4

Analysis of Magnetic Rayleigh-Taylor (MRT) instability

4.1 Introduction

In previous Chapters, we have examined a direct energy conversion scheme to convert plasma kinetic energy in an Inertial Fusion Energy system into pulsed electrical energy. Preliminary numerical studies [52, 82] described in Chapters 2 and 3 indicate that the proposed system is promising in terms of overall conversion efficiency. However, such a plasma, expanding across a magnetic field, is subject to the Magnetic Rayleigh Taylor (MRT) instability. The growth of MRT instability on the surface of the plasma, around the time of stagnation, is evident from the results presented in Chapter 3. A detailed analysis of such instabilities forms the subject of this chapter.

The MRT instability occurs when an electrically conducting fluid, e.g. plasma, is decelerated or supported by the magnetic field. The classical linear MRT growth rate [83] is defined as, $\gamma_L = (kg)^{1/2}$ for $kL_n \ll 1$ and $\gamma_L = (g/L_n)^{1/2}$ for $kL_n \gg 1$;
where $k$ is the wave number, $g$ is the deceleration, $L_n \sim \left[ \frac{\partial \ln(n)}{\partial x} \right]^{-1}$ is the density scale length of the plasma and $n$ is the plasma density. For efficient operation of the proposed MFC system [52,82], the instability amplitude must be small so that the irregular surface caused by growth of the MRT instability does not disturb the smooth compression of the magnetic field between the plasma and solenoid. Large amplitude flute modes and plasma jetting can damage the cavity wall [33].

Numerical and experimental studies on plasma expansion in an external magnetic field and the analysis of interchange instabilities in space and laboratory plasmas can be found in Refs. [40,84–97] (also see references therein). The majority of the above-mentioned works examine plasma expansion in a uniform unconfined background magnetic field where there is negligible compression of the magnetic field. In the MFC system, however, the magnetic field outside the plasma increases due to magnetic flux compression.

Previous work related to plasma energy conversion and including the role of MFC has been reported in Ref. [33]. There are, however, two major differences between that work and the present work. Firstly, Ref. [33] analyses a different plasma parameter range, starting with an initial radius of $\sim 1$ m and system dimensions of $\sim 14$ m in radius. Since the pickup coil is located at a radius of $\sim 9$ m, a low initial magnetic field is sufficient to stop the plasma close to the coil radius. Therefore, a magnetic field of $\sim 0.57$ T is used in Refs. [33]. In our last study [52,82], described in Chapter 2 and 3, we had examined the case of a much smaller, practically-relevant system having a coil radius $\sim 1.5$ m, higher-pressure plasma ($\sim 10^7$ Pa) with an initial radial expansion velocity $\sim 10^7$ m/s, which requires a higher magnetic field (5 T) to extract enough energy from the plasma.

Secondly, the simulation results given in Ref. [33] start with an unperturbed initial plasma state, so that instabilities are seeded by numerically-produced per-
The purpose of this study is to numerically analyze, using MHD fluid simulations, the MRT instability on the surface of the plasma liner and its implications for the proposed MFC system.

4.2 MHD model and Computational Scheme

The equations of the MHD model used in this work have been described in Chapter 3. In Chapter 2, we have developed a pure Lagrangian MHD scheme self consistently coupled with external circuit equations to solve the governing equations. That scheme, however, is not suitable for the present study as large material deformations are expected. In such situations, use of a purely Lagrangian scheme leads to severe mesh distortion. Consequently, in Chapter 3, we have formulated an Eulerian MHD scheme with volume-of-fluid material interface tracking [60] to handle large plasma deformations in the MFC system. The method was successful in analysing large deformation plasma dynamics in the proposed MFC system. However, for the present study, it demands a prohibitively large number of cells in the simulation. This is due to the order of magnitude difference between the
different scale lengths involved in the system, such as MFC system dimensions of the order of few meters, plasma initial perturbation amplitude $\alpha_{\text{in}}$ of the order of few $\mu$m and wavelength $\lambda$ ranging from few mm to cm. Note that for numerical convergence with respect to the mesh size, at least 10–20 cells per $\lambda$ are required. This demands an extremely large number of cells in the simulation. Hence neither of the two foregoing techniques can be used for analysing MRT instabilities in an MFC system. In this chapter, we report on the development and use of an unstructured Lagrangian scheme [98] for such problems.

This unstructured Lagrangian scheme [98] helps control artificial grid distortion and ‘hourglass’-type motion. Further, to stabilize the grid, a node based tensor viscosity [100] and an artificial grid distortion control algorithm [99] are used. This allows us to simulate plasma evolution till the stagnation or turn-around time $t_s$ without numerical instabilities. Approximately at this time, the inductive energy across the load reaches a maximum [52,82]. Therefore, in the present work, we are only interested in studying the evolution of the MRT instability till the stagnation time. We have obtained a substantial reduction of the overall computational time with the help of an unstructured Lagrangian scheme, since the total number of cells required in the simulation are considerably reduced.

