Chapter 5

A Common Fixed Points in Cone and rectangular cone Metric Spaces

In this chapter we have establish fixed point theorems in cone metric spaces and rectangular cone metric space. In the first part we have proved some generalization of common fixed point on cone metric space with $w$-distance. Also proved a common fixed point theorem for a pair of weakly compatible self mappings defined on a cone metric space under different contractive conditions. In the second part we have proved some generalization of common fixed point on cone rectangular metric spaces.

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5.1.1 Introduction

In 2007 Huang and Zhang [72] introduced the notation of cone metric space as natural generalization of the notion of metric space $(X,d)$, where metric $d : X \times X \to \mathbb{R}^+$ is replaced by the metric $d : X \times X \to E$, where $E$ is real Banach space. This theory generalized by see in, [3, 4, 5, 123, 49, 116].

Section (5.1.2) is devoted to the basic operations related to the cone, cone metric space and some basic properties of cone metric spaces with $w$-distance and some generalization result of H.Lakzian F.Arabyani,[76] are recalled in this section (5.1.3). A fixed
point theorem of multifunctions proved in cone metric spaces and we generalized some result of S. Rezapour, [124] in section (5.1.4). In section (5.1.5) applying notion of weakly compatible maps we have proved fixed point theorem in cone metric spaces.

5.1.2 Preliminary Notes

Definition 5.1.1. [72] Let $E$ always be a real Banach space and $P$ a subset of $E$. $P$ is called a cone if

(i) $P$ is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b$,
(iii) $P \cap (-P) = 0$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to $P$ by

$x \leq y$ if and only if $y - x \in P$. The notation $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int} P$, where $\text{int} P$ denotes the interior of $P$ i.e. $P^0$.

Definition 5.1.2. The cone $P$ is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$
\|x\| \leq M \|y\|.
$$

The least positive number satisfying above is called the normal constant of $P$. It is clear that $M \geq 1$.

In the following, let $E$ be a normed linear space, $P$ be a cone in $E$ satisfying $\text{int}(P) \neq \emptyset$, and '$\leq$' denote the partial ordering on $E$ with respect to $P$.

Definition 5.1.3. [72] Let $X$ be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,
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(c) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a cone metric on \( X \), and \((X, d)\) is called a cone metric space.

**Example 5.1.1.** [72] Let \( E = \mathbb{R}^2, P = \{(x, y) \in E | x, y \geq 0\}, X = \mathbb{R} \) and \( d : X \times X \to E \) defined by \( d(x, y) = (|x - y|, \alpha|x - y|) \), where \( \alpha \geq 0 \) is constant. Then \((X, d)\) is a cone metric space.

**Example 5.1.2.** [123] Let \( E = \mathbb{R}^3, P = \{(x, y, z) \in E | x, y, z \geq 0\}, X = \mathbb{R} \) and \( d : X \times X \to E \) defined by \( d(x, y) = (\alpha|x - y|, \beta|x - y|, \gamma|x - y|) \), where \( \alpha, \beta, \gamma \) are positive constants. Then \((X, d)\) is a cone metric space.

**Example 5.1.3.** [85] Let \( E = \ell^p, (0 < p < 1), P = \{\{x_n\}_{n \geq 1} \in E | x_n \geq 0, \text{ for all } n\}, (X, \rho) \) a metric space and \( d : X \times X \to E \) defined by \( d(x, y) = \{\rho(x_n, y_n)^\frac{1}{p}\}_{n \geq 1} \). Then \((X, d)\) is a cone metric space.

**Definition 5.1.4.** [72] Let \((X, d)\) be a cone metric space, \( x \in X \) and \( \{x_n\}_{n \geq 1} \) a sequence in \( X \). Then

(a) \( \{x_n\}_{n \geq 1} \) is said to converge to \( x \) whenever for every \( c \in E \) with \( 0 \ll c \) there is a natural number \( N \) such that \( d(x_n, x) \ll c \) for all \( n \geq N \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \).

(b) \( \{x_n\}_{n \geq 1} \) is said to be a Cauchy sequence whenever for every \( c \in E \) with \( 0 \ll c \) there is a natural number \( N \) such that \( d(x_n, x_m) \ll c \), for all \( n, m \geq N \).

(c) \((X, d)\) is called a complete cone metric space if every Cauchy sequence is convergent in \( X \).

**Definition 5.1.5.** Let \((X, d)\) be a cone metric space and \( B \subseteq X \).

(i) A point \( b \) in \( B \) is called an interior point of \( B \) whenever there exists a point \( p, 0 \ll p \), such that

\[ N(b, p) \subseteq B, \]
where \( N(b,p) = \{ y \in X | d(y,b) \ll p \} \).

(ii) A subset \( A \subseteq X \) is called open if each element of \( A \) is an interior point of \( A \).

