Chapter 6

Discrete time $Geo/E_d/1$

Queues with Postponed work and Protected stages

In this chapter, we study a discrete time $Geo/E_d/1$ queue with postponed work with the service of each customer having $n$ stages of which first $v$ stages are unprotected. Till now we assumed that all the arriving customers are alike. But here we categorise the customers in to high and low priority customers. If the buffer has at least one customer, the low priority ones are postponed and high priority ones wait in the buffer. When the buffer is full, the system will not permit further arrivals of high pri-

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Some results of this chapter are included in the following papers.
1. A.Krishnamoorthy, C.B.Ajayakumar, Discrete time $Geo/E_d/1$ Queues with Postponed work and Protected stages (Communicated).

141
Chapter 6. Discrete time Geo/Ea/1 Queues with Postponed work and Protected stages

ority customers. But at that time, a low priority customer can enter the pool with a specified probability or it is also lost from the system with complementary probability.

This is highly practical since in many cases, time is a constraint for the existence of finite capacity queues and the server will be interested to make maximum gain. So he naturally turns to high priority customers and the service of low priority customers will be postponed. If the buffer is full, no further high priority customers join it. At that time, even the low priority customer may not accept the offer of postponement. So the priority based postponement is desirable from the system point of view.

However the postponed work are transferred to the buffer for immediate service with a specified probability at a service completion epoch if the number in the buffer at that time is less than a pre-assigned lower level. But during the service of that postponed work, the buffer size may rise to a pre-assigned higher level and so the server will be compelled to preempt the service of the low priority customer. But the server cannot preempt the work if it is on protected stages of service. We discuss two models in this chapter. The preempted work from unprotected stages will be lost for ever from the system in model-1. This is considered as a negative arrival. In queues with negative arrivals, it is assumed that customer(s) in service is removed due to such an arrival. However in this model, a new arrival hitting a pre-assigned higher level in the buffer takes the role of negative arrival since it decreases a low priority customer in the system. This is also common in practical cases. Actually the undergoing work gets damaged without completing the service. Sudden death of a patient in an operation theatre is an example of such a negative arrival.
In model-2, the preempted work from unprotected stages is considered as an interruption and such an interrupted work is again postponed and wait at the head of the pool for the next chance of transfer. But in the interruption period, if the number of continuously served higher priority work from the buffer attains a pre-assigned number at a service completion epoch, the interrupted customer is transferred to the buffer for immediate service and no further interruption is allowed to that customer in service. The service to the interrupted customer is repeated when it is taken for service again.

6.1 Model-1: With negative arrivals

6.1.1 Mathematical formulation

Consider a Geo/$E_d$/1 queue with finite buffer of capacity $K$. The time between two successive arrivals is governed by a geometrical law with parameter $\alpha$ and the service time of each customer is ruled by a discrete Erlang distribution having $n$ stages of which first $v$ stages are unprotected and the remaining $n - v$ stages are protected. It is described by an irreducible $PH$-representation $(\beta, S)$ of order $n$ where

$$\beta = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1 \times n};$$
Chapter 6. Discrete time Geo/Ea/1 Queues with Postponed work and Protected stages

\[ S = \begin{bmatrix} s_{11} & s_{12} \\ s_{22} & s_{23} \\ \vdots & \vdots \\ s_{n0} \end{bmatrix}_{n \times n} ; \quad S^0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s_{n0} \end{bmatrix}_{n \times 1} \]

Arriving customers are classified into two categories; high priority customers with arrival probability \( p_1 \) and low priority customers with arrival probability \( p_2 \). If a higher priority customer on arrival finds the buffer not full, he joins the same. Otherwise he leaves the system permanently. If a low priority customer on arrival sees the buffer empty, he enters the buffer for immediate service. If the buffer has at least one customer, he proceeds to a pool of postponed work having infinite capacity. But if the buffer is full, he joins the pool only with a specified probability \( \gamma \) (\( 0 \leq \gamma < 1 \)). With probability \( 1 - \gamma \), such customers do not join the system. So clearly the pool will occupy only low priority customers. But the buffer will be occupied by high priority customers and at most one transferred low priority customer.

When at the end of a service, if there are postponed customers, the system operates as follows. If the buffer is empty, the one ahead of all waiting in the pool gets transferred to the buffer for immediate service. If the buffer contains \( y \) jobs, where \( 1 \leq y \leq L - 1; \ 2 \leq L \leq K - 1 \), at a service completion epoch, then with probability \( p \), the head of the queue in the pool is transferred to the buffer for immediate service. With probability \( q = 1 - p \), no such transfer takes place. When at the end of a service, if the buffer is empty, and the pool has no work, the server becomes idle.

When a pool work is on service in unprotected stages, if the buffer
6.1. Model-1: With negative arrivals

size rises to a pre-assigned number \( M + 1 \) such that \( L \leq M \leq K - 1 \) at an arrival epoch, the server will preempt the service of the lower priority customer. The preempted work from unprotected stages will be lost for ever from the system. Here the event which causes to decrease the number of customers by 1, without an actual service completion, is the arrival of a high priority customer to rise the buffer level from \( M \) to \( M + 1 \). So here this is the negative arrival(event). We emphasize that if the pool work at server is on protected stages, such an arrival does not act as a negative arrival as there is no preemption for the low priority customer. Following a negative arrival, the server will perform the buffer work. A diagrammatic representation of model-1 is given in figure 6.1.

Fig 6.1: \( Geo/E_d/1 \) queue with postponed work and Negative arrival

In this discrete time queueing system, time axis is divided in to intervals of equal length called slots, and where all queueing activities take place at the slot boundaries. An arrival and a departure also can happen at a slot boundary. In other words, two or more distinct events can also take place at slot boundaries in the discrete time set up. Let the time axis
be marked by 0, 1, 2, ..., m, ... For mathematical clarity, we assume that departures occur in the interval \((m-, m)\) and arrivals occur in the interval \((m, m+)\). That is, departures occur at the moment immediately before the slot boundaries and arrivals occur at the moment immediately after the slot boundaries. The model is studied as a quasi birth-death (QBD) process and a solution of the classical matrix geometric type is obtained (see [45] and [38]). We define the state space of the QBD and exhibit the structure of its transition probability matrix.

The state space consists of all tuples of the form \((i, j, b, h)\) where \(i\) denotes the number of postponed work in the pool having infinite capacity; \(j\) denotes the number of jobs in the finite buffer including the unit in service; \(b\) denotes the status of the system where

\[
    b = \begin{cases} 
    0, & \text{buffer work is in progress} \\
    1, & \text{pool work is being served}
    \end{cases}
\]

and \(h\) denotes the stage of service in progress at that instant.

Consider the boundary level \(i = 0\). We denote the empty system \((0, 0, 0, 0)\) by 0.

If \(1 \leq j \leq K\) and \(b = 0\) then \(h = 1, 2, ..., n\).

If \(1 \leq j \leq M\) and \(b = 1\) then \(h = 1, 2, ..., n\).

If \(M + 1 \leq j \leq K\) and \(b = 1\) then \(h = v + 1, ..., n\). So the boundary level \(i = 0\) constitute \(N_1 = 1 + 2Mn + (K-M)(2n-v)\) states.

Now consider the level \(i \neq 0\).
If 1 ≤ j ≤ K and b = 0 then h = 1, 2, ..., n.

If 1 ≤ j ≤ M and b = 1 then h = 1, 2, ..., n.

If M + 1 ≤ j ≤ K and b = 1 then h = v + 1, ..., n. So there are

\[ N_2 = 2Mn + (K - M)(2n - v) \]

states are there in the level \( i \neq 0 \).

The transition probability matrix is

\[
P = \begin{bmatrix}
    B_1 & B_0 \\
    B_2 & A_1 & A_0 \\
    & A_2 & A_1 & A_0 \\
    & & \ddots & \ddots & \ddots
\end{bmatrix}
\]

where the matrix \( B_0 \) is of dimension \( N_1 \times N_2 \), \( B_1 \) is square matrix of order \( N_1 \) and \( B_2 \) is of dimension \( N_2 \times N_1 \). \( A_0, A_1 \) and \( A_2 \) are square of order \( N_2 \). Each of these matrices is itself highly structured.