A typical unstructured mesh used in the simulation is given in Fig. 4.1. Only one quarter of the system is simulated due to symmetry. Details of the unstructured Lagrangian scheme can be found in Refs. [98–100], and essential details are given in Appendix B. Similarly, the details of the MHD scheme can be found in Ref. [52, 60, 82] and are omitted here for the sake of brevity.
4.3 Initial conditions

The initial plasma parameters are taken from earlier published data for a D-3He plasma. The plasma energy $E_p$ and mass $m_p$ used in this study are 280 MJ and 4.4 mg respectively [25–27, 31, 32]. A 5 Tesla seed magnetic field is used in the simulation and the system parameters are the same as described in Chapters 2 and 3. Initially, therefore, the plasma undergoes free expansion across the $B$ (see Chapters 2 and 3 and Refs. [25–27, 52, 82]). Therefore, we have started our simulation with an initial plasma radius of $\sim$0.2 m. Initial radial profiles for the plasma density, temperature and velocity are generated using a separate 2D simulation without considering the effect of $B$ (free expansion up to a radius equal to 0.2 m). The initial conditions thus obtained are shown in Fig. 4.2 as a function of plasma radius.

The overall computational approach in this work is summarized below. We have analyzed the evolution of MRT instability in two steps.

1. In the first step, we have applied random amplitude perturbations. The
instability has been seeded by adding a random fraction of the initial amplitude $\alpha_m$ to the plasma radius on the outer surface. In this method, the non-linear evolution of different modes can be studied simultaneously. Also, it helps to identify the dominant modes in the spectrum. However, this study is subject to the limitation that the shortest wavelength that can be studied is restricted to the mesh-size used in the $\theta$ direction near the surface of the plasma.

2. In the second step, we have used a single-mode sinusoidal perturbation. The initial wavelength $\lambda_m$ of this perturbation is varied typically around the wavelength of the dominant modes found in the previous analysis (random perturbations). The perturbation is imposed by defining the outer radius as

$$R(x, y) = R_0 + \alpha_m \sin(2\pi r/\lambda_m)$$

where $r = \sqrt{x^2 + y^2}$ and $\alpha_m$ are the radius and perturbation amplitude re-
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In the random seed analysis, the mesh-size is typically governed by the shortest wavelength that has to be studied. For single mode analysis, we have used a mesh-size ranging from $\lambda/20$ to $\lambda/12$, which is found sufficient to yield numerical convergence with respect to the mesh-size.

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The simulation results are analyzed using a Fast Fourier Transform (FFT) technique. The Fourier spectrum of modes in the plasma liner at different times is obtained as follows. The simulation yields $R(\theta, t)$ on the outer surface of the plasma. We subtract the $R(\theta, t)$ from the average outer radius to yield the deviations $\Delta R(\theta, t)$. A fast Fourier transform is performed on these values to yield the Fourier spectrum.

4.4.1 Random perturbation

The plasma has an initial radius $r_p = 0.2$ m. A random perturbation is imposed on the outer surface of the plasma, as described in Sec. 4.3 with $\alpha_n = 5 \mu$m. We have used 160 cells in the $\theta$ direction ($n_\theta = 160$) near the surface of the plasma. Since only one quarter of the system is simulated, we have a spherical plasma liner with a circumference $C = \pi r_p/2$. The mode number corresponding to a wavelength $\lambda$ is given by $n = C/\lambda$. Therefore, the shortest and largest wavelength that can be studied with this mesh-size are $2C/n_\theta \sim 4$ mm and $C \sim 0.3$ m respectively. Note that these values change along with the instantaneous plasma radius $r_p(t)$ (geometric divergence effect).

The early phase is characterized by plasma expansion, a comparatively low
value of acceleration $g$ and hence a low growth rate $\gamma$. A high pressure plasma region is created near the surface [52,82] as the outer surface slows down due to $B$ and the inner region catches up with the outer surface, as shown in Fig. 4.3(a). This pressure build up near the plasma surface tends to smooth out the perturbations. Hence, for a very short initial period, the amplitudes of all the modes decrease. As the plasma expands further, the $B$ outside the plasma increases due to MFC and hence the interface deceleration $g$ increases, as shown in Fig. 4.3(b). This in turn increases the growth rate of the modes.

