\rho}_{\rm out} = \Gamma(\hat{\rho}_{\rm in})$ is classical *for every* input state $\hat{\rho}_{\rm in}$, i.e., if and only if the diagonal 'weight' function of every output state is a genuine probability distribution.

We first derive the *nonclassicality-based* canonical forms for Gaussian channels [15]. The available classification by Holevo and collaborators is *entanglement-based*, and so it is not suitable for our purpose, since the notion of nonclassicality breaking has a more restricted invariance. A nonclassicality breaking Gaussian channel Γ preceded by any Gaussian unitary $\mathcal{U}(\mathcal{S})$ is nonclassicality breaking if and only if Γ itself is nonclassicality

	Canonical form	NB	EB	CP
I	$(\kappa\,\mathbb{1},\ \text{diag}(a,b))$	$(a-1)(b-1)\geq\kappa^4$	$ab\geq(1+\kappa^2)^2$	$ab\geq(1-\kappa^2)^2$
II	$(\kappa\,\sigma_3,\ \text{diag}(a,b))$	$(a-1)(b-1)\geq\kappa^4$	$ab\geq(1+\kappa^2)^2$	$ab\geq(1+\kappa^2)^2$
III	$(\text{diag}(1,0),\ Y)$,	$a,\ b\geq 1$, a, b being eigenvalues of Y	$ab\geq 1$	$ab\geq 1$
	$(\text{diag}(0,0),\ \text{diag}(a,b))$	$a,\ b\geq 1$	$ab\geq 1$	$ab\geq 1$

Table 1: Showing the nonclassicality breaking (NB), entanglement breaking (EB) and complete-positivity (CP) conditions for the three canonical forms.

breaking. In contradistinction, the nonclassicality breaking aspect of Γ and that of $\mathcal{U}(\mathcal{S})\,\Gamma$ [Γ followed the Gaussian unitary $\mathcal{U}(\mathcal{S})$] are not equivalent in general. They are equivalent if and only if $\mathcal{S}$ is in the intersection $Sp(2n,\,R)\cap SO(2n,\,R)$ of symplectic phase space rotations, or passive elements in the quantum optical sense [16]. The canonical forms and the corresponding necessary and sufficient conditions for nonclassicality breaking, entanglement breaking and complete-positivity are listed in Table 1.

For all three canonical forms we show that a nonclassicality breaking channel is necessarily entanglement breaking. There are channel parameter ranges wherein the channel is entanglement breaking but not nonclassicality breaking, but the nonclassicality of the output state is of a ‘weak’ kind in the following sense : For every entanglement breaking channel, there exists a particular value of squeeze-parameter r_0, depending only on the channel parameters and not on the input state, so that the entanglement breaking channel followed by unitary squeezing of extent r_0 always results in a nonclassicality breaking channel. It is in this precise sense that nonclassicality breaking channels and entanglement breaking channels are essentially one and the same.

Squeezing is not the only form of nonclassicality. Our result not only says that the output of an entanglement breaking channel could at the most have squeezing-type nonclassicality, it further says that the nonclassicality of *all* output states can be removed by a *fixed* unitary squeezing transformation.

Finally, in **Chapter 6**, we briefly summarise the conclusions of each of the chapters, and explore possible avenues and prospects for future directions of study.

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