We use the following matrices in the sequel.

\[
E = S^0 \beta = \begin{bmatrix}
    0 & 0 \\
    s_n & 0
\end{bmatrix}_{n \times n};
\]

\[
t_1 = \begin{bmatrix}
    \alpha p_1 \beta & \alpha p_2 \beta
\end{bmatrix}_{1 \times 2n}; \\
\]

\[
t_2 = \begin{bmatrix}
    (1 - \alpha) S^0
\end{bmatrix}_{2n \times 1};
\]
Chapter 6. Discrete time Geo/Ea/1 Queues with Postponed work and Protected stages

\[ V_1 = \begin{bmatrix} (1 - \alpha)S + \alpha p_1 E & \alpha p_2 E \\ \alpha p_1 E & (1 - \alpha)S + \alpha p_2 E \end{bmatrix}_{2n \times 2n} \]

\[ V_2 = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}_{2n \times 2n} \quad ; \quad V_3 = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}_{2n \times 2n} \]

\[ V_4 = \begin{bmatrix} (1 - \alpha)S + \alpha p_1 E & \bar{0} \\ \alpha p_1 E & (1 - \alpha)S \end{bmatrix}_{2n \times 2n} \]

\[ V_5 = \begin{bmatrix} (1 - \alpha)S + \alpha p_1 E & \bar{0} \\ \alpha p_1 E + \alpha p_1 t_3 & (1 - \alpha)S \end{bmatrix}_{2n \times 2n} \quad ; \quad t_3 = \begin{bmatrix} e_v & 0 \\ 0 & 0 \end{bmatrix}_{n \times n} \]

where \( e_v \) is a column vector of ones of order \( v \);

\[ t_4 = \begin{bmatrix} s_{(v+1)(v+1)} & s_{(v+1)(v+2)} \\ & \ddots \\ & \ddots \\ & & s_{nn} \end{bmatrix}_{(n-v) \times (n-v)} \]

\[ t_5 = \begin{bmatrix} \bar{0} \\ t_4 \end{bmatrix}_{n \times (n-v)} \quad ; \quad t_6 = \begin{bmatrix} \bar{0} & \bar{0} \\ s_{n0} & \bar{0} \end{bmatrix}_{(n-v) \times n} \]
6.1. Model-1: With negative arrivals

\[ V_6 = \begin{bmatrix} S & 0 \\ 0 & t_5 \end{bmatrix}_{2n \times (2n-v)} ; \quad V_7 = \begin{bmatrix} E & 0 \\ t_6 & 0 \end{bmatrix}_{(2n-v) \times 2n} ; \]

\[ V_8 = \begin{bmatrix} (1 - \alpha)S + \alpha p_1 E & 0 \\ \alpha p_1 t_6 & (1 - \alpha) t_4 \end{bmatrix}_{(2n-v) \times (2n-v)} ; \]

\[ V_9 = \begin{bmatrix} S & 0 \\ 0 & t_4 \end{bmatrix}_{(2n-v) \times (2n-v)} ; \quad V_{10} = \begin{bmatrix} E & 0 \\ t_6 & 0 \end{bmatrix}_{(2n-v) \times (2n-v)} ; \]

\[ V_{11} = \begin{bmatrix} (1 - \alpha) S + \alpha p_1 E & 0 \\ \alpha p_1 t_6 & (1 - \alpha p_2 \gamma) t_4 \end{bmatrix}_{(2n-v) \times (2n-v)} ; \]

\[ V_{12} = \begin{bmatrix} 0 & E \\ 0 & E \end{bmatrix}_{2n \times 2n} ; \quad V_{13} = \begin{bmatrix} (1 - \alpha)S & \alpha p_2 E \\ 0 & (1 - \alpha) S + \alpha p_2 E \end{bmatrix}_{2n \times 2n} ; \]

\[ V_{14} = \begin{bmatrix} (1 - \alpha) S + q \alpha p_1 E & \alpha p_2 E \\ q \alpha p_1 E & (1 - \alpha) S + p p_2 E \end{bmatrix}_{2n \times 2n} . \]

The matrix \( B_1 \) corresponds to the transition from the level 0 to 0 is
Chapter 6. Discrete time Geo/Em/1 Queues with Postponed work and Protected stages

given below:

$$B_1 = \begin{bmatrix} 1 - \alpha & t_1 \\ t_2 & V_1 & \Phi_1 \\ \Delta_1 & V_2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
V_5 & \Phi_2 \\
\Delta_2 & V_8 & \Phi_3 \\
\Delta_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
V_8 & \Phi_3 \\
\Delta_3 & V_{11} \end{bmatrix}_{N_1 \times N_1}$$

where $\Phi_1 = \alpha p_1 V_2$ corresponds to the transition of the buffer size from $j$ to $j + 1$ for $j = 1, 2, ..., M - 1$, $\Phi_2 = \alpha p_1 V_6$ corresponds to the transition of the buffer size from $M$ to $M + 1$ and $\Phi_3 = \alpha p_1 V_9$ corresponds to the transition of the buffer size from $j$ to $j + 1$ for $j = M + 1, M + 2, ..., K - 1$. $\Delta_1 = (1 - \alpha)V_3$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = 2, 3, ..., M$, $\Delta_2 = (1 - \alpha)V_7$ corresponds to the transition of the buffer size from $M + 1$ to $M$ and $\Delta_3 = (1 - \alpha)V_{10}$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = M + 2, M + 3, ..., K$. Also $V_1$ corresponds to the transition of the buffer size from 1 to 1, $V_4$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = 2, 3, ..., M - 1$, $V_5$ corresponds to the transition of the buffer size from $M$ to $M$, $V_8$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = M + 1, M + 2, ..., K - 1$ and $V_{11}$ corresponds to the transition of the buffer size from $K$ to $K$. 
The matrix $B_0$ is given by

$$B_0 = \begin{bmatrix}
\Phi_4 & \Delta_4 & \cdots & & \\
\Delta_4 & \Phi_4 & & & \\
& \ddots & \ddots & \ddots & \\
& & \Delta_5 & \Phi_5 & \\
& & & \Delta_6 & \ddots \\
& & & & \ddots & \Phi_5 \\
& & & & & \Delta_6 & \Phi_6
\end{bmatrix}_{N_1 \times N_2}$$

where $\Phi_4 = \alpha p_2 V_2$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = 1, 2, ..., M$, $\Phi_5 = \alpha p_2 V_9$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = M + 1, M + 2, ..., K - 1$ and $\Phi_6 = \alpha p_2 \gamma V_9$ corresponds to the transition of the buffer size from $K$ to $K$. Also $\Delta_4 = \alpha p_2 V_3$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = 2, 3, ..., M$, $\Delta_5 = \alpha p_2 V_7$ corresponds to the transition of the buffer size from $M + 1$ to $M$ and $\Delta_6 = \alpha p_2 V_{10}$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = M + 2, M + 3, ..., K$.

The matrix $B_2$ is given by

$$B_2 = \begin{bmatrix}
\Delta_7 & \Phi_7 & \\
\Delta_8 & \Phi_8 & \\
& \ddots & \ddots \\
& & \Delta_8 & \Phi_8
\end{bmatrix}_{N_2 \times N_1}$$

where $\Phi_7 = \alpha p_1 V_{12}$ corresponds to the transition of the buffer size from
1 to 2 and $\Phi_8 = p\alpha p_1 V_{12}$ corresponds to the transition of the buffer size from $j$ to $j+1$ for $j = 2, 3, ..., L$. Also $\Delta_7 = (1 - \alpha)V_{12}$ corresponds to the transition of the buffer size from 1 to 1 and $\Delta_8 = (1 - \alpha)pV_{12}$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = 2, 3, ..., L$.

The matrix $A_1$ is given by

$$
A_1 = \begin{bmatrix}
V_{13} & \Phi_9 \\
\Delta_9 & V_{14} & \ddots \\
& \ddots & \ddots & \ddots \\
& & \Delta_{10} & V_4 & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & V_5 & \Phi_{10} \\
& & & & \Delta_{11} & V_8 & \Phi_{11} \\
& & & & & \Delta_{12} & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
& & & & & & & V_{11}
\end{bmatrix}_{N_2 \times N_2}
$$

where $V_{13}$ corresponds to the transition of the buffer size from 1 to 1, $V_{14}$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = 2, 3, ..., L$, $V_4$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = L + 1, L + 2, ..., M - 1$, $V_5$ corresponds to the transition of the buffer size from $M$ to $M$, $V_8$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = M + 1, M + 2, ..., K - 1$, $V_{11}$ corresponds to the transition of the buffer size from $K$ to $K$.$\Phi_9 = \alpha p_1 V_2$ corresponds to the transition of the buffer size from $j$ to $j+1$ for $j = 1, 2, ..., M - 1$. $\Phi_{10} = \alpha p_1 V_6$ corresponds to the transition of the buffer size from $M$ to $M + 1$, $\Phi_{11} = \alpha p_1 V_9$ corresponds to the transition of the buffer size from $j$
to $j + 1$ for $j = M + 1, M + 2, ..., K - 1$. Also $\Delta_9 = (1 - \alpha)qV_3$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = 2, 3, ..., L$, $\Delta_{10} = (1 - \alpha)V_3$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = L + 1, ..., M$, $\Delta_{11} = (1 - \alpha)V_7$ corresponds to the transition of the buffer size from $M + 1$ to $M$ and $\Delta_{12} = (1 - \alpha)V_{10}$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = M + 2, M + 3, ..., K$.