**Remark 5.1.1.** [152] Let \( E \) be an ordered Banach (normed) space. Then \( c \) is an interior point of \( P \), if and only if \([-c,c]\) is neighborhood of 0, where \( [c,c] = \{ y \in E | -c \leq y \leq c \} \) and \( N(0,c) = \{ y \in E | -c \ll y \ll c \} \subseteq [c,c] \).

**Remark 5.1.2.** [158]

1. If \( u \leq v \) and \( v \ll w \), then \( u \ll w \).

   Indeed \( w - u = (w - v) + (v - u) \geq w - v \), implies
   \[ [-w + u, w - u] \supseteq [-(w - v), (w - v)]. \]

2. If \( u \ll v \) and \( v \ll w \), then \( u \ll w \).

   Indeed \( w - u = (w - v) + (v - u) \geq w - v \), implies
   \[ [-w + u, w - u] \supseteq [-(w - v), (w - v)]. \]

3. If \( 0 \leq u \ll c \) for each \( c \in P^0 \), then \( u = 0 \).

4. If \( a \leq b + c \) for each \( c \in P^0 \), then \( a \leq b \).

5. If \( 0 \leq x \leq y \) in \( E \) and \( 0 \leq a \) in \( \mathbb{R} \), then \( 0 \leq ax \leq ay \).

6. If \( 0 \leq x_n \leq y_n \) for each \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = x \), \( \lim_{n \to \infty} y_n = y \), then \( 0 \leq x \leq y \).

7. If \( 0 \leq d(x_n,x) \leq b_n \) and \( b_n \to 0 \), for any \( c \in intP \), there exist \( N \in \mathbb{N} \) such that

   \( \forall n \geq N, \) then \( d(x_n,x) \ll c \), where \( \{x_n\} \) is a sequence and \( x \) is a given point in \( X \).

8. If \( E \) is real Banach space with a cone \( P \) and if \( a \leq \lambda a \) where \( a \in P \) and \( 0 < \lambda < 1 \), then \( a = 0 \).

**Proof:** The condition \( a \leq \lambda a \) means \( \lambda a - a \in P \), that is \(- (1 - \lambda) a \in P \). Since \( a \in P \) and \( 1 - \lambda > 0 \), then also \( (1 - \lambda) a \in P \). Thus we have \( (1 - \lambda) a \in P \cap (-P) \) and \( a = 0 \).
9. If \( c \in P^0 \), \( 0 \leq a_n \) and \( a_n \to 0 \), then there exist natural number \( N \) such that for all \( n > N \) we have \( a_n \ll c \).

**Proof:** Let \( 0 \ll c \) be given. Choose a symmetric neighborhood \( V \) such that \( c+V \subseteq P \). Since \( a_n \to 0 \), there is positive number \( N \) such that \( a_n \in V = -V \) for \( n > N \).

This means that \( c \pm a_n \in c+V \subseteq \text{int} P \) for \( n > N \), that is \( a_n \ll c \).

**Definition 5.1.6.** The cone \( P \) is called regular if every increasing sequence which is bounded from above is convergent. That is if \( \{x_n\}_{n \geq 1} \) is a sequence such that \( x_1 \leq x_2 \leq \cdots \leq y \) for some \( y \in E \), then there is \( x \in E \) such that \( \lim_{n \to 0} ||x_n - x|| = 0 \).

Equivalently the cone \( P \) is regular if and only if every decreasing sequence which is bounded from below is convergent.

**Lemma 5.1.4.** [72] \textit{Every regular cone is normal.}

**Lemma 5.1.5.** [72] \textit{There is not normal cone with normal constant} \( M < 1 \).

### 5.1.3 A Fixed Point Theorem in Cone Metric Spaces with \( w \)-Distance

Huang and Zhang [72] introduced cone metric space without \( w \)-distance. The metric space with \( w \)-distance was introduced by Osama Kada, Tomonari Suzuki, Wataru Takahashi and Nakoshi Shioji [106, 107]. We compose these concept together and generalized idea of cone metric space with \( w \)-distance.

**Definition 5.1.7.** [76] Let \( X \) be a cone metric space with metric \( d \). Then a mapping \( p : X \times X \to E \) is called \( w \)-distance on \( X \) if the following satisfy:

(a) \( 0 \leq p(x, y) \) for all \( x, y \in X \),
(b) \( p(x, z) \leq p(x, y) + p(y, z) \) for all \( x, y, z \in X \),
(c) \( p(x, \cdot) \to E \) is lower semi-continuous for all \( x \in X \),
(d) for any $0 \ll \alpha$, there exists $0 \ll \beta$ such that $p(z, x) \ll \beta$ and $p(z, y) \ll \beta$ imply $d(x, y) \ll \alpha$, where $\alpha, \beta \in E$.