Fig. 4.4 shows snapshots of the Fourier spectrum at different times for this case. The initial figure at $t = 0$ shows a spectrum with comparatively larger amplitudes for longer wavelength modes. Fig. 4.4(b), corresponding to $t \sim 0.036$ $\mu$s, shows that the amplitudes of all modes become comparable, with an average amplitude of $\sim 30$ $\mu$m. During this time, the amplitudes of short $\lambda$ modes are increased by $\sim 10$ times while the amplitudes of longer $\lambda$ modes essentially remain constant. This is because of the faster growth of shorter wavelength modes (comparatively higher $\gamma$). However, these short wavelength modes also tend to saturate earlier. For $t \geq 0.05$ $\mu$s, all the modes grow with nearly equal $\gamma$, with a comparatively higher-amplitude spectrum in the intermediate wavelength range (modes 20–50). This is due to the non-linear evolution of the modes. For the present system dimensions and plasma parameters, the time scales of magnetic field diffusion and thermal conduction are much greater than the typical plasma expansion time [52, 82]. Similarly, we have also neglected the plasma ion viscosity [52, 82]. Therefore, the non-linear evolution of the modes might be the consequence of other non-linear effects such as mode saturation [101], interaction of different modes (mode coupling) and harmonic mode generation.

In order to understand coupling between different modes, we have used a cross-
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Figure 4.3: (a) Plasma pressure (normalized to $10^7$ Pa) vs radius at different times during the initial phase of the plasma expansion. The plasma forms a shell like geometry [52, 82]. (b) The temporal evolution of $g$ of the outer surface (normalized to $10^{14}$ ms$^{-2}$) during the initial phase of plasma expansion.

correlation ($f_{cr}$) analysis, as explained in Ref. [102]. First the FFT spectrum is obtained for a large number of time points. This yields a time-series of the amplitudes of different modes in the spectrum. We have calculated $f_{cr}$ between these time series using the following expression [103].

$$f_{cr} = \frac{\sum (\alpha^n_j - \bar{\alpha}_j) \times \sum (\alpha^n_k - \bar{\alpha}_k)}{\sqrt{\sum (\alpha^n_j - \bar{\alpha}_j)^2} \sqrt{\sum (\alpha^n_k - \bar{\alpha}_k)^2}}$$  \hspace{1cm} (4.1)

Here, $\alpha^n_j$ and $\alpha^n_k$ are the time series for modes $j$ and $k$, $\bar{\alpha}_j$ and $\bar{\alpha}_k$ are the respective average values. Fig. 4.5 shows $f_{cr}$ for a few important modes ($n = 22$, 34, 38 and 50) with other modes in the spectrum. The results indicate strong cross-correlation between different modes. This coupling causes the transfer of energy between different modes. A detailed study would be required to analyze this non-linear mode coupling between different modes. This, however, lies beyond the scope of this thesis. The main objective of this multi-mode analysis is to qualitatively identify the dominant modes and the corresponding wavelength regime in the amplitude spectrum. This dominant wavelength range is next explored using
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single mode analysis with an initial sinusoidal perturbation.

It is noteworthy that since the number of modes \( (n) \) are fixed, the \( \lambda \) corresponding to a mode number increases with time due to the plasma expansion \( (r_p \) and hence \( C \) changes with respect to time). Thus the shortest \( \lambda \) (highest \( n \)) that can be studied increases with time due to geometric divergence. Fig. 4.6 shows the temporal variation of \( \lambda \) for \( n = 80 \). The \( \lambda \) for this mode changes from 4 mm to 22.5 mm. That is, as the plasma expands radially outwards, the shorter wavelength spectrum is continuously eliminated even though the number of modes/cells are fixed. Therefore, we have repeated the analysis with \( n = 600 \). This allows study of the shortest \( \lambda \) \( (n = 300) \) that varies from ~0.1–5.8 mm. This study will also help us to see the sensitivity of the results with respect to the mesh-size.

Fig. 4.7 shows snapshots of the Fourier spectrum at different times for this case. Similar to the earlier case, the FFT spectrum has comparatively larger initial amplitudes for longer \( \lambda \) modes and as the plasma expands the shorter \( \lambda \) modes in the spectrum evolve faster and saturate. A progressive transition to longer \( \lambda \) regime is observed. Similar trends have been observed in the simulations of Z-pinch implosions [104]. The wavelength regime of the dominant modes and their amplitudes are consistent with the previous results obtained for \( n = 160 \). This means that there is no significant change in the results by increasing the number of cells and hence by including shorter wavelength modes (0.1–5.8 mm) in the simulation.