The matrix $A_0$ is given by

$$
A_0 = \begin{bmatrix}
\Phi_{12} & \Delta_{13} & \cdots & \cdots & \Delta_{14} \\
\Delta_{13} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \ddots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
\Delta_{14} & \cdots & \cdots & \cdots & \Phi_{13} \\
\Delta_{16} & \Phi_{14} & \cdots & \cdots & \cdots
\end{bmatrix}_{N_2 \times N_2}
$$

where

The matrix $A_2$ is given by

$$
A_2 = \begin{bmatrix}
\Delta_{17} & \Phi_{15} & \cdots & \cdots & \cdots \\
\Delta_{18} & \Phi_{16} & \cdots & \cdots & \cdots \\
\vdots & \cdots & \ddots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
\Delta_{18} & \Phi_{16} & \cdots & \cdots & \cdots
\end{bmatrix}_{N_2 \times N_2}$$
Chapter 6. Discrete time Geo/Ea/1 Queues with Postponed work and Protected stages

where $\Phi_{15} = \alpha p_1 V_{12}$ corresponds to the transition of the buffer size from 1 to 2 and $\Phi_{16} = p \alpha p_1 V_{12}$ corresponds to the transition of the buffer size from $j$ to $j+1$ for $j = 2, 3, \ldots, L$. Also $\Delta_{17} = (1 - \alpha)V_{12}$ corresponds to the transition of the buffer size from 1 to 1 and $\Delta_{18} = (1 - \alpha)pV_{12}$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = 2, 3, \ldots, L$.

6.1.2 Stability criterion

**Theorem 6.1.1.** The system is stable if and only if

$$
\alpha p_2 \sum_{j=1}^{K-1} \sum_{b=0}^{1} \sum_{h=1}^{n} \pi_{j, b, h} + \alpha p_2 \gamma \sum_{b=0}^{1} \sum_{h=1}^{n} \pi_{K, b, h} < \frac{1}{N_2} \sum_{l=1}^{m} m_{1l}
$$

where $N_2 = 2Mn + (K - M)(2n - v)$ and $\pi$ is the unique solution to $\pi A = \pi; \pi e = 1$ for $A = A_0 + A_1 + A_2$.

**Proof.** Let $G_{l'}$ be the conditional probability that the QBD process, starting in the state $l = (i, j, b, h)$ (for $i > 1$) where $1 \leq j \leq K; 0 \leq b \leq N; 1 \leq h \leq m$ at time $t = 0$ reaches the state $l' = (i - 1, j', b', h')$ where $1 \leq j' \leq K; 0 \leq b' \leq N; 1 \leq h' \leq m$, for the first time, in a finite time. That is

$$
G_{l'} = P[\tau < \infty : \chi(\tau) = l' | \chi(0) = l]
$$

where $\tau$ is the first passage time from the level $i$ to the level $i - 1$. Because of the structure of $Q$, the probability $G_{l'}$ does not depend on $i$. The matrix with elements $G_{l'}$ is denoted by $G$.

Suppose the matrix $A = A_0 + A_1 + A_2$ is irreducible. Then the necessary
and sufficient condition for the positive recurrence of the process is that
the matrix $G$ is stochastic. For this, the condition $\pi A_2 e > \pi A_0 e$ must
be satisfied where $\pi$ is the stationary probability vector associated with
$A = A_0 + A_1 + A_2$. That is, it is the unique solution to $\pi A = \pi$, $\pi e = 1$
and $A = A_0 + A_1 + A_2$. The quantity $\rho = \frac{\pi A_0 e}{\pi A_2 e}$ is called the traffic intensity
of the QBD process. $G$ is obtained as the minimal non negative solution
to the matrix quadratic equation

$$ G = A_2 + A_1 G + A_0 G^2. $$

This is obvious. On the left-hand side, $G$ records the distribution of the
first state visited in $l'$ conditioned on the initial state being in $l$. In the
right-hand side, these visits to $l'$ are decomposed in to three groups; the
first term corresponds to the case where the QBD directly moves from $i$ to
$i - 1$ in one transition with probabilities recorded in $A_2$; as for the second
term, with probabilities recorded in $A_1$, the QBD remains in $l$ from where
it still has to move eventually to $l'$, with probabilities recorded in $G$; finally
for the last term, with probabilities recorded in $A_0$, the QBD moves up
to $i + 1$ from where it still has to move eventually to $l$, with probabilities
recorded in $G$ and then to $l'$ again with probabilities recorded in $G$.

Let $m_1 = [m_{1i}]$ denote the column vector of dimension $K(N + 1)m$
where $m_{1i}$ denotes the mean first passage time from the level $i$ ($i > 1$)
to the level $i - 1$ given that the first passage time started in the state
$l$. We have $G = (I - A_1)^{-1} A_2 + (I - A_1)^{-1} A_0 G^2$. Consequently $m_1 = [I - A_1 - A_0(I + G)]^{-1} e$.

For the system stability, the rate of drift from level $i$ to level $i - 1$
should be greater than that to level $i + 1$. The rate of drift from level
Chapter 6. Discrete time Geo/Eđ/1 Queues with Postponed work and Protected stages

$i$ to the level $i + 1$ is given by $\alpha p_2 \sum_{j=1}^{K-1} \sum_{b=0}^{1} \sum_{h=1}^{n} \pi_{j bh} + \alpha p_2 \gamma \sum_{b=0}^{1} \sum_{h=1}^{n} \pi_{Kh}$. It follows that the condition $\pi A_0 e < \pi A_2 e$ is equivalent to the given stability criterion.

So by an appropriate choice of $\gamma$, that is by postponing a fraction of overflowing customers, one can obtain a stable system even if arrival rate is greater than service rate.

6.1.3 Stationary distribution

Since the model is studied as a QBD process, its stationary distribution, if it exists, has a matrix geometric solution. Assume that the stability criterion is satisfied. Let the stationary vector $x$ of $P$ be partitioned by the levels in to subvectors $x_i$ for $i \geq 0$. Then $x_i$ has the matrix geometric form

$$x_i = x_1 R^{i-1}$$

for $i \geq 2$ where $R$ is the minimal non negative solution to the matrix equation

$$A_0 + RA_1 + R^2 A_2 = R$$

and the vectors $x_0, x_1$ are obtained by solving the equations

$$x_0 (B_1 - I) + x_1 B_2 = 0$$

$$x_0 B_0 + x_1 (A_1 - I + RA_2) = 0$$
subject to the normalising condition

\[ x_0 e + x_1 (I - R)^{-1} e = 1 \quad (6.5) \]

From the above discussion it is clear that to determine \( x \), a key step is the computation of the rate matrix \( R \). For this purpose, we use logarithmic reduction algorithm as in section 5.1.2 in chapter 5. We again partition \( x_i \) by sublevels as

\[ x_0 = (x_{00}, x_{01}, x_{02}, \ldots, x_{0M}, x_0(M+1), \ldots, x_{0K}) \]

and

\[ x_i = (x_{i1}, x_{i2}, \ldots, x_{iM}, x_i(M+1), \ldots, x_{iK}) \]

where \( i \geq 1 \) and \( x_{00} \) is a scalar and \( x_{0j} = (x_{0j0}, x_{0j1}) \), where if \( 1 \leq j \leq M \), then both \( x_{0j0} \) and \( x_{0j1} \) are vectors of order \( n \) and if \( M+1 \leq j \leq K \), then \( x_{0j0} \) is a vector of order \( n \) and \( x_{0j1} \) is a vector of order \( n - v \). Also

\[ x_{ij} = (x_{ij0}, x_{ij1}) \]

where \( i \geq 1 \) and if \( 1 \leq j \leq M \), then both \( x_{ij0} \) and \( x_{ij1} \) are vectors of order \( n \) and if \( M+1 \leq j \leq K \), then \( x_{ij0} \) are vectors of order \( n \) and \( x_{ij1} \) are vectors of order \( n - v \).
Chapter 6. Discrete time Geo/E$_d$/1 Queues with Postponed work and Protected stages

6.1.4 Performance characteristics

1. The probability that there are $i$ customers in the pool is

$$a_i = \sum_{j=1}^{M} \sum_{b=0}^{1} \sum_{h=1}^{n} x_{ijbh} + \sum_{j=M+1}^{K} \left( \sum_{h=1}^{n} x_{ij0h} + \sum_{h=v+1}^{n} x_{ij1h} \right)$$

for $i > 0$ and

$$a_0 = x_{00} + \sum_{j=1}^{M} \sum_{b=0}^{1} \sum_{h=1}^{n} x_{0jbh} + \sum_{j=M+1}^{K} \left( \sum_{h=1}^{n} x_{0j0h} + \sum_{h=v+1}^{n} x_{0j1h} \right).$$

2. The probability that there are $j$ customers in the buffer (including the one in service) is

$$b_j = \begin{cases} 
  x_{00}, & \text{if } j = 0 \\
  \sum_{b=0}^{1} \sum_{h=1}^{n} x_{0jbh} + \sum_{i=1}^{\infty} \sum_{b=0}^{1} \sum_{h=1}^{n} x_{ijbh}, & \text{if } 1 \leq j \leq M
\end{cases}$$
6.1. Model-1: With negative arrivals

Fig 6.3: $p_1$ versus $\theta_{\text{lost}}$ and $\theta_{\text{TR}}$

and for $M + 1 \leq j \leq M$,

$$b_j = \sum_{h=1}^{n} x_{0j0h} + \sum_{h=v+1}^{n} x_{0j1h} + \sum_{i=1}^{\infty} \left( \sum_{h=1}^{n} x_{ij0h} + \sum_{h=v+1}^{n} x_{ij1h} \right).$$