**Definition 5.1.8.** [76] Let $X$ be a cone metric space with metric $d$, let $p$ be a $w$-distance on $X$, $x \in X$ and $\{x_n\}$ a sequence in $X$, then

(a) $\{x_n\}$ is called a $p$-Cauchy sequence whenever for every $\alpha \in E$, $0 \ll \alpha$, there is a positive integer $N$ such that, for all $m, n \geq N, p(x_m, x_n) \ll \alpha$.

(b) A sequence $\{x_n\}$ in $X$ is called a $p$-convergent to a point $x \in X$ whenever for every $\alpha \in E$, $0 \ll \alpha$, there is a positive integer $N$ such that, for all $n \geq N, p(x_n, x) \ll \alpha$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

(c) $(X, d)$ is a complete cone metric space with $w$-distance if every Cauchy sequence is $p$-convergent.

**Lemma 5.1.6.** [76] Let $X$ be a cone metric space with metric $d$, let $p$ be a $w$-distance on $X$ and let $f$ be a function from $X$ into $E$ that $0 \leq f(x)$ for any $x \in X$. Then an into function $q : X \times X \to E$ given by $q(x, y) = f(x) + p(x, y)$ for each $(x, y) \in X \times X$ is also a $w$-distance.

**Example 5.1.7.** [76] (i) Let $(X, d)$ be a metric space. Then $p = d$ is a $w$-distance on $X$.

(ii) Let $X$ be a norm linear space with Euclidean norm. Then the mapping $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \|x\| + \|y\|$, for all $x, y \in X$ is a $w$-distance on $X$.

(iii) Let $X$ be a norm linear space with Euclidean norm. Then the mapping $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \|y\|$, for all $x, y \in X$ is a $w$-distance on $X$.

Finally, note that the relations $intP + intP \subseteq intP$ and $\lambda intP \subseteq intP$ ($\lambda > 0$).

**Example 5.1.8.** Let $E = l^1$, $P = \{\{x_n\}_{n \geq 1} \in E | x_n \geq 0, \text{ for all } n\}$, $(X, p)$ a metric space and $d : X \times X \to E$ defined by $d(x, y) = \{\frac{p(x, y)}{2^n}\}_{n \geq 1}$. Then $(X, d)$ is a cone metric space.
if set \( p = d \), then \((X, d)\) is cone metric space with \( w \)-distance \( p \) and the normal constant of \( P \) is equal to \( M = 1 \).

**Theorem 5.1.9.** [76] Let \((X, d)\) be a cone metric space with \( w \)-distance \( p \) on \( X \) and the mapping \( T : X \rightarrow X \) there exist \( r \in [0, 1/2) \) such that

\[
p(Tx, T^2x) \leq rp(x, Tx),
\]

for every \( x \in X \) and

\[
\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0,
\]

for every \( y \in X \) with \( y \neq Ty \). Then there is \( z \in X \) such that \( z = Tz \). Moreover, if \( P \) be a normal cone with normal constant \( M \) and \( v = T(v) \), then \( p(v, v) = 0 \).

**Theorem 5.1.10.** Let \((X, d)\) be a complete cone metric space with \( w \)-distance \( p \). Let \( P \) be a normal cone on \( X \). Suppose a mapping \( T : X \in X \) satisfy the contractive condition

\[
p(Tx, Ty) \leq r[p(Tx, x) + p(Ty, y)],
\]

for all \( x \in X \), where \( r \in [0, 1/2) \) is a constant. Then, \( T \) has a unique fixed point in \( X \). For each \( x \in X \), the iterative sequence \( \{T^n(x)\}_{n \geq 1} \) converges to the fixed point.

**Proof.** Let \( x \in X \) and define \( x_{n+1} = Tx_n \). Then

\[
x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots, x_n = T^nx_0.
\]

\[
p(Tx_0, T^2x_0) = p(Tx_0, T(Tx_0))
\]

\[
\leq r\{p(Tx_0, x_0) + p(T^2x_0, Tx_0)\}
\]

\[
(1 - r)p(Tx_0, T^2x_0) \leq rp(Tx_0, x_0)
\]

\[
\leq \left(\frac{r}{1 - r}\right)p(Tx_0, x_0).
\]
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Similarly,

\[ p(T^2x_0, T^3x_0) \leq \left( \frac{r}{1-r} \right)^2 p(Tx_0, x_0). \]

Thus in general, if \( n \) is positive integer

\[ p(T^n x_0, T^{n+1} x_0) \leq h^n p(Tx_0, x_0), \]

where \( h = \frac{r}{1-r} \), and \( 0 \leq r < \frac{1}{2} \).