Let us now consider the effect of initial amplitude. We have already obtained results for \( \alpha_n = 5 \mu m \). We have repeated the analysis with two different amplitudes, viz., \( \alpha_n = 50 \mu m \) and 0.5 mm, a variation by two orders of magnitude. Fig. 4.8 shows the Fourier spectrum obtained for these two cases. The initial spectrum obtained for these two cases are as shown in the first plot of Fig. 4.4 with a
Figure 4.4: Spectral evolution of perturbations in plasma liner at different times, starting with an initial random perturbation. We have drawn an envelope over the wave structure, adapted from [104], (dashed blue line) to easily identify the dominant modes and corresponding wavelength regime. Figures (a)—(f) are at t=0, 0.04 µs, 0.044 µs, 0.048 µs, 0.058 µs and 0.081 µs, respectively. The corresponding average plasma radii are 0.2 m, 0.78 m, 0.9 m, 0.95 m, 1.0 m and 1.2 m respectively.
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Figure 4.5: Cross-correlation factor ($f_{cr}$) for few dominant modes ($n = 22, 34, 38$ and $50$) with other modes in the spectrum.

multiplication factor of 10 and 100 respectively. The dominant modes and their $\lambda$ regime obtained for these two cases are in good agreement with the previous case ($\alpha_n = 5 \, \mu m$). However, with $\alpha_n = 0.5 \, mm$, the spectrum amplitudes of the dominant modes are $\sim 60-80$ times higher than the short $\lambda$ modes ($n > 60$) compared to $\sim 2-3$ times for the previous case. This is because the higher initial amplitude of short $\lambda$ modes leads to their reaching saturation faster than for lower initial amplitudes.

The spectral evolution obtained clearly demonstrates its complex nature for the case with random initial perturbations. The large number of modes make it difficult to follow individual mode evolution and distinguish between the various factors influencing the $\gamma$. As mentioned in the beginning, the primary objective of this random seed perturbation analysis is to find out the dominant modes and their $\lambda$ regime in the spectrum. These wavelengths are then subsequently used in the
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Figure 4.6: Temporal variation of $\lambda$ for mode number 80. The $\lambda$ for this mode changes from 4 mm to 2.25 cm.

single mode analysis. Inspecting Fig. 4.4, it is clear that towards the stagnation time the dominant modes shift towards intermediate wavelengths $\lambda = 3.7-8$ cm (mode number $n = 20-50$ for $n_\theta = 160$). Also, the higher wavelength modes ($\lambda$ ranging from 10–100 cm) have comparatively lower amplitudes from $t = 0.05 \mu s$ to $t_s$. The dominant wavelength regime obtained in this analysis is consistent with the observations in our earlier work using a 2D Eulerian MHD scheme [82]. In Ref. [82], the instabilities grow from numerical perturbations and the $\lambda$ of the dominant mode observed towards $t_s$ is $\sim 6.8$ cm [82].

The evolution of the dominant modes to intermediate wavelength regime is the consequence of non-linear mode saturation and/or mode coupling effects. Other effects, such as $B$ diffusion into the plasma, thermal conduction and viscosity, are not significant for the present MFC system parameters [52, 82]. The shorter wavelength modes, which grow more rapidly, saturate at an earlier time. With saturation, their growth rate becomes close to a constant value [101], so that they are eventually overtaken by intermediate $\lambda$ modes. For longer $\lambda$ modes ($n = 1-10$ for $n_\theta = 160$), the growth rate $\gamma \propto 1/\sqrt{\lambda}$ is slower than the $\gamma$ of the intermediate modes. Apart from these, the non-linear coupling of different modes which occurs
Figure 4.7: Spectral evolution of plasma liner at different times from an initial random amplitude perturbation (with $n_0 = 600$). We have drawn an envelope over the wave structure (dashed blue line) to easily identify the dominant modes and corresponding wavelength regime.
Figure 4.8: Fourier spectrum obtained for two different values of $\alpha_n$ (random initial perturbation). The initial spectrum obtained for these two cases are as shown in the first plot of Fig. 4.4 with a multiplication factor of 10 and 100 respectively.

simultaneously, as seen by the cross-correlation analysis, contributes to the growth of other modes in the spectrum. This makes the numerical picture more difficult to interpret. Therefore, in the next section, we have described the instability analysis with a single-mode initial perturbation. The initial wavelength ($\lambda_{in}$) of this perturbation is varied typically around the wavelength of the dominant modes found in this section.

### 4.4.2 Single mode sinusoidal perturbation

The initial perturbation is taken to be sinusoidal, as described in Sec. 4.3. $\lambda_{in}$ is varied from 6.9 mm—6.28 cm ($n = 5–45$). These modes are chosen in such a way that the corresponding $\lambda_{in}$ lies within the dominant $\lambda$ regime found in the previous study. For each $\lambda_{in}$, four values of $\alpha_{in}$ are used; $\lambda_{in}/1000$, $\lambda_{in}/100$, $\lambda_{in}/50$ and $\lambda_{in}/10$. Note that for the cases with $\alpha_{in} \sim \lambda_{in}/10$, the mode amplitude and the wavelength are comparable. This means that $\alpha_{in}$ is close to the mode saturation limit [101]. This value, however, is included in the test cases by considering the fact that the $\lambda$ of a given mode increases due to plasma expansion. Also, we have observed that for a short initial period of time, the $\alpha$ decreases—this is examined
in later sections. Apart from these, the following fact also need to be considered. In reality the plasma expands from a radius of \(\sim 200 \mu m\) after fusion energy release is completed. However, as mentioned above, we have started our simulation with a radius of 0.2 m (1000 times expansion). Therefore, modes with comparable \(\alpha\) and \(\lambda\) would pre-exist on the plasma surface.