3. The mean number of pooled customers is

$$\mu_{\text{POOL}} = \sum_{i=1}^{\infty} ia_i = x_1(I - R)^{-2}e.$$

4. The mean buffer size is

$$\mu_{\text{BUFFER}} = \sum_{j=1}^{K} jb_j.$$

5. The probability that an arriving customer enters service immediately is $b_0$.

6. The rate at which the lower priority customer who finds the buffer
Fig 6.4: $p_1$ versus the probability of negative arrival

full leave the system without entering pool is

$$\theta_{\text{LOST}} = \alpha p_2 (1 - \gamma) b_K.$$  

7. The rate at which pooled customers transfer in to the buffer for immediate service is

$$\theta_{TR} = \sum_{i=1}^{\infty} \sum_{b=0}^{1} x_{i1b} s_{n0} + \sum_{i=1}^{\infty} \sum_{j=2}^{L} \sum_{b=0}^{1} x_{ijb} p_s n_0.$$  

8. Probability for a negative arrival (the rate at which negative arrival
occurs) is
\[ N_R = \sum_{i=0}^{\infty} \sum_{h=1}^{v} x_{iM1h} \alpha p_1. \]

6.1.5 Numerical results

To illustrate the performance of the system, we present the following numerical results. A lower priority customer encountering the buffer full, will be inclined to join the pool with higher value of \( \gamma \) if the value of \( L \) and \( p \) are larger. On the other hand, \( \gamma \) inversely varies with \( K \). Based on this, we can take \( \gamma = \frac{Lp}{K} \). The impact of \( p_1 \) (the probability of higher priority customer) on various measures with \( K = 7, L = 4, M = 5, n = 4, v = 2, \alpha = 0.24, p = 0.5, \gamma = \frac{Lp}{K} \),
Chapter 6. Discrete time Geo/E_d/1 Queues with Postponed work and Protected stages

Fig 6.6: $p$ versus $\theta_{\text{LOST}}$ and $\theta_{\text{TR}}$

\[ S = \begin{bmatrix} 0.001 & 0.999 \\ 0.001 & 0.999 \\ 0.0015 & 0.9985 \\ 0.001 \end{bmatrix} \quad \text{and} \quad S^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.999 \end{bmatrix} \]

are numerically computed and shown in figures 6.2, 6.3 and 6.4. As $p_1$ increases the mean pool size decreases due to the decrease of lower priority customers. At the same time, the mean buffer size increases. Hence the transfer rate from the pool to the buffer decreases. As the buffer becomes full, loss rate of lower priority customers starts to increase. As the buffer size approaches $M$, probability of a negative arrival increases at first and then decreases when it rises above $M$.

The effect of $p$ on various measures with $K = 7, L = 4, M = 5, n = 4, v = 2, \alpha = 0.24, p_1 = 0.8, \gamma = \frac{Lp}{K}$ and for the same $S$ and $S^0$ mentioned above, is computed and shown in figures 6.5, 6.6 and 6.7. Here also as $p$ increases, transfer rate increases. So the mean pool size decreases for a high value of $p_1$. Then the buffer size increases. Also probability of loss
6.2. Model-2: With service interruptions under $N$-policy

Fig 6.7: $p$ versus the probability of negative arrival

of lower priority customers decreases due to the effect of the dependence of $p$ on $\gamma$. As $p$ increases probability of a negative arrival increases as expected.

6.2 Model-2: With service interruptions under $N$-policy

Here we discuss the discrete time version of the model discussed in chapter 4, with several additional features such as protected stages of service and priority of customers.
6.2.1 Mathematical formulation

In model-1, when a pooled customer is on service, if the buffer size rises to a pre-assigned number $M + 1$ such that $L \leq M \leq K - 1$, at an arrival epoch, the server will preempt the pool work in progress in unprotected stages and the preempted work will be lost for ever from the system. But in this model, the preempted work is considered to get interrupted. This interrupted pool work is postponed and stay as the head of the queue in the pool for getting next chance of transfer. From the epoch of interruption, the server will serve customers from the buffer and the counting of the number of continuously served customers from the buffer starts. When it reaches $N$ ($N > 0$) at a service completion epoch, the interrupted pooled customer gets transferred to the buffer for immediate service and further interruption is not allowed for such a work. The server will repeat the interrupted work when it is considered again. All other assumptions are same as that of model-1. A diagrammatic representation of the model-2 is given in figure 6.8.

![Diagram](image)

Fig 6.8: Geo/$E_d$/1 queue with postponed work and Service interruption

The state space consists of all tuples of the form $(i, j, b, r, h)$ where $i$
denotes the number of postponed work in the pool having infinite capacity; $j$ denotes the number of jobs in the finite buffer including the unit in service; $b$ denotes the status of the system where

$$b = \begin{cases} 
0 & \text{buffer work is in progress} \\
1 & \text{pool work being served} 
\end{cases}$$

If $b = 0$, $r$ denotes the number of continuously served customers from the buffer including the work at server and If $b = 1$, $r$ denotes the number of continuously served customers from the buffer only, during the period of interruption with $r \neq 0$; $r = 0$ indicates that the head of the pool work is not an interrupted one; $h$ denotes the stage of service in progress at that epoch.

Consider the boundary level $i = 0$. We denote the empty system $(0,0,0,0)$ by 0.

If $1 \leq j \leq K$ and $b = 0$ then $r = 0$ and $h = 1,2,\ldots,n$.

If $1 \leq j \leq M$ and $b = 1$ then $r = 0,1,2,\ldots,N$, and $h = 1,2,\ldots,n$.

If $M + 1 \leq j \leq K$, $b = 1$ and $r = 0,1,2,\ldots,N - 1$ then $h = v + 1,\ldots,n$.

If $M + 1 \leq j \leq K$, $b = 1$ and $r = N$ then $h = 1,2,\ldots,n$.

So the boundary level $i = 0$ constitute $\aleph_1 = 1 + M(N + 2)n + (K - M)[2n + N(n - v)]$ states.

Now consider the level $i \neq 0$.

If $1 \leq j \leq K$ and $b = 0$ then $r = 0,1,2,\ldots,N$ and $h = 1,2,\ldots,n$. 
Chapter 6. Discrete time Geo/E_d/1 Queues with Postponed work and Protected stages

If \( 1 \leq j \leq M \) and \( b = 1 \) then \( r = 0, 1, 2, ..., N \) and \( h = 1, 2, ..., n \).

If \( M + 1 \leq j \leq K \), \( b = 1 \) and \( r = 0, 1, 2, ..., N - 1 \) then \( h = v + 1, ..., n \).

If \( M + 1 \leq j \leq K \), \( b = 1 \) and \( r = N \) then \( h = 1, 2, ..., n \).

So there are \( \aleph_2 = 2Mn(N + 1) + (K - M)[N(n - v) + (N + 2)n \]) states are there in the level \( i \neq 0 \).

The transition probability matrix is

\[
P = \begin{bmatrix}
B_1 & B_0 & \,
B_2 & A_1 & A_0 & \\
A_2 & A_1 & A_0 & \\
& A_2 & A_1 & A_0 & \\
& & & \ddots & \ddots & \ddots
\end{bmatrix}
\]

where the matrix \( B_0 \) is of dimension \( \aleph_1 \times \aleph_2 \), \( B_1 \) is square matrix of order \( \aleph_1 \) and \( B_2 \) is of dimension \( \aleph_2 \times \aleph_1 \). \( A_0, A_1 \) and \( A_2 \) are square of order \( \aleph_2 \). Each of these matrices is itself highly structured.

We use the following matrices in the sequel. \( \beta, S, S^0 \) are all same as that of model-1 and \( E = S^0 \beta \).

\[
S^* = \begin{bmatrix}
S(v+1)(v+1) & S(v+1)(v+2) & \,
S(v+2)(v+2) & \ddots & \\
& \ddots & \ddots & \\
& & & S_{nn}
\end{bmatrix}_{(n-v) \times (n-v)};
\]

\[
S^* = \begin{bmatrix}
S(v+1)(v+1) & S(v+1)(v+2) & \,
S(v+2)(v+2) & \ddots & \\
& \ddots & \ddots & \\
& & & S_{nn}
\end{bmatrix}_{(n-v) \times (n-v)};
\]

\[
S^* = \begin{bmatrix}
S(v+1)(v+1) & S(v+1)(v+2) & \,
S(v+2)(v+2) & \ddots & \\
& \ddots & \ddots & \\
& & & S_{nn}
\end{bmatrix}_{(n-v) \times (n-v)};
\]

\[
S^* = \begin{bmatrix}
S(v+1)(v+1) & S(v+1)(v+2) & \,
S(v+2)(v+2) & \ddots & \\
& \ddots & \ddots & \\
& & & S_{nn}
\end{bmatrix}_{(n-v) \times (n-v)};
\]
6.2. Model-2: With service interruptions under $N$-policy