Let \( m, n \in \mathbb{N} \) and \( m > n \), we have to show sequence \( \{ T^n x_0 \} \) is Cauchy

\[
p(T^n x_0, T^m x_0) \leq p(T^n x_0, T^{n+1} x_0) + p(T^{n+1} x_0, T^{n+2} x_0) + \cdots + p(T^{m-1} x_0, T^m x_0) \]
\[
\leq (h^n + h^{n+1} + \cdots + h^{m-1}) p(Tx_0, x_0) \]
\[
< h^n (1 + h + h^2 + \cdots) p(Tx_0, x_0) \]
\[
= \frac{h^n}{1-h} p(Tx_0, x_0),
\]

Let \( c \in E \) with \( 0 \ll c \). Choose natural number \( n_1 \) such that

\[
\frac{h^n}{1-h} p(Tx_0, x_0) \ll \frac{c}{2},
\]

for all \( n \geq n_1 \). Thus \( p(T^n x_0, T^m x_0) \ll c \) for all \( m > n \). Therefore \( \{ T^n x_0 \} \) is Cauchy in \( X \). Since \( X \) is complete space, then there exist \( x^* \in X \) such that \( x_n \to x^* \), as \( n \to \infty \).

Choose natural number \( n_2 \) such that \( p(T^n x_0, x^*) \ll \frac{c(1-r)}{2} \) for all \( n \geq n_2 \). Hence

\[
p(Tx^*, x^*) \leq p(Tx^*, T^n x_0) + p(T^n x_0, x^*) \]
\[
\leq r \{ p(Tx^*, x^*) + p(T^n x_0, T^{n-1} x_0) \} + p(T^n x_0, x^*) \]
\[
\leq h p(T^n x_0, T^{n-1} x_0) + \frac{1}{1-r} p(T^n x_0, x^*) \]
\[
\leq \frac{h^n}{1-r} p(Tx_0, x_0) + \frac{1}{1-r} p(T^n x_0, x^*) \]
\[
\ll \frac{c}{2} + \frac{c}{2}
\]
\[
= c
\]
for $n \geq n_2$. Thus $p(Tx^*, x^*) \ll \frac{c}{m}$ for all $m \geq 1$. So $\frac{c}{m} - p(Tx^*, x^*) \in P$ for all $m \geq 1$.
Since $\frac{c}{m} \to 0$ as $m \to \infty$ and $P$ is closed so $-p(Tx^*, x^*) \in P$. But $p(Tx^*, x^*) \in P$.
Therefore $p(Tx^*, x^*) = 0$, so $Tx^* = x^*$.

Suppose $y^*$ be another fixed point of $T$, then
\[
p(x^*, y^*) = p(Tx^*, Ty^*)
= r[p(Tx^*, x^*) + p(Ty^*, y^*)]
= 0.
\]
Hence $x^* = y^*$, thus fixed point of $T$ is unique. \hfill \Box

5.1.4 A Fixed Point Theorem of Multifunctions in Cone Metric Spaces

The theory of common fixed points of multifunctions is a generalization of the theory of fixed point of mappings in some sense. The study of fixed point theorems for multivalued mappings was initiated by Kakutani, in 1941, in finite dimensional spaces and was extended to infinite dimensional Banach spaces by Bohnenblust and Karlin, in 1950, and to locally convex spaces by Ky Fan, in 1952.

There are many articles about fixed point theory and fixed points of multifunctions (for example, [31, 125]). Here we have given generalization of S. Rezapour, [124] result about common fixed points of two multifunctions on the cone metric spaces with normal constant $M = 1$. Most of familiar cones are normal with normal constant $M = 1$. But for each $k > 1$ there are cones with normal constant $M > k$. First we state the following two lemmas.

Lemma 5.1.11. [124] Let $(X, d)$ be a cone metric space, $P$ a normal cone with normal constant $M = 1$ and $A$ a compact set in $(X, \tau_c)$, then for every $x \in X$ there exists $a_0 \in A$
such that

\[ ||d(x, a)|| = \inf_{a \in A} ||d(x, a)||. \]

**Lemma 5.1.12.** [124] Let \((X, d)\) be a cone metric space, \(P\) a normal cone with normal constant \(M = 1\) and \(A, B\) are two compact sets in \((X, \tau_c)\), Then

\[ \sup_{x \in B} d'(x, A) < \infty, \]

where \(d'(x, A) = \inf_{a \in A} ||d(x, a)||\) for each \(x\) in \(X\).

**Definition 5.1.9.** [125], Let \((X, d)\) be a cone metric space, \(P\) a normal cone with normal constant \(M = 1\), \(H_c(X)\) the set of all compact subsets of \((X, \tau_c)\) and \(A \in H_c(X)\). By using above Lemma, we can define

\[ h_A : H_c(X) \rightarrow [0, \infty) \quad \text{and} \quad d_H : H_c(X) \times H_c(X) \rightarrow [0, \infty). \]

by

\[ h_A(B) = \sup_{x \in A} d'(x, B) \quad \text{and} \quad d_H(A, B) = \max\{h_A(B), h_B(A)\}, \]

respectively.