Fig. 4.9 shows the density profile near the plasma surface and the corresponding Lagrangian mesh towards \(t = t_s\) for the case with \(\lambda_{in} = 3.14\) cm \((n = 10)\) and \(\alpha_{in} = \lambda_{in}/10\). The plasma forms a shell like geometry near the stagnation time. This is consistent with the observations in Refs. [52, 82]. The evolution of the FFT spectrum amplitude is shown in Fig. 4.10. Inspecting the plots, it is clear that apart from the fundamental imposed mode, the evolution of other harmonic modes \((nk \text{ where } n = 2, 3, \ldots)\) with \(\lambda_n = \lambda_{in}/n\) are also taking place. We have observed that, in all the test cases presented here, the evolution of harmonic modes (the non-linear phase of instability growth) occur significantly when \(\alpha_{in} \sim \lambda_{in}/10\).

In order to understand the mode coupling and harmonic mode generation in detail, we have performed a multi-mode analysis by imposing two fundamental modes having different \(\lambda\). Initially two fundamental modes, numbers 10 and 40, are imposed with the same initial amplitude \(\alpha_{in} \sim 500 \mu m\) for each mode, i.e. \(\alpha_{in} \sim \lambda_{in}/65\) for mode 10 and \(\alpha_{in} \sim \lambda_{in}/15\) for mode 40. The right side plot in Fig. 4.10, shows the FFT spectrum (normalized to the highest amplitude in the spectrum) at \(t \sim 0.04\) ms for this case. For single mode perturbation with \(\alpha_{in} = \lambda_{in}/10\) and \(n = 40\), only one harmonic mode \((n = 80)\) is appeared at \(t = 0.04\) ms, see Fig. 4.10(b). The amplitude of this harmonic mode is \(\sim 20\%\) of the fundamental mode. Also, note that the FFT spectrum obtained in the single mode analysis for \(n = 10\) with \(\alpha_{in} = \lambda_{in}/50\) shows negligible amplitude for its harmonic modes at \(t = 0.04\) ms. However, with multi-mode perturbation (modes
10 and 40), the evolution of other modes due to the interaction between different primary and harmonic modes are observed. The appearance of a mode $n = 30$ which is the difference of fundamental modes is also observed. Note that for single mode perturbation with $n = 40$, the spectrum amplitude of the modes 20 and 30 were negligible.

Further, to observe the mode coupling between two fundamental short wavelength modes (say $n_1$ and $n_2$), we have repeated the multi-mode analysis for two different sets of fundamental modes $(n_1, n_2) = (30, 40) \& (60, 80)$ with $\alpha_{in} \sim 500 \mu m$. The FFT spectrum ($\alpha/\alpha_{peak}$) obtained for these two cases are shown in Fig. 4.11. Note the appearance of inverse cascade modes 10 and 20 corresponding to the difference $n_2 - n_1$ for these two cases respectively (the generation of inverse cascade modes in a Z-pinch implosion system with multi-mode perturbation analysis is reported in Ref. [104]). That is the short wavelength modes upon saturation generate higher wavelength modes along with other short wavelength harmonic modes. Note that the amplitude of harmonic modes are found to be insignificant for all the cases of multi-mode analysis when the value of $\alpha_{in}$ for each mode is set equal to $\sim 5 \mu m$ (this is true even with increased spatial resolution). That is the evolution of harmonic modes and their interactions with both the primary and other harmonic modes are found to be significant only when the value of $\alpha_{in}$ is comparable to $\lambda_m$. This implies upon saturation these modes evolve non-linearly with the generation of harmonic and inverse cascade modes.

Our aim with this multi-mode perturbation study was to qualitatively analyze the non-linear evolution of the modes (particularly the short $\lambda$ modes) when $\alpha$ is comparable to $\lambda$ (close to mode saturation). It is clear that the non-linear evolution of the modes upon saturation is characterized by the generation of harmonic modes and mode coupling. Detailed analysis of the evolution of these harmonic modes
for various initial conditions in terms of number fundamental modes, their initial amplitudes and wavelength lies beyond the scope of this work. For these studies, more computational efforts with increased spatial resolution to resolve the highest harmonic mode [104] in the spectrum are required.

In Fig. 4.12, we have shown the temporal evolution of the perturbation amplitude $\alpha$ and the wavelength $\lambda$. As mentioned earlier, $\alpha$ decreases till $t \sim 0.03 \mu s$ due to the geometric divergence effect, comparatively low growth rate $\gamma_L \sim (2\pi g / \lambda)^{1/2}$ ($g$ is comparatively low during the initial phase of the expansion) and the high pressure region created near the surface of the plasma. However, as the plasma expands further, the $B$ outside the plasma and hence $g$ increases due to MFC. This increases the $\gamma$ value. Therefore, at a time $\sim 0.03 \mu s$ the $\alpha$ begins to grow.