\[
\begin{align*}
  u_1 &= \begin{bmatrix} \beta & 0 \end{bmatrix} \times (N+1)n ; \\
  u_2 &= e_{N+1} \otimes S^0; \\
  u_3 &= \begin{bmatrix} E & 0 \end{bmatrix} \times (N+1)n ; \\
  u_4 &= e_{N+1} \otimes E; \\
  u_5 &= \begin{bmatrix} (1-\alpha)S + \alpha p E & \cdots \\
  \alpha p E & (1-\alpha)S \\
  \vdots & \ddots \\
  \alpha p E & (1-\alpha)S \end{bmatrix}^{(N+1)n \times (N+1)n} ; \\
  u_6 &= I_{N+1} \otimes S ; \\
  u_7 &= \begin{bmatrix} S & 0 \end{bmatrix} \times (N+1)n ; \\
  u_8 &= \begin{bmatrix} e_{N+1} \otimes E & 0 \end{bmatrix}^{(N+1)n \times (N+1)n} ; \\
  u_9 &= \begin{bmatrix} 0 \\
  S^* \end{bmatrix}^{n \times (n-v)} ; \\
  u_{10} &= \begin{bmatrix} I_N \otimes u_9 & 0 \\
  0 & 0 \end{bmatrix}^{(N+1)n \times [N(n-v)+n]} ; \\
  u_{11} &= \begin{bmatrix} 0 \\
  I_N \otimes F \end{bmatrix}^{(N+1)n \times (N+1)n} ; \\
  u_{12} &= \begin{bmatrix} 0 & 0 \\
  s_{n0} & 0 \end{bmatrix}^{(n-v) \times n}.
\end{align*}
\]
Chapter 6. Discrete time Geo/Ed/1 Queues with Postponed work and Protected stages

\[ u_{13} = \begin{bmatrix} e_N \otimes u_{12} \\ E \end{bmatrix} \in (N(n-v)+n) \times n \]

\[ u_{14} = \begin{bmatrix} I_N \otimes S^* & 0 \\ 0 & S \end{bmatrix} \in (N(n-v)+n) \times (N(n-v)+n) \]

\[ u_{15} = \begin{bmatrix} e_N \otimes u_{12} & 0 \\ E & 0 \end{bmatrix} \in (N(n-v)+n) \times (N+1)n \]

\[ u_{16} = \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} \in (N+1)n \times (N+1)n \]

\[ u_{17} = \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} \in (N+1)n \times (N(n-v)+n) \]

\[ u_{18} = I_{N+1} \otimes E \]

\[ u_{19} = \begin{bmatrix} I_N \otimes pE & 0 \\ 0 & E \end{bmatrix} \in (N+1)n \times (N+1)n \]

\[ u_{20} = \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & I_{N-1} \otimes E \\ 0 & 0 & 0 \end{bmatrix} \in (N+1)n \times (N+1)n \]

\[ H_1 = \begin{bmatrix} \alpha p_1 \beta & \alpha p_2 u_1 \end{bmatrix} \in (N+2)n \times 1 \]

\[ H_2 = \begin{bmatrix} S^0 \\ t_2 \end{bmatrix} \in (N+2)n \times 1 \]
6.2. Model-2: With service interruptions under $N$-policy

\[ H_3 = \begin{bmatrix} (1 - \alpha)S + \alpha p_1 E & \alpha p_2 u_3 \\ \alpha p_1 u_4 & u_5 \end{bmatrix}_{(N+2)n \times (N+2)n} \]

\[ H_4 = \begin{bmatrix} \alpha p_1 S & 0 \\ 0 & \alpha p_1 u_6 \end{bmatrix}_{(N+2)n \times (N+2)n}; \quad H_5 = \begin{bmatrix} E & 0 \\ u_4 & 0 \end{bmatrix}_{(N+2)n \times (N+2)n} \]

\[ H_6 = \begin{bmatrix} (1 - \alpha)S + \alpha p_1 E & 0 \\ \alpha p_1 u_4 & (1 - \alpha)u_6 \end{bmatrix}_{(N+2)n \times (N+2)n} \]

\[ H_7 = \begin{bmatrix} \alpha p_1 S & 0 \\ 0 & \alpha p_1 u_{10} \end{bmatrix}_{(N+2)n \times (N+2)n} \]

\[ H_8 = \begin{bmatrix} E & 0 \\ u_{13} & 0 \end{bmatrix}_{[N(n-v)+2n] \times (N+2)n} \]

\[ H_9 = \begin{bmatrix} (1 - \alpha)S + \alpha p_1 E & 0 \\ \alpha p_1 u_{13} & (1 - \alpha)u_{14} \end{bmatrix}_{[N(n-v)+2n] \times [N(n-v)+2n]} \]

\[ H_{10} = \begin{bmatrix} \alpha p_1 S & 0 \\ 0 & \alpha p_1 u_{14} \end{bmatrix}_{[N(n-v)+2n] \times [N(n-v)+2n]} \]
Chapter 6. Discrete time Geo/E$_d$/1 Queues with Postponed work and Protected stages

\begin{align*}
H_{11} &= \begin{bmatrix} E & \bar{0} \\ u_{13} & \bar{0} \end{bmatrix}_{[N(n-v)+2n] \times [N(n-v)+2n]} ; \\
H_{12} &= \begin{bmatrix} (1-\alpha p_2 \gamma)S + \alpha p_1 E & \bar{0} \\ \alpha p_1 u_{13} & (1-\alpha p_2 \gamma)u_{14} \end{bmatrix}_{[N(n-v)+2n] \times [N(n-v)+2n]} ; \\
H_{13} &= \begin{bmatrix} \bar{0} & u_{18} \\ \bar{0} & u_8 \end{bmatrix}_{2(N+1)n \times (N+2)n} ; \\
H_{14} &= \begin{bmatrix} (1-\alpha)u_6 & \alpha p_2 u_{18} \\ \bar{0} & (1-\alpha)u_6 + \alpha p_2 u_8 \end{bmatrix}_{2(N+1)n \times 2(N+1)n} ; \\
H_{15} &= \begin{bmatrix} u_6 & \bar{0} \\ \bar{0} & u_6 \end{bmatrix}_{2(N+1)n \times 2(N+1)n} ; \\
H_{16} &= \begin{bmatrix} \bar{0} & u_{19} \\ \bar{0} & p u_8 \end{bmatrix}_{2(N+1)n \times (N+2)n} ; \\
H_{17} &= \begin{bmatrix} u_{20} & \bar{0} \\ u_8 & \bar{0} \end{bmatrix}_{2(N+1)n \times 2(N+1)n} ; \\
H_{18} &= \begin{bmatrix} (1-\alpha)u_6 + q \alpha p_1 u_{20} & \alpha p_2 u_{18} \\ q \alpha p_1 u_8 & (1-\alpha)u_6 + \alpha p_2 u_8 \end{bmatrix}_{2(N+1)n \times 2(N+1)n} .
\end{align*}
6.2. Model-2: With service interruptions under \(N\)-policy

\[
H_{19} = \begin{bmatrix}
\tilde{0} & u_{16} \\
\tilde{0} & \tilde{0}
\end{bmatrix}_{2(N+1)n \times (N+2)n} \quad ; \quad H_{20} = \begin{bmatrix}
\tilde{0} & u_{17} \\
\tilde{0} & \tilde{0}
\end{bmatrix}_{2(N+1)n \times [N(n-v)+2n]}
\]

\[
H_{21} = \begin{bmatrix}
(1 - \alpha)u_6 + \alpha p_1 u_{20} & \alpha p_2 u_{16} \\
\alpha p_1 u_8 & (1 - \alpha)u_6
\end{bmatrix}_{2(N+1)n \times (N+1)n}
\]

\[
H_{22} = \begin{bmatrix}
u_6 & \tilde{0} \\
\tilde{0} & u_{10}
\end{bmatrix}_{2(N+1)n \times [(N+2)n+N(n-v)]}
\]

\[
H_{23} = \begin{bmatrix}
\alpha p_2 u_6 & \tilde{0} \\
\alpha p_1 t_{11} & \alpha p_2 u_6
\end{bmatrix}_{2(N+1)n \times (N+1)n}
\]

\[
H_{24} = \begin{bmatrix}
\tilde{0} & u_{17} \\
\tilde{0} & \tilde{0}
\end{bmatrix}_{[(N+1)n+N(n-v)+n] \times [n(n-v)+2n]}
\]

\[
H_{25} = \begin{bmatrix}
u_{20} & \tilde{0} \\
u_{15} & \tilde{0}
\end{bmatrix}_{[(N+1)n+N(n-v)+n] \times [2(N+1)n]}
\]

\[
H_{26} = \begin{bmatrix}
(1 - \alpha)u_6 + \alpha p_1 u_{20} & \alpha p_2 u_{17} \\
\alpha p_1 u_{15} & (1 - \alpha)u_{14}
\end{bmatrix}
\]

having order \([N + n + N(n - v) + n] \times [(N + 1)n + N(n - v) + n]\) ;
Chapter 6. Discrete time Geo/E\(_d\)/1 Queues with Postponed work and Protected stages

\[
H_{27} = \begin{bmatrix}
    u_6 & 0 \\
    0 & u_{14}
\end{bmatrix} \quad \text{having order } \left((N+1)n + N(n - v) + n\right) \times \left((N+1)n + N(n - v) + n\right)
\]

\[
H_{28} = \begin{bmatrix}
    u_{20} & 0 \\
    u_{15} & 0
\end{bmatrix} \quad \text{having order } \left((N+1)n + N(n - v) + n\right) \times \left((N+1)n + N(n - v) + n\right)
\]