**Remark 5.1.3.** [125], Let \((X, d)\) be a cone metric space with normal constant \(M = 1\). Define \(\rho : X \times X \rightarrow [0, \infty)\) by \(\rho(x, y) = ||d(x, y)||\). Then, \((X, \rho)\) is a metric space. This implies that for each \(A, B \in H_c(X)\) and \(x, y \in X\), we have the following relations

(i) \(d'(x, A) \leq ||d(x, y)|| + d'(y, A)\),

(ii) \(d'(x, A) \leq d'(x, B) + h_B(A)\), and

(iii) \(d'(x, A) \leq ||d(x, y)|| + d'(y, B) + h_B(A)\).

**Theorem 5.1.13.** Let \((X, d)\) be a complete cone metric space with normal constant \(M = 1\) and \(T_1, T_2 : X \rightarrow H_c(X)\) two multifactons satisfying the relation

\[ d_H(T_1 x, T_2 y) \leq \alpha d'(x, T_1 x) + \beta d'(y, T_2 y) \]
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for all \( x, y \in X \), where \( \alpha + \beta < 1/2 \) and \( \alpha > 0, \beta > 0 \) are constants. Then, \( T_1 \) and \( T_2 \) have a common fixed point, that is, there exists \( x \in X \) such that \( x \in T_1x \) and \( x \in T_2x \).

\[ \text{Proof.} \] Let \( x_0 \in X \) be given. By Lemma 5.1.11, there is \( x_1 \in T_1x_0 \) such that
\[
d'(x_0, T_1x_0) = ||d(x_0, x_1)||.
\]
Also, there is \( x_2 \in T_2x_1 \) such that
\[
d'(x_1, T_2x_1) = ||d(x_1, x_2)||.
\]
Thus, we obtain a sequence \( \{x_n\}_{n \geq 1} \) in \( X \) such that \( x_{2n-1} \in T_1x_{2n-2}, x_{2n} \in T_2x_{2n-1}, \)
\[
d'(x_{2n-2}, T_1x_{2n-2}) = ||d(x_{2n-2}, x_{2n-1})||,
\]
and
\[
d'(x_{2n-1}, T_2x_{2n-1}) = ||d(x_{2n-1}, x_{2n})||,
\]
for all \( n \geq 1 \). Thus, for all \( n \geq 1 \) we have
\[
||d(x_{2n}, x_{2n+1})|| = d'(x_{2n}, T_1x_{2n})
\]
\[
\leq h_{T_2x_{2n-1}}(T_1x_{2n})
\]
\[
\leq d_H(T_2x_{2n-1}, T_1x_{2n})
\]
\[
\leq \alpha d'(x_{2n}, T_1x_{2n}) + \beta d'(x_{2n-1}, T_2x_{2n-1})
\]
\[
\leq \alpha ||d(x_{2n}, x_{2n+1})|| + \beta ||d(x_{2n-1}, x_{2n})||
\]
\[
||d(x_{2n}, x_{2n+1})|| \leq \left( \frac{\beta}{1 - \alpha} \right) ||d(x_{2n-1}, x_{2n})||
\]
\[
\leq c ||d(x_{2n-1}, x_{2n})||,
\]
for all \( n \geq 1 \), \( \alpha + \beta < 1 \) and \( c = \max\{\frac{\beta}{1 - \alpha}, \frac{\alpha}{1 - \beta}\} \). Also
\[
||d(x_{2n-1}, x_{2n})|| = d'(x_{2n-1}, T_2x_{2n-1})
\]
\[
\begin{aligned}
&\leq h_{T_1x_{2n-2}}(T_2x_{2n-1}) \\
&\leq d_H(T_1x_{2n-1}, T_2x_{2n-1}) \\
&\leq \alpha ||d(x_{2n-2}, x_{2n-1})|| + \beta ||d(x_{2n-1}, x_{2n})|| \\
||d(x_{2n-1}, x_{2n})|| &\leq \left( \frac{\alpha}{1-\beta} \right) ||d(x_{2n-2}, x_{2n-1})|| \\
&\leq c||d(x_{2n-2}, x_{2n-1})||,
\end{aligned}
\]

for all \( n \geq 1 \). This implies that for all \( m \geq 1 \)
\[
||d(x_m, x_{m+1})|| \leq c||d(x_{m-1}, x_m)|| \\
\leq c^2||d(x_{m-2}, x_{m-1})|| \\
\vdots \\
\leq c^m||d(x_0, x_1)||.
\]

Then for \( n > m \) we have
\[
||d(x_n, x_m)|| \leq \sum_{i=m+1}^{n} ||d(x_i, x_{i-1})|| \\
\leq (c^{n-1} + c^{n-2} + \cdots + c^m)||d(x_0, x_1)|| \\
\leq \left( \frac{c^m}{1-c} \right)||d(x_0, x_1)|| \rightarrow 0, \quad n, m \rightarrow \infty.
\]

Hence, \( \{x_n\}_{n \geq 1} \) is a Cauchy sequence in \( X \). Thus, there exists \( x^* \in X \) such that \( x_n \rightarrow x^* \).