The temporal evolution of the $\alpha$ predicted by the linear theory with a constant growth rate $\gamma_L$ is also shown in Fig. 4.12. Since, both the $\lambda$ and $g$ varies with time, we have used their time averaged values ($\lambda_{avg} \sim 10$ cm, $g_{avg} \sim 3 \times 10^{14}$ m/s$^2$) for calculating the growth rate $\gamma_L \sim (2\pi g_{avg} / \lambda_{avg})^{1/2} \sim 1.4 \times 10^8$ s$^{-1}$. Note that $\lambda_{avg}$ and $g_{avg}$ are also averaged along the $\theta$ direction. It is clear from the figure that the assumption of linear growth for $\alpha$ from $t = 0$ to $t_s$ with a growth rate $\gamma_L$ tends to overestimate the final amplitude by orders of magnitude. The magnetic deceleration $g$ and hence the $\gamma$ value, as mentioned earlier, becomes significant at a time $\sim 0.03 \mu s$. Therefore, we have also plotted the temporal evolution of $\alpha$ starting from $t \sim 0.03 \mu s$ by using both $\gamma_L$ (shifted line in the Fig. 4.12) and $\gamma_L(t) \sim (2\pi g(t) / \lambda(t))^{1/2}$, where $g(t)$ and $\lambda(t)$ are the instantaneous values of the interface deceleration and wavelength respectively. These plots are close to the simulation result, except towards the stagnation time $t_s$, where the simulation result shows a non-linear evolution. Clearly, the assumption of linear growth from $t = 0$ produce a much larger amplitude than is observed computationally.
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The temporal evolution of $\alpha$ for a few other cases of sinusoidal perturbation ($n = 5, 25$ and 45) with $\alpha_m = \lambda_m/1000$ is shown in Fig. 4.13. The evolution of $\alpha$ starting from $t = 0.03 \mu s$ is found to be in agreement with the predictions of linear theory. However, similar to the earlier case, non-linear evolution of the modes is observed towards the time $t = t_s$. For short wavelength modes, this onset of non-linearity occurs at an earlier time. For example the onset of non-linearity for mode $n = 45$ occurs at a time $t \sim 0.05 \mu s$, whereas for mode $n = 5$ this occurs at a time $t \sim 0.07 \mu s$. In short, the growth of the modes near stagnation time, although exponential in nature, occurs at a lower rate than that predicted by linear theory.

![Density profile](image)

Figure 4.9: Density ($\times 10^5$ kg/m$^3$) profile near the plasma surface towards $t = t_s$ for the case with $\lambda_m = 3.14$ cm ($n = 10$) and $\alpha_m = \lambda_m/10$. Right side plot is the corresponding Lagrangian Mesh with a mesh-size $\sim \lambda/12$.

For a plasma of given mass, its radial expansion velocity increases with its initial energy $E_p$. Therefore, the plasma energy determines (for fixed $B$) the stagnation time, plasma stopping radius and hence the growth of the modes. Typical plasma energy $E_p$ and mass $m_p$ reported for inertial fusion plasmas vary from 140–300 MJ and 1.2–6 mg respectively [25–27,31,32,52,82]. Fig. 4.15 shows the results of a sample calculation ($n = 10$, $\alpha_m = \lambda/1000$) with two different values for $E_p$ (140 and 280 MJ) with $m_p \sim 4.4$ mg. No significant difference in the final amplification factor (at $t \sim t_s$) is observed between these two cases despite having different
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Figure 4.10: The left side plot shows the spectrum amplitude (mm) at different times (single mode analysis) with $\lambda_{in} = 3.14 \text{ cm}$ ($n = 10$) and $\alpha_{in} = \lambda_{in}/10$. The FFT spectrum shows the evolution of harmonic modes. Right side plot shows the normalized spectrum amplitude ($\alpha/\alpha_{peak}$) at $t \sim 0.04 \mu s$ obtained in the multi-mode analysis (modes 10 & 40, $\alpha_{in} = 500 \mu m$) and single mode analysis ($n = 40$, $\alpha_{in} = \lambda/10$).

operational time, interface deceleration and plasma stopping radius. The decrease in $g$ (hence $\gamma$) and the increase in overall operational time (more growth) makes the final $\alpha/\alpha_{in}$ factor nearly the same for this particular system parameters with $B \sim 5 \text{ T}$. A similar trend is observed for other cases with different $n$, $\alpha_{in}$ and $m_p$. A more comprehensive analysis with different plasma mass, energy (different fusion yield), mode number, $\alpha_{in}$ and initial $B$ lies beyond the scope of this work.