\[
H_{29} = \begin{bmatrix}
    (1 - \alpha_2 \gamma)u_6 & \alpha_2 \gamma u_{17} \\
    \alpha_1 u_{15} & (1 - \alpha_2 \gamma)u_{14}
\end{bmatrix}
\]

\[
H_{30} = \begin{bmatrix}
    u_7 & 0 \\
    0 & u_6
\end{bmatrix} \quad \text{having order } \left((N+1)n + N(n - v) + n\right) \times \left((N+1)n + N(n - v) + n\right)
\]

\[
H_{31} = \begin{bmatrix}
    u_3 & 0 \\
    u_8 & 0
\end{bmatrix} \quad \text{having order } \left((N+1)n + N(n - v) + n\right) \times \left((N+1)n + N(n - v) + n\right)
\]

\[
H_{32} = \begin{bmatrix}
    0 & 0 \\
    u_{11} & 0
\end{bmatrix} \quad \text{having order } \left((N+1)n + N(n - v) + n\right) \times \left((N+1)n + N(n - v) + n\right)
\]

\[
H_{33} = \begin{bmatrix}
    u_3 & 0 \\
    u_{15} & 0
\end{bmatrix} \quad \text{having order } \left((N(n - v) + 2n) \times 2(N + 1)n\right)
\]

\[
H_{34} = \begin{bmatrix}
    u_7 & 0 \\
    0 & u_{14}
\end{bmatrix} \quad \text{having order } \left((N(n - v) + 2n) \times 2(N + 1)n\right)
\]

\[
H_{35} = \begin{bmatrix}
    u_3 & 0 \\
    u_{15} & 0
\end{bmatrix} \quad \text{having order } \left((N(n - v) + 2n) \times 2(N + 1)n\right)
\]
6.2. Model-2: With service interruptions under $N$-policy

\[ H_{36} = \begin{bmatrix} \bar{0} & u_{18} \\ \bar{0} & u_8 \end{bmatrix}_{2(N+1)n \times 2(N+1)n} \]

\[ H_{37} = \begin{bmatrix} \bar{0} & u_{19} \\ \bar{0} & pu_{8} \end{bmatrix}_{2(N+1)n \times 2(N+1)n} ; \quad H_{38} = \begin{bmatrix} \bar{0} & u_{16} \\ \bar{0} & \bar{0} \end{bmatrix}_{2(N+1)n \times 2(N+1)n} \]

\[ H_{39} = \begin{bmatrix} \bar{0} & u_{17} \\ \bar{0} & \bar{0} \end{bmatrix}_{2(N+1)n \times [(N+1)n+N(n-v)+n]} \]

\[ H_{40} = \begin{bmatrix} \bar{0} & u_{17} \\ \bar{0} & \bar{0} \end{bmatrix}_{[(N+1)n+N(n-v)+n] \times [(N+1)n+N(n-v)+n]} \]

The matrix $B_1$ corresponds to the transition from the level 0 to 0 is given below:

\[ B_1 = \begin{bmatrix}
1 - \alpha & H_1 \\
\Omega_1 & H_3 & H_4 \\
\Omega_2 & H_6 & \ddots \\
& \ddots & \ddots & H_7 \\
& \Omega_3 & H_9 & H_{10} \\
& \Omega_4 & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
\end{bmatrix}_{\aleph_1 \times \aleph_1} \]
where $\Omega_1 = (1 - \alpha)H_2 \text{ corresponds to the transition of the buffer size from 1 to 0, } \Omega_2 = (1 - \alpha)H_5 \text{ corresponds to the transition of the buffer size from } j \text{ to } j - 1 \text{ for } j = 2, 3, ..., M, \text{ and } \Omega_3 = (1 - \alpha)H_8 \text{ corresponds to the transition of the buffer size from } M + 1 \text{ to } M, \Omega_4 = (1 - \alpha)H_{11} \text{ corresponds to the transition of the buffer size from } j \text{ to } j - 1 \text{ for } j = M + 2, M + 3, ..., K. \text{ Also } H_1 \text{ corresponds to the transition of the buffer size from 0 to 1, } H_4 \text{ corresponds to the transition of the buffer size from } j \text{ to } j + 1 \text{ for } j = 1, 2, ..., M - 1, H_7 \text{ corresponds to the transition of the buffer size from } M \text{ to } M + 1, H_{10} \text{ corresponds to the transition of the buffer size from } j \text{ to } j + 1 \text{ for } j = M + 1, M + 2, ..., K - 1 \text{ and } H_3 \text{ corresponds to the transition of the buffer size from 1 to 1, } H_6 \text{ corresponds to the transition of the buffer size from } j \text{ to } j \text{ for } j = 2, 3, ..., M, \text{ and } H_9 \text{ corresponds to the transition of the buffer size from } j \text{ to } j \text{ for } j = M + 1, M + 2, ..., K - 1, H_{12} \text{ corresponds to the transition of the buffer size from } K \text{ to } K.$

The matrix $B_0$ is given by

$$B_0 = \begin{bmatrix}
\Theta_1 & \Theta_2 & \cdots & \Theta_4 \\
\Omega_5 & \Theta_2 & \cdots & \\
\cdots & \cdots & \cdots & \\
\Omega_6 & \Theta_3 & \cdots & \\
\cdots & \cdots & \cdots & \\
\Omega_7 & \cdots & \cdots & \\
\Omega_7 & \cdots & \cdots & \\
\end{bmatrix}_{n_1 \times n_2}$$

where $\Omega_5 = \alpha p_2 H_{31} \text{ corresponds to the transition of the buffer size from } j \text{ to } j - 1 \text{ for } j = 2, 3, ..., M, \Omega_6 = \alpha p_2 H_{33} \text{ corresponds to the transition of the buffer size from }$
of the buffer size from $M + 1$ to $M$ and $\Omega_7 = \alpha p_2 H_{35}$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = M + 2, M + 3, \ldots, K$. Also $\Theta_1 = \alpha p_2 H_{30}$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = 1, 2, \ldots, M - 1$, $\Theta_2 = \alpha p_2 H_{30} + \alpha p_2 H_{32}$ corresponds to the transition of the buffer size from $M$ to $M$, $\Theta_3 = \alpha p_2 H_{34}$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = M + 1, M + 2, \ldots, K - 1$ and $\Theta_4 = \alpha p_2 \gamma H_{34}$ corresponds to the transition of the buffer size from $K$ to $K$.

The matrix $B_2$ is given by

$$
B_2 = \begin{bmatrix}
\Omega_8 & \Theta_5 \\
\Omega_9 & \Theta_6 \\
\vdots & \ddots \\
\Omega_{10} & \Theta_7 \\
\vdots & \ddots \\
\Theta_8 \\
\Omega_{11} & \Theta_9 \\
\vdots & \ddots \\
\end{bmatrix}_{n_2 \times n_1}
$$

where $\Omega_8 = (1 - \alpha)H_{13}$ corresponds to the transition of the buffer size from 1 to 1, $\Omega_9 = (1 - \alpha)H(16)$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = 2, 3, \ldots, L$, and $\Omega_{10} = (1 - \alpha)H_{19}$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = L + 1, L + 2, \ldots, M$, $\Omega_{11} = (1 - \alpha)H_{24}$ corresponds to the transition of the buffer size from $j$ to $j$ for $j = M + 1, M + 2, \ldots, K - 1$ and $\Omega_{12} = (1 - \alpha)p_2 \gamma H_{24}$ corresponds to the
transition of the buffer size from $K$ to $K$. Also $\Theta_5 = \alpha p_1 H_{13}$ corresponds to the transition of the buffer size from 1 to 2, $\Theta_6 = \alpha p_1 H_{16}$ corresponds to the transition of the buffer size from $j$ to $j + 1$ for $j = 2, 3, ..., L$, $\Theta_7 = \alpha p_1 H_{19}$ corresponds to the transition of the buffer size from $j$ to $j + 1$ for $j = L + 1, L + 2, ..., M - 1$, $\Theta_8 = \alpha p_1 H_{20}$ corresponds to the transition of the buffer size from $M$ to $M + 1$ and $\Theta_9 = \alpha p_1 H_{24}$ corresponds to the transition of the buffer size from $j$ to $j + 1$ for $j = M + 1, M + 2, ..., K - 1$.