Now by using Remark 5.1.3, we have
\[
d'(x^*, T_1x^*) \leq d'(x^*, T_2x_{2n-1}) + h_{T_2x_{2n-1}}(T_1x^*) \\
\leq d'(x^*, T_2x_{2n-1}) + d_H(T_2x_{2n-1}, T_1x^*) \\
\leq ||d(x^*, x_{2n})|| + \alpha d'(x_{2n-1}, T_2x_{2n-1}) + \beta d'(x^*, T_1x^*),
\]

for all \( n \geq 1 \). Hence,
\[
d'(x^*, T_1x^*) \leq \frac{\alpha}{1-\beta} d'(x_{2n-1}, T_2x_{2n-1}) + \frac{1}{1-\beta} d(x^*, x_{2n})
\]
for all \( n \geq 1 \). Therefore, \( d'(x^*, T_1x^*) = 0 \). By Lemma 5.1.12, \( x^* \in T_1x^* \). On the other hand, similarly we have

\[
d'(x^*, T_2x^*) \leq d'(x^*, T_1x^*) + h_{T_1x_2n}(T_2x^*)
\]

\[
\leq d'(x^*, T_1x^*) + d_H(T_1x_2n, T_2x^*)
\]

\[
\leq \|d(x^*, x_{2n+1})\| + \alpha d'(x_{2n}, T_1x^*) + \beta d'(x^*, T_2x^*),
\]

for all \( n \geq 1 \). Hence,

\[
d'(x^*, T_2x^*) \leq \frac{\alpha}{1 - \beta} d'(x_{2n}, T_1x^*) + \frac{1}{1 - \beta} \|d(x^*, x_{2n+1})\|
\]

\[
\leq cd'(x_{2n}, T_1x^*) + \frac{1}{1 - \beta} \|d(x^*, x_{2n+1})\|
\]

\[
= cd(x_{2n}, T_1x^*) + \frac{1}{1 - \beta} \|d(x^*, x_{2n+1})\|
\]

for all \( n \geq 1 \). Therefore, \( d'(x^*, T_2x^*) = 0 \). By Lemma 5.1.12, \( x^* \in T_2x^* \). Therefore, \( x^* \) is a common fixed point of \( T_1 \) and \( T_2 \).

**Theorem 5.1.14.** Let \( (X, d) \) be a complete cone metric space with normal constant \( M = 1 \) and \( T_1, T_2 : X \rightarrow H_c(X) \) two multifunctions satisfy the relation

\[
d_H(T_1x, T_2y) \leq \alpha d'(y, T_1x) + \beta d'(x, T_2y)
\]

for all \( x, y \in X \), where \( \alpha + \beta < 1/2 \) and \( \alpha, \beta > 0 \) are a constants. Then, \( T_1 \) and \( T_2 \) have a common fixed point.

**Proof.** A similar argument to that of the proof of Theorem 5.1.13 shows that there exists a Cauchy sequence \( \{x_n\}_{n \geq 1} \) in \( X \) such that

\[
x_{2n-1} \in T_1x_{2n-2}, x_{2n} \in T_2x_{2n-1},
\]
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\[ d'(x_{2n-2}, T_1x_{2n-2}) = ||d(x_{2n-2}, x_{2n-1})||, \]

and

\[ d'(x_{2n-1}, T_2x_{2n-1}) = ||d(x_{2n-1}, x_{2n})||, \]

for all \( n \geq 1 \). Thus, there exists \( x^* \in X \) such that \( x_n \to x^* \). Now by using Remark 5.1.3, we have

\[
d'(x^*, T_1x^*) \leq d'(x^*, T_2x_{2n-1}) + h_{T_2x_{2n-1}}(T_1x^*)
\]

\[
\leq d'(x^*, T_2x_{2n-1}) + d_H(T_2x_{2n-1}, T_1x^*)
\]

\[
\leq ||d(x^*, x_{2n})|| + \alpha d'(x^*, T_2x_{2n-1}) + \beta d'(x_{2n-1}, T_1x^*)
\]

\[
\leq ||d(x^*, x_{2n})|| + \alpha ||d(x^*, x_{2n})|| + \beta ||d(x^*, T_1x^*)|| + \beta ||d(x^*, x_{2n-1})||,
\]

for all \( n \geq 1 \). Hence,

\[
d'(x^*, T_1x^*) \leq \left(\frac{1 + \alpha}{1 - \beta}\right)||d(x^*, x_{2n})|| + \left(\frac{\beta}{1 - \beta}\right)||d(x^*, x_{2n-1})||,
\]