The results can now be summarized. Fig. 4.14 shows the values of maximum $\alpha/\alpha_{in}$ and $\alpha$ obtained at $t = 0.09 \mu s$ (close to the stagnation time) for different values of initial mode number $n$ and perturbation amplitude $\alpha_{in}$. The final amplitude amplification factor $\alpha/\alpha_{in}$ obtained is typically higher for the cases with lower $\alpha_{in}$ values. Also, small wavelength modes have comparatively higher $\alpha/\alpha_{in}$ value. For a given $n$, the final $\alpha/\alpha_{in}$ value obtained (at $t \sim t_s$), when $\alpha_{in} \sim \lambda/10$, is found to be much lower than the $\alpha/\alpha_{in}$ values obtained for $\alpha_{in} \sim \lambda/100$ and $\lambda/1000$. This difference increases towards higher $n$ (mode saturation for short $\lambda$ modes happens at an earlier time). However, inspecting the actual amplitude ($\alpha$) variation, com-
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Figure 4.11: The normalized spectrum amplitude ($\alpha/\alpha_{\text{peak}}$) obtained in the multi-mode perturbation analysis with $\alpha_{\text{in}} \sim 500 \mu m$. The left and right side plots are for modes 30 & 40 at $t = 0.04 \mu s$ and modes 60 & 80 at $t = 0.03 \mu s$ respectively.

Figure 4.12: The evolution of perturbation amplitude $\alpha$ (single mode analysis) for $n = 10$ with $\alpha_{\text{in}} = \lambda_{\text{in}}/10$. Here, $\alpha_L$ is the analytical variation assuming constant linear growth rate $\gamma_L \sim (2\pi g_{\text{avg}}/\lambda_{\text{avg}})_{1/2}$. Similarly $\alpha_L(t)$ is obtained by using instantaneous values for $\gamma_L(t) \sim (2\pi g(t)/\lambda(t))_{1/2}$.

Comparatively larger $\alpha$ values (despite having lower $\alpha/\alpha_{\text{in}}$ values) are observed for the cases with $\alpha_{\text{in}} \sim \lambda/10$. Furthermore, as the mode number $n$ increases (shorter wavelength modes), the $\alpha$ values tend to decrease for $\alpha_{\text{in}} \sim \lambda/10$.

It is worth mentioning here that the conversion of plasma energy into electrical energy across a resistive load, during several expansion and compression cycles of the plasma [25–27], are for an unperturbed initial plasma with $B \leq 0.6$ T. Such operation would be inefficient/challenging for the present system parameters.
since the plasma outer surface, after the first expansion phase, would have high-amplitude perturbations. During the next implosion phase of the plasma, after the turn-around, these perturbations grow further and may generate plasma jetting or extremely large amplitude wave structures, which could affect the smooth implosion of the plasma (the compression phase) and damage the cavity wall. Therefore, further studies are required to explore the concept of plasma energy recovery across a resistive load with several expansion and compression phases [25–27].
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Figure 4.15: Temporal evolution of spectrum amplitude (single mode analysis with \( n = 10 \)) and plasma outer surface radius (both normalized to their initial values) with \( m_p \sim 4.4 \text{ mg} \) and \( E_p = 140 \) and 280 MJ.

The implications for the proposed MFC system (for the present plasma and system parameters) from the results summarized above are as below: The instability amplitudes are not large enough to severely disturb smooth compression of \( B \) for initial perturbations with \( \alpha_{in} \leq \lambda_{in}/10 \). Comparatively large wave structures are observed for the short wavelength modes with \( \alpha_{in} \sim \lambda_{in}/10 \).

Next, it is desirable to determine the threshold value of initial amplitude beyond which instability growth would significantly degrade operation due to large flute structures and jetting. Hence we have next conducted the instability analysis with \( \alpha_{in} \geq \lambda_{in} \) for modes 5, 10, 20 and 40. The present Lagrangian scheme, however, has failed to simulate the plasma dynamics till the stagnation time for \( \alpha_{in} \geq \lambda_{in} \) due to large plasma deformation (plasma jetting). Fig. 4.16 shows such a situation for \( n = 10 \) with \( \alpha_{in} = \lambda_{in} \).

Beyond this time point, the Lagrangian scheme fails due to severe mesh tangling. Therefore, we have continued the analysis with an Eulerian MHD model described in Chapter 3. It is necessary to validate the Eulerian MHD model before we can believe its prediction of MRT instability growth rate. This has been
done by comparing the final $\alpha$ predicted by both models for different perturbation wavelengths (modes $5-40$) with $\alpha_{in} \sim \lambda_{in}/10$. The differences in the final $\alpha$ thus obtained were not more than $\sim 6\%$ of the peak value, the Eulerian model predicting higher growth.