$$A_1 = \begin{bmatrix} H_{14} & \Theta_{10} \\ \Omega_{12} & H_{18} & \ddots \\ & \ddots & \ddots \\ & & \Omega_{13} & H_{21} & \ddots \\ & & & \ddots & \ddots & \Theta_{10} \\ & & & & H_{21} & \Theta_{11} \\ & & & & \Omega_{14} & H_{26} & \Theta_{12} \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & \Omega_{15} & \ddots & \ddots & \Theta_{12} \\ & & & & \Omega_{15} & H_{29} \end{bmatrix}_{\aleph_2 \times \aleph_2}$$

where $\Omega_{12} = (1 - \alpha)qH_{17}$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = 2, 3, ..., L$, $\Omega_{13} = (1 - \alpha)H_{17}$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = L + 1, L + 2, ..., M$ and $\Omega_{14} = (1 - \alpha)H_{25}$ corresponds to the transition of the buffer size from $M + 1$ to $M$, $\Omega_{15} = (1 - \alpha)H_{28}$ corresponds to the transition of the buffer size from $j$ to $j - 1$ for $j = M + 2, M + 3, ..., K$. Also $\Theta_{10} = \alpha p_1 H_{15}$ corresponds to the transition of the buffer size from $j$ to $j + 1$ for $j = 1, 2, ..., M - 1$, ...
\[ \Theta_{11} = \alpha p_1 H_{22} \] corresponds to the transition of the buffer size from \( M \) to \( M + 1 \), \( \Theta_{12} = \alpha p_1 H_{27} \) corresponds to the transition of the buffer size from \( j \) to \( j + 1 \) for \( j = M + 1, M + 2, \ldots, K - 1 \), \( H_{14} \) corresponds to the transition of the buffer size from 1 to 1, \( H_{18} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = 2, 3, \ldots, L \), \( H_{21} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = L + 1, L + 2, \ldots, M \), \( H_{26} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = M + 1, M + 2, \ldots, K - 1 \), \( H_{29} \) corresponds to the transition of the buffer size from \( K \) to \( K \).

\[
A_0 = \begin{bmatrix}
\Theta_{13} & & \\
\Omega_{16} & \ddots & \\
& \ddots & \ddots \\
& & \Omega_{17} & V_{23} & \\
& & \ddots & \Omega_{18} & \Theta_{14} \\
& & & \Omega_{19} & \ddots \\
& & & & \Theta_{14} \\
& & & & \Omega_{19} \end{bmatrix}_{\aleph_2 \times \aleph_2}
\]

where \( \Omega_{16} = q \alpha p_2 H_{17} \) corresponds to the transition of the buffer size from \( j \) to \( j - 1 \) for \( j = 2, 3, \ldots, L \), \( \Omega_{17} = \alpha p_2 H_{17} \) corresponds to the transition of the buffer size from \( j \) to \( j - 1 \) for \( j = L + 1, L + 2, \ldots, M \), \( \Omega_{18} = \alpha p_2 H_{25} \) corresponds to the transition of the buffer size from \( M + 1 \) to \( M \) and \( \Omega_{19} = \alpha p_2 H_{28} \) corresponds to the transition of the buffer size from \( j \) to \( j - 1 \) for \( j = M + 2, M + 3, \ldots, K \). Also \( \Theta_{13} = \alpha p_2 H_{15} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = 1, 2, \ldots, M - 1 \), \( H_{23} \) corresponds to the
transition of the buffer size from \( M \) to \( M \), \( \Theta_{14} = \alpha p_2 H_{27} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = M + 1, M + 2, \ldots, K - 1 \) and \( \Theta_{15} = \alpha p_2 \gamma H_{27} \) corresponds to the transition of the buffer size from \( K \) to \( K \).

\[
A_2 = \begin{bmatrix}
\Omega_{20} & \Theta_{16} \\
\Omega_{21} & \Theta_{17} \\
& \ddots & \ddots \\
& \Omega_{22} & \Theta_{18} \\
& & \ddots & \ddots \\
& & \Omega_{23} & \Theta_{20} \\
& & & \ddots & \ddots \\
& & & \Omega_{24}
\end{bmatrix}_{8 \times 8}
\]

where \( \Omega_{20} = (1 - \alpha) H_{36} \) corresponds to the transition of the buffer size from 1 to 1, \( \Omega_{21} = (1 - \alpha) H_{37} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = 2, 3, \ldots, L \), and \( \Omega_{22} = (1 - \alpha) H_{38} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = L + 1, L + 2, \ldots, M \), \( \Omega_{23} = (1 - \alpha) H_{40} \) corresponds to the transition of the buffer size from \( j \) to \( j \) for \( j = M + 1, M + 2, \ldots, K - 1 \) and \( \Omega_{24} = (1 - \alpha) p_2 \gamma H_{40} \) corresponds to the transition of the buffer size from \( K \) to \( K \). Also \( \Theta_{16} = \alpha p_1 H_{36} \) corresponds to the transition of the buffer size from 1 to 2, \( \Theta_{17} = \alpha p_1 H_{37} \) corresponds to the transition of the buffer size from \( j \) to \( j + 1 \) for \( j = 2, 3, \ldots, L \), \( \Theta_{18} = \alpha p_1 H_{38} \) corresponds to the transition of the buffer size from \( j \) to \( j + 1 \) for \( j = L + 1, L + 2, \ldots, M - 1 \), \( \Theta_{19} = \alpha p_1 H_{39} \) corresponds to the transition
of the buffer size from $M$ to $M+1$ and $\Theta_{20} = \alpha p_1 H_{40}$ corresponds to the transition of the buffer size from $j$ to $j+1$ for $j = M+1, M+2, ..., K-1$.

### 6.2.2 Stability criterion

**Theorem 6.2.1.** The system is stable if and only if

$$
\alpha p_2 \left( \sum_{j=1}^{K-1} \sum_{r=0}^{N} \sum_{h=1}^{n} \pi_{j0rh} + \sum_{j=1}^{M} \sum_{r=0}^{N} \sum_{h=1}^{n} \pi_{j1rh} \right)
$$

$$+ \alpha p_2 \sum_{j=M+1}^{K-1} \left( \sum_{r=0}^{N-1} \sum_{h=v+1}^{n} \pi_{j1rh} + \sum_{h=1}^{n} \pi_{j1Nh} \right)
$$

$$+ \alpha p_2 \gamma \left( \sum_{r=0}^{N-1} \sum_{h=v+1}^{N} \pi_{K1rh} + \sum_{h=1}^{n} \pi_{K1Nh} + \sum_{r=0}^{N} \sum_{h=1}^{n} \pi_{K0rh} \right)
$$

$$+ \alpha p_1 \sum_{r=0}^{N-1} \sum_{h=1}^{m} \pi_{M1rh} < \frac{1}{\sum_{l=1}^{N_2} m_{1l}}
$$

where $N_2 = 2Mn(N + 1) + (K - M) [N(n - v) + (N + 2)n$ and $\pi$ is the unique solution to $\pi A = \pi; \pi e = 1$ for $A = A_0 + A_1 + A_2$.

**Proof.** Let $G_{ll'}$ be the conditional probability that the QBD process starting in the state $l = (i, j, b, h)$ (for $i > 1$) where $1 \leq j \leq K; 0 \leq b \leq N; 1 \leq h \leq m$ at time $t = 0$ reaches the state $l' = (i - 1, j', b', h')$ where $1 \leq j' \leq K; 0 \leq b' \leq N; 1 \leq h' \leq m$, for the first time, in a finite time. That is

$$G_{ll'} = P[\tau < \infty : \chi(\tau) = l' | \chi(0) = l]$$
Chapter 6. Discrete time \( Geo/E_a/1 \) Queues with Postponed work and Protected stages

where \( \tau \) is the first passage time from the level \( i \) to the level \( i - 1 \). Because of the structure of \( Q \), the probability \( G_{l'l'} \) does not depend on \( i \). The matrix with elements \( G_{l'l'} \) is denoted by \( G \).

Suppose the matrix \( A = A_0 + A_1 + A_2 \) is irreducible. Then the necessary and sufficient condition for the positive recurrence of the process is that the matrix \( G \) is stochastic. For this, the condition \( \pi A_2 e > \pi A_0 e \) must be satisfied where \( \pi \) is the stationary probability vector associated with \( A = A_0 + A_1 + A_2 \). That is, it is the unique solution to \( \pi A = \pi \), \( \pi e = 1 \) and \( A = A_0 + A_1 + A_2 \). The quantity \( \rho = \frac{\pi A_0 e}{\pi A_2 e} \) is called the traffic intensity of the QBD process. \( G \) is obtained as the minimal non negative solution to the matrix quadratic equation

\[
G = A_2 + A_1 G + A_0 G^2.
\]

This is obvious. On the left-hand side, \( G \) records the distribution of the first state visited in \( l' \) conditioned on the initial state being in \( l \). In the right-hand side, these visits to \( l' \) are decomposed in to three groups; the first term corresponds to the case where the QBD directly moves from \( i \) to \( i - 1 \) in one transition with probabilities recorded in \( A_2 \); as for the second term, with probabilities recorded in \( A_1 \), the QBD remains in \( l \) from where it still has to move eventually to \( l' \), with probabilities recorded in \( G \); finally for the last term, with probabilities recorded in \( A_0 \), the QBD moves up to \( i + 1 \) from where it still has to move eventually to \( l \), with probabilities recorded in \( G \) and then to \( l' \) again with probabilities recorded in \( G \).

Let \( m_1 = [m_{1i}] \) denotes the column vector of dimension \( K(N + 1)m \) where \( m_{1i} \) denotes the mean first passage time from the level \( i \) \( (i > 1) \) to the level \( i - 1 \) given that the first passage time started in the state
6.2. Model-2: With service interruptions under $N$-policy

1. We have $G = (I - A_1)^{-1}A_2 + (I - A_1)^{-1}A_0G^2$. Consequently $m_1 = [I - A_1 - A_0(I + G)]^{-1}e$.

For the system stability, the rate of drift from level $i$ to level $i - 1$ should be greater than that to level $i + 1$. It follows that the condition $\pi A_0 e < \pi A_2 e$ is equivalent to the given stability criterion.