for all \( n \geq 1 \). Therefore, \( d'(x^*, T_1x^*) = 0 \). By Lemma 5.1.12, \( x^* \in T_1x^* \). Also, we have

\[
d'(x^*, T_2x^*) \leq d'(x^*, T_1x_{2n}) + h_{T_1x_{2n}}(T_2x^*)
\]

\[
\leq d'(x^*, T_1x_{2n}) + d_H(T_1x_{2n}, T_2x^*)
\]

\[
\leq ||d(x^*, x_{2n+1})|| + \alpha d'(x^*, T_1x_{2n}) + \beta d'(x_{2n}, T_2x^*)
\]

\[
\leq ||d(x^*, x_{2n+1})|| + \alpha ||d(x^*, x_{2n+1})|| + \beta ||d(x^*, T_2x^*)|| + \beta ||d(x^*, x_{2n})||,
\]

for all \( n \geq 1 \). Hence,

\[
d'(x^*, T_2x^*) = \left(\frac{1 + \alpha}{1 - \beta}\right)||d(x^*, x_{2n+1})|| + \left(\frac{\beta}{1 - \beta}\right)||d(x^*, x_{2n})||,
\]

for all \( n \geq 1 \). Therefore, \( d'(x^*, T_2x^*) = 0 \). By Lemma 5.1.12, \( x^* \in T_2x^* \). Therefore, \( x^* \) is a common fixed point of \( T_1 \) and \( T_2 \).

\[ \square \]

5.1.5 A Common Fixed Point For Weakly Compatible Maps

In this section, a common fixed point theorem is proved for a pair of weakly compatible self mappings defined on a cone metric space under a contractive condition.
5.1 Cone Metric Space

**Definition 5.1.10.** [69] Let $f$ and $g$ be self mappings of a set $X$. If $w = fx = gx$ for some $x$ in $X$, then $x$ is called a coincidence point of $f$ and $g$, and $w$ is called a point of coincidence of $f$ and $g$.

**Definition 5.1.11.** [69] Two self mappings $f$ and $g$ of a set $X$ are said to be weakly compatible if they commute at their coincidence points, that is, if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

**Proposition 5.1.15.** [3] Let $f$ and $g$ be weakly compatible self mappings of a set $X$. If $f$ and $g$ have a unique point of coincidence, that is, $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.

**Theorem 5.1.16.** Let $(X,d)$ be a cone metric and $P$ be a normal cone with normal constant $K$. Let the mappings $f, g : X \to X$ satisfy the following conditions:

(i) $f(X) \subset g(X)$,

(ii) $g(X)$ is complete subspace of $X$,

(iii) $d(fx, fy) \leq \alpha d(fx, gy) + \beta d(fy, gx) + \gamma d(fx, gx) + \delta d(fy, gy)$,

where $\alpha, \beta, \gamma, \delta > 0$ and $2\alpha + 2\beta + \delta + \gamma < 1$, then $f$ and $g$ have a unique coincidence point in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

**Proof.** Let $x_0$ be an arbitrary point in $X$. Then, since $f(X) \subset g(X)$, we choose a point $x_1$ in $X$ such that $f(x_0) = g(x_1)$. Continuing this process, having chosen $x_n$ in $X$, we obtain $x_{n+1}$ in $X$ such that $f(x_n) = g(x_{n+1})$. Then

$$d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1})$$

$$\leq \alpha d(fx_n, gx_{n-1}) + \beta d(fx_{n-1}, gx_n) + \gamma d(fx_n, gx_n) + \delta d(fx_{n-1}, gx_{n-1})$$

$$\leq \alpha \{d(fx_n, gx_n) + d(gx_n, gx_{n-1})\} + \beta \{d(fx_{n-1}, gx_{n+1}) + d(gx_{n+1}, gx_n)\}$$
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\[ + \gamma d(fx_n, gx_n) + \delta d(fx_{n-1}, gx_{n-1}) \]
\[ = \alpha \{d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})\} + \beta \{d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_n)\} \]
\[ + \gamma d(gx_{n+1}, gx_n) + \delta d(gx_n, gx_{n-1}) \]
\[ = (\alpha + 2\beta + \gamma)d(gx_{n+1}, gx_n) + (\alpha + \delta)d(gx_n, gx_{n-1}). \]

Hence,

\[ d(gx_{n+1}, gx_n) \leq hd(gx_n, gx_{n-1}), \]

where

\[ h = \frac{\alpha + \delta}{1 - (\alpha + 2\beta + \gamma)} \]

with \( 2\alpha + 2\beta + \gamma + \delta < 1. \)

