INTRODUCTION

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The collection $\Sigma(X)$ of all topologies on a fixed non-empty set $X$ is a partially ordered set under the natural partial order of set inclusion. The discrete and indiscrete topologies are the smallest and largest elements in $\Sigma(X)$ respectively.

The intersection of an arbitrary collection of topologies on the set $X$ is a topology on $X$. So under the partial ordering, any subset of the collection of all topologies on the set $X$ has an infimum, namely their set theoretic intersection. Hence the collection $\Sigma(X)$ of all topologies on the set $X$ is a complete lattice under the
The two lattice operations join and meet of the lattice $\Sigma(X)$ can be described as follows. The meet of a collection of topologies on $X$ is simply their set theoretic intersection. The join of a collection of topologies is the topology generated by the union of all the topologies in the collection.

The study of lattice of topologies and its sublattices is one of the interesting areas in point set topology. As mentioned earlier, G. Birkhoff [6] and R. Vaidynathaswamy [32] initiated the study of the lattice theoretic properties of $\Sigma(X)$ and some of its sublattices. Following them many authors studied the lattice properties of the lattice of topologies from different perspectives.
If \( L \) is a lattice, and \( a \) and \( b \) are elements of \( L \), then \( b \) is said to be an upper neighbour of \( a \) if \( a < b \), and \( a \leq c \leq b \), \( c \in L \) implies that \( a = c \) or \( c = b \). An element \( a \) is said to be a lower neighbour of an element \( b \) in a lattice, when the element \( b \) is an upper neighbour of \( a \).

An atom in a lattice is any element ‘\( a \)’ in the lattice which is an upper neighbour of the smallest element. Dually a dual atom in a lattice is any element in the lattice which is lower neighbour of the largest element.

A lattice such that every element other than the smallest can be written as the join of atoms smaller than or equal to the element is called an atomic lattice. A dually atomic lattice is defined dually.
There are atoms in the lattice $\Sigma(X)$. The atoms of the lattice $\Sigma(X)$ are precisely the topologies of the form $\{\emptyset, X, A\}$ where $A$ is a proper nonempty subset of the set $X$. In the lattice $\Sigma(X)$ every element other than the indiscrete topology can be written as a join of atoms smaller than or equal to it. Hence the lattice of topologies $\Sigma(X)$ is an atomic lattice [32].

If $\tau$ is a topology on $X$ and if $A$ is a nonempty subset of $X$, then the simple expansion of the topology $\tau$ by the set $A$ is the join $\tau \vee \{\emptyset, X, A\}$ in the lattice of topologies. It is denoted by $\tau(A)$. The topology $\tau(A)$ can be represented in the following form

$$\tau(A) = \{ U \cup (V \cap A) \mid \text{where } U \text{ and } V \text{ are open in the topology } \tau \} [12].$$
N. Levine [17] initiated the investigation of the properties of $\tau (A)$ in relation with the properties of the topology $\tau$ and the set $A$.

In the lattice $\Sigma(X)$, there are dual atoms also. The dual atoms of $\Sigma(X)$, which are the lower neighbours of the discrete topology, are called ultra topologies. The ultra topologies in $\Sigma(X)$ are described below using ultra filters on $X$.

A filter on a set $X$ is a nonempty collection of nonempty subsets of $X$ such that the intersection of two elements in it is again a member of the collection and a superset of a member of the collection is also a member of the collection.

There are two types of filters. When the intersection of all the members in a filter is non-empty, it
is called a fixed filter. When this intersection is empty, it is called free filter.

The collection of all filters on a fixed set is ordered under the natural order of set inclusion. There are maximal members in this collection and these maximal members are called ultrafilters.

Fixed ultra filters are also called principal ultra filters. The principal ultrafilters are of the form $\mathcal{U}(a) = \{ A \subseteq X : a \in A \}$ for an element $a$ in $X$.

The ultratopologies are topologies of the form $\emptyset( X \setminus \{x\}) \cup \mathcal{U}$, for $x \in X$ where $\mathcal{U}$ is an ultrafilter on $X$ such that $\{x\}$ is not a member of $\mathcal{U}$. This ultratopology is denoted as $\mathcal{I}(x, \mathcal{U})$. This ultra filter $\mathcal{U}$ is called the associated ultra filter of the ultra topology $\mathcal{I}(x, \mathcal{U})$. In the lattice $\Sigma(X)$ every non discrete
topology can be written as the greatest lower bound of the ultratopologies finer than it. So the lattice of topologies is dually atomic [11].

Corresponding to simple expansion of a topology, the simple reduction of topology \( \tau \) is defined as the greatest lower bound with an ultra topology[22].