Figure 4.16: Initial plasma configuration (blue color plot) and the plasma configuration at $t = 0.033\mu s$ obtained in the single mode analysis for $n = 10$ with $\alpha_{in} = \lambda_{in}$.

Fig. 4.17 shows snapshots of MRT instability evolution, using Eulerian simulation, at different times, for $n = 10$ with $\alpha_{in} = \lambda_{in}$. Zoomed in plots of plasma density and pressure near the surface of the plasma at $0.09\mu s$ are shown in Fig. 4.18 and Fig. 4.19 respectively. Large plasma deformation, plasma jetting, flute breaking (towards $t_s$) etc are observed. Also, for this case, the instability amplitude is large enough to reach the coil inner surface before the stagnation time, leading to inefficient flux compression. In order to quantify the decrease in the flux compression efficiency, we have plotted the value of $\eta = B_f/B_{t_s}$ vs mode number for different values of $\alpha_{in}$. Here, $B_f$ and $B_{t_s}$ are the $B$ at a time when the plasma instability amplitude (jetting) first reaches the coil inner surface and the peak magnetic field obtained ($\sim 9.5$ T) at $t_s$ assuming ideal operation, respectively.

Fig. 4.20 shows the $\eta$ calculated for modes 5, 10, 20 and 40 with two different
values of $\alpha_{in}$ ($\lambda_{in}$ and $2\lambda_{in}$). For modes $5-10$, the decrease in efficiency is $\sim 15-20\%$ when $\alpha_{in} \sim \lambda_{in}$ and $\sim 30-40\%$ when $\alpha_{in} \sim 2\lambda_{in}$. However, for a given $\alpha_{in}$ (related to $\lambda_{in}$), the decrease in $\eta$ is found to be smaller for short $\lambda$ modes. In general, a loss of efficiency $\sim 20\%$ is expected for longer $\lambda$ modes ($n \leq 20$) and short $\lambda$ modes ($n > 20$) when $\alpha_{in} \sim \lambda_{in}$ and $\alpha_{in} \sim 2\lambda_{in}$ respectively.

Figure 4.17: The snap shots of MRT instability evolution at different times (left and right side plots are at 0.035$\mu$s and 0.09$\mu$s respectively.) with an Eulerian MHD scheme for $n = 10$ with $\alpha_{in} = \lambda_{in}$.

Figure 4.18: The plasma density (normalized to $1.0 \times 10^{-5}$ kg/m$^3$) at 0.09 $\mu$s obtained with an Eulerian MHD scheme for $n = 10$ with $\alpha_{in} = \lambda_{in}$. The plot is shown only near the surface of the plasma for better clarity.
4.5 Conclusions of this study

Two-dimensional MHD simulations of random, single and multi-mode perturbation growth in an MFC system driven by a fusion plasma sphere have been carried out for different initial perturbation amplitudes and wavelengths. The simulation takes into account the effects of magnetic flux compression and geometric divergence due to spherical plasma expansion.

In the random seed perturbation analysis, we have found that the dominant modes in the spectrum show a progressive transition from the short-wavelength to the intermediate-wavelength regime, $\lambda \sim 4\text{–}8 \text{ cm}$ – this is consistent with the observations in Ref. [82]. The cross-correlation analysis indicates the mode coupling between dominant modes and other modes in the spectrum.

The multi-mode (sinusoidal) analysis, with two different fundamental modes, and with $\alpha_{in} \sim 500 \mu\text{m}$, shows the appearance of higher harmonics of the individual modes, as well as the shorter wavelength ($n_1 + n_2$) and higher wavelength inverse cascade ($n_2 - n_1$) modes created by non-linear interaction of fundamental and
harmonic modes.

In the case of single-mode perturbation, the modes continue to grow exponentially with nearly constant $\gamma$ and make a transition into the non-linear phase (mode saturation). That is the amplitude growth of the modes towards stagnation time, although exponential in nature, is lower than the growth predicted by linear theory. We also note that extremely large flute structures and plasma jetting, which could damage or reach the cavity-wall/coil and to severely disturb the smooth compression of the magnetic field, are not seen during the time period of our interest, viz., the first expansion phase of the plasma. This means that it is feasible to have efficient flux compression during the first expansion phase in the proposed system, for perturbation amplitudes $\alpha_{in} \leq \lambda_{in}/10$. However, for $\alpha_{in} \geq \lambda_{in}$, the instability amplitudes are large enough, especially for longer $\lambda$ modes, to cause plasma jetting leading to significant reduction in the flux compression efficiency.

There are a number of remaining issues that need to be addressed to obtain a complete description of the evolution of MRT instability in MFC systems driven by fusion plasma. The extension of the present work to three dimensions may give a better understanding of mode coupling and non-linear evolution. Future work
should quantify the effect of MRT instability in terms of conversion efficiency by using MHD models that are coupled with external circuit equations [52].