So by an appropriate choice of $\gamma$, that is by postponing a fraction of overflowing customers, one can obtain a stable system even if arrival rate is greater than service rate.

6.2.3 Stationary distribution

Since the model is studied as a QBD process, its stationary distribution, if it exists, has a matrix geometric solution. Assume that the stability criterion is satisfied. Let the stationary vector $x$ of $P$ be partitioned by the levels in to subvectors $x_i$ for $i \geq 0$. Then $x_i$ has the matrix geometric form

$$x_i = x_1 R^{i-1}$$

for $i \geq 2$ where $R$ is the minimal non negative solution to the matrix equation

$$A_0 + RA_1 + R^2 A_2 = R$$

(6.7)
and the vectors \( x_0, x_1 \) are obtained by solving the equations

\[
\begin{align*}
    x_0(B_1 - I) + x_1B_2 &= 0 \quad (6.8) \\
    x_0B_0 + x_1(A_1 - I + RA_2) &= 0 \quad (6.9)
\end{align*}
\]

subject to the normalising condition

\[
x_0e + x_1(I - R)^{-1}e = 1 \quad (6.10)
\]

From the above discussion it is clear that to determine \( x \), a key step is the computation of the rate matrix \( R \). Here also we use logarithmic reduction algorithm as in section 5.1.2 in chapter 5. We again partition \( x_i \) by sublevels as

\[
x_0 = (x_{00}, x_{01}, x_{02}, \ldots, x_{0M}, x_{0(M+1)}, \ldots, x_{0K})
\]

and

\[
x_i = (x_{i1}, x_{i2}, \ldots, x_{iM}, x_{i(M+1)}, \ldots, x_{iK})
\]

where \( i \geq 1 \) and \( x_{00} \) is a scalar and \( x_{0j} = (x_{0j0}, x_{0j1}) \), where if \( 1 \leq j \leq M \), then \( x_{0j0} \) are vectors of order \( n \) and \( x_{0j1} \) are vectors of order \( (N+1)n \) and if \( M + 1 \leq j \leq K \), then \( x_{0j0} \) are vectors of order \( n \) and \( x_{0j1} \) are vectors of order \( N(n - v) + n \). Also

\[
x_{ij} = (x_{ij0}, x_{ij1})
\]

where \( i \geq 1 \) and if \( 1 \leq j \leq M \), then \( x_{ij0} \) and \( x_{ij1} \) are vectors of order \( (N+1)n \) and if \( M + 1 \leq j \leq K \), then \( x_{ij0} \) are vectors of order \( (N+1)n \) but \( x_{ij1} \) are vectors of order \( N(n - v) + n \).
6.2.4 Performance characteristics

1. The probability that there are \( i \) customers in the pool is

\[
a_i = \sum_{j=1}^{M} \sum_{b=0}^{1} \sum_{r=0}^{N} \sum_{h=1}^{n} x_{ijbrh} 
\]

\[
+ \sum_{j=M+1}^{K} \left( \sum_{r=0}^{N} \sum_{h=1}^{n} x_{ij0rh} + \sum_{r=0}^{N-1} \sum_{h=v+1}^{n} x_{ij1rh} + \sum_{h=1}^{n} x_{ij1Nh} \right) 
\]

for \( i > 0 \) and

\[
a_0 = x_{000} + \sum_{j=1}^{M} \left( \sum_{h=1}^{n} x_{0j00h} + \sum_{r=0}^{N} \sum_{h=1}^{n} x_{0j1rh} \right) 
\]

\[
+ \sum_{j=M+1}^{K} \left( \sum_{h=1}^{n} x_{0j00h} + \sum_{r=0}^{N-1} \sum_{h=v+1}^{n} x_{0j1rh} + \sum_{h=1}^{n} x_{0j1Nh} \right) 
\]

2. The probability that there are \( j \) customers in the buffer (including the one in service) is

\[
b_j = \sum_{h=1}^{n} x_{0j00h} + \sum_{r=0}^{N} \sum_{h=1}^{n} x_{0j1rh} + \sum_{i=1}^{\infty} \sum_{b=0}^{1} \sum_{r=0}^{N} \sum_{h=1}^{n} x_{ijbrh} 
\]

\[
+ \sum_{i=1}^{\infty} \left( \sum_{r=0}^{N} \sum_{h=1}^{n} x_{ij0rh} + \sum_{r=0}^{N-1} \sum_{h=v+1}^{n} x_{ij1rh} + \sum_{h=1}^{n} x_{ij1Nh} \right) 
\]

for \( 1 \leq j \leq M \),

\[
b_j = \sum_{h=1}^{n} (x_{0j00h} + x_{0j1Nh}) + \sum_{r=0}^{N-1} \sum_{h=v+1}^{n} x_{0j1rh} 
\]

\[
+ \sum_{i=1}^{\infty} \left( \sum_{r=0}^{N} \sum_{h=1}^{n} x_{ij0rh} + \sum_{r=0}^{N-1} \sum_{h=v+1}^{n} x_{ij1rh} + \sum_{h=1}^{n} x_{ij1Nh} \right) 
\]
Chapter 6. Discrete time Geo/Ea/1 Queues with Postponed work and Protected stages

for $M + 1 \leq j \leq K$ and $b_0 = x_{00}$.

3. The mean number of pooled customers is

$$\mu_{POOL} = \sum_{i=1}^{\infty} ia_i = x_1(I - R)^{-2}e.$$  

4. The mean buffer size is

$$\mu_{BUFFER} = \sum_{j=1}^{K} jb_j.$$  

5. The probability that an arriving customer enters service immediately is $b_0$.

6. The rate at which the lower priority customer who find the buffer full leave the system without entering pool (mean number of customers not joining the system per unit time) is

$$\theta_{LOST} = \alpha p_2(1 - \gamma)b_K.$$  

7. The rate at which pooled customers transfer into the buffer for immediate service is

$$\theta_{TR} = \sum_{i=1}^{\infty} \sum_{b=0}^{1} \sum_{r=0}^{N} x_{i1brn}S_{n0} + \sum_{i=1}^{\infty} \sum_{j=2}^{L} \sum_{b=0}^{1} \sum_{r=0}^{N-1} x_{ijbrn}P_{S_{n0}}$$

$$+ \sum_{i=1}^{\infty} \sum_{j=1}^{K} x_{ij0Nn}S_{n0}.$$
6.2. Model-2: With service interruptions under $N$-policy

8. Interruption rate is

$$I_R = \sum_{i=0}^{\infty} \sum_{r=0}^{N-1} \sum_{h=1}^{v} x_{iM1rh} \alpha p_1.$$ 

6.2.5 Numerical results

![Graphs showing numerical results](image1)

**Fig 6.9:** $p$ versus $\mu_{POOL}$ and $\mu_{BUFFER}$

![Graphs showing numerical results](image2)

**Fig 6.10:** $p$ versus $\theta_{LOST}$ and $\theta_{TR}$
In this section, we illustrate the performance of the system by considering some numerical results. A lower priority customer encountering the buffer full, will be inclined to join the pool with higher $\gamma$ if the $L$ and $p$ values are larger. On the other hand $\gamma$ inversely varies with $K$ and $N$. To model this situation, we take $\gamma = \frac{Lp}{K} + \frac{1}{N}$. But the relationship is feasible for those values of $L, p, K$ and $N$ such that $0 \leq \gamma \leq 1$.

The effect of $p$ on various measures with $K = 7, L = 4, M = 5, n = 4, N = 3, v = 2, \alpha = 0.2, p_1 = 0.8, \gamma = \frac{Lp}{K} + \frac{1}{N}$, 

$$S = \begin{bmatrix} 0.001 & 0.999 \\ 0.001 & 0.999 \\ 0.0015 & 0.9985 \\ 0.001 & \end{bmatrix} \quad \text{and} \quad S^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.999 \end{bmatrix}$$

is computed and shown in figures 6.9 and 6.10. As $p$ increases, transfer rate increases. So the mean pool size decreases for a high value of $p_1$. Then the buffer size increases. Also probability of loss of lower priority customers decreases due to the effect of the dependence of $p$ on $\gamma$. 

Fig 6.11: $N$ versus $\mu_{POOL}$ and $\mu_{BUFFER}$
Fig 6.12: $N$ versus $\theta_{LOST}$ and $\theta_{TR}$

The impact of $N$ on various measures with $K = 7, L = 4, M = 5, n = 4, v = 2, \alpha = 0.2, p_1 = 0.8, p = 0.5, \gamma = \frac{L_p}{K} + \frac{1}{N}$ and for the same $S$ and $S^0$ mentioned above, is shown in figures 6.11 and 6.12. As $N$ increases $\mu_{POOL}, \mu_{BUFFER}, \theta_{TR}$ decrease monotonically whereas $\theta_{LOST}$ increases monotonically. This is due to the fact that by our assumption $\gamma$ varies inversely as $N$ and as a result, loss rate increases and inflow rate to the pool decreases as $N$ increases. So the transfer rate of the interrupted customer from the pool to the buffer decreases, and thus the mean buffer size decreases.
Chapter 6. Discrete time $Geo/E_d/1$ Queues with Postponed work and Protected stages