Also the collection of all \( T_1 \) topologies forms a sublattice of \( \Sigma(X)[7] \). This sub lattice is denoted by \( \Lambda(X) \). The sublattice \( \Lambda(X) \) has properties very similar to that of \( \Sigma(X) \). It has a least element. It is called the cofinite topology on \( X \). The non-empty closed sets of this topology are the finite subsets of \( X \). This topology is sometimes called the minimum \( T_1 \) topology on \( X \). The greatest element of the lattice of \( T_1 \) topologies on \( X \) is also the discrete topology on \( X \).
There are atoms and dual atoms in the lattice \( \Lambda(X) \). The atoms of \( \Lambda \) are unions of the singletons of the set \( X \) with the co-finite topology. Since there are topologies on infinite sets without singleton open sets, this lattice is not atomic.

There are dual atoms in this lattice when \( X \) is infinite. The dual atoms of \( \Lambda(X) \) are of the form 
\[ \mathcal{I}(x, \mathcal{U}) = \varnothing(X \setminus \{x\}) \cup \mathcal{U}, \text{ for } x \in X \text{ where } \mathcal{U} \text{ is a non-principal ultrafilter on } X. \] This lattice is also dually atomic.

There are many interesting sublattices of the lattice of the topologies. As noted earlier, the lattice of all \( T_1 \) topologies is one of the important sublattices. Other important sublattices of the lattice of the topologies are the lattice of all principal topologies, the lattice of all regular topologies, the lattice of all completely regular
topologies, the lattice of all countably accessible topologies etc. (See [13], [14], [15]).

The properties upper semimodularity and lower semimodularity are defined in terms of the neighbours of an element. Specifically a lattice $L$ is called upper semimodular if for all elements $a, b$ in $L$ such that $a$ is a lower neighbour of $a \land b$ implies $a \lor b$ is an upper neighbour of $b$. The lattice $L$ is lower semimodular if its dual lattice is upper semimodular.

Most of the works appeared on the lattice of topologies on a fixed set discuss properties like complementation, distributivity, modularity, lattice morphisms etc. But authors like R. E. Larson, W. J. Thron, R. Valent and others. (See [35], [31], [34], [21] etc.) deviated from this main stream of study and they attempted to study properties like upper modularity,
lower modularity, upper and lower semi modularity, embedding different types of lattices in the lattice of topologies, existence of upper and lower neighbours etc.

The study of upper or lower neighbours is closely connected with study of properties like upper semimodularity, lower semimodularity. Hence a study of upper or lower neighbours a topology are important.

In their survey article R. E. Larson and Susan J. Andima [15] observed the following. "If X is an infinite set and P is any topological property, then the collection of topologies in the lattice of the topologies possessing the property P may be identified simply from the lattice structure of the lattice of the topologies. This follows from the theorem that for an infinite X, the group of lattice automorphisms of the lattice of the topologies is isomorphic to the symmetric group on X."
Therefore the only automorphisms of the lattice of the topologies for infinite \( X \) are those which simply permute the elements of \( X \). Therefore any automorphism of the lattice of the topologies must map all the topologies in the lattice of the topologies onto homeomorphic images. 

*Thus the topological properties of the elements the lattice of the topologies must be determined by the position of topologies in the lattice of the topologies.*” This means study of the neighbours of topologies is important and this thesis is an attempt in this direction.

We can view this problem from another perspective also. The collection of subsets of a topological space containing a point in their interior is a filter. We call this filter as the neighbourhood filter of that point with respect to the topology. The topology of a set can be characterised using the neighbourhood filters of its points. So two different topologies have different
neighbourhood filters at least one point. In other words altering neighbourhood filters means altering topologies. This thesis attempts to study how to alter the neighbourhood filter of a point or minimum number of points. A typical case is the neighbourhood filters of all points except one point are the same. Here our intention is to study how to alter neighbourhoods of minimum number of points.

In the first chapter we discuss upper or lower neighbours for general lattices. A theorem and its dual about neighbours are proved in general lattices. Using the theorem it is proved that in an atomic modular lattice every element has an upper neighbour and in a dually atomic modular lattice every element has a lower neighbour. The results are in such a general fashion that they are also applicable to some of the sublattices of the lattice of topologies. Using the above results we
characterise the lower neighbours of ultra topologies as the intersection of two ultra topologies.

In the second chapter we study upper neighbours in the Lattice of Topologies. An equivalence relation on the non open sets is defined on the collection of all non open subsets. Then an order relation is defined among the equivalence classes. Using this order relation the upper neighbours in Lattice of Topologies are characterised. Using these results we determine that a special interval in the lattice of topologies is a Boolean Algebra generated by a subset.

In the third chapter we conduct a similar study of lower neighbours in the lattice of topologies. The results in this chapter are generalisations of results given by Larson and Thron from the lattice of $T_1$ topologies to Lattice of Topologies.
In the fourth chapter we consider topologies without upper neighbours. We generalise some existing results and prove that no countably accessible topology possesses upper neighbours in the Lattice of Topologies. We also compare the Lattice of Topologies with the lattice of Čech Closure Operators in this context.