Now for \( n > m, \) we get

\[ d(gx_n, gx_m) \leq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \ldots + d(gx_{m+1}, gx_m) \]
\[ \leq (h^{n-1} + h^{n-2} + \ldots + h^m)d(gx_1, gx_0) \]
\[ \leq \frac{h^m}{1 - h}d(gx_1, gx_0). \]

Using the normality of cone \( P, \) implies that

\[ \|d(gx_n, gx_m)\| \leq \frac{h^m}{(1 - h)}K\|d(gx_1, gx_0)\|. \]

Then \( d(gx_n, gx_m) \to 0 \) as \( n, m \to \infty \) and so \( \{g(x_n)\} \) is a Cauchy sequence in \( X. \) Since \( g(X) \) is a complete subspace of \( X, \) so there exists \( q \) in \( g(X) \) such that \( g(x_n) \to q, \) as \( n \to \infty. \) Consequently, we can find \( p \) in \( X \) such that \( g(p) = q. \) Thus,

\[ d(gx_n, fp) = d(fx_{n-1}, fp) \]
\[ \leq \alpha d(fx_{n-1}, gp) + \beta d(fp, gx_{n-1}) + \gamma d(fx_{n-1}, gx_{n-1}) + \delta d(fp, gp) \]
\[ \leq \alpha d(fx_{n-1}, gp) + \beta[d(fp, gx_n) + d(gx_n, gx_{n-1})] + \gamma d(fx_{n-1}, gx_{n-1}) + \] \[ \delta[d(fp, gx_n) + d(gx_n + gp)] \]
\[ (1 - \beta - \gamma)d(gx_n, fp) \leq \alpha d(fx_{n-1}, q) + \beta d(gx_n, gx_{n-1}) + \gamma d(fx_{n-1}, gx_{n-1}) + \delta d(gx_n, q) \]
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\[
\leq \left( \frac{1}{(1 - \beta - \gamma)} \right) \left[ \alpha d(fx_{n-1}, q) + \beta d(gx_n, gx_{n-1}) + \gamma d(fx_{n-1}, gx_{n-1}) + \delta d(gx_n, q) \right]
\]

which, using the normality of cone \( P \), implies that

\[
\|d(gx_n, fp)\| \leq \left( \frac{K}{(1 - \beta - \gamma)} \right) \left[ \|\alpha d(fx_{n-1}, q) + \beta d(gx_n, gx_{n-1}) + \gamma d(fx_{n-1}, gx_{n-1}) + \delta d(gx_n, q)\| \right]
\]

\[
\leq \left( \frac{K}{(1 - \beta - \gamma)} \right) \left[ \alpha \|d(fx_{n-1}, q)\| + \beta \|d(gx_n, gx_{n-1})\| + \gamma \|d(fx_{n-1}, gx_{n-1})\| + \delta \|d(gx_n, q)\| \right]
\]

\[
= 0, \quad \text{as} \quad n \to \infty.
\]

Hence, \( d(gx_n, fp) \to 0 \), as \( n \to \infty \). Also, we have \( d(gx_n, gp) \to 0 \), as \( n \to \infty \). The uniqueness of a limit in a cone metric space implies that \( f(p) = g(p) \). Again, we show that \( f \) and \( g \) have a unique point of coincidence. For this, if possible, there exists another point \( t \) in \( X \) such that \( f(t) = g(t) \). Then, we have

\[
d(gt, gp) = d(ft, fp) \leq \alpha d(ft, gp) + \beta d(fp, gt) + \gamma d(ft, gt) + \delta d(fp, gp),
\]

by the normality of cone \( P \), implies that \( \|d(gt, gp)\| = 0 \), and so, we have \( gt = gp \). Finally, using the Proposition (5.1.15), we conclude that \( f \) and \( g \) have a unique common fixed point.

\[\square\]

**Example 5.1.17.** Let \( E = I^2 \), for \( I = [0, 1] \), \( P = \{(x, y) \in E | x, y \geq 0\} \subset I^2 \), \( d : I \times I \to E \) such that \( d(x, y) = (|x - y|, \xi |x - y|) \), where \( \xi > 0 \) is a constant. Define \( fx = \frac{\xi x}{1+\xi} \), for all \( x \in I \) and \( gx = \xi x \) for all \( x \in I \). Then, for \( \xi = 1 \), both the mappings \( f \) and \( g \) are weakly compatible and satisfy all the conditions of the above theorem with \( x = 0 \) as a unique common fixed point.
5.2 Cone Rectangular Metric Space

In this section we introduce fixed point theorems in cone rectangular metric spaces and proved some results on Banach contraction principle in a complete normal cone rectangular metric space also generalize the Kannans fixed point theorem in a cone rectangular metric. At the last we have proved result for expansive onto mappings on cone rectangular metric spaces.

5.2.1 Introduction

Azam, Arshad and Beg, [7] introduced the notion of cone rectangular metric spaces by replacing the triangular inequality of a cone metric space by a rectangular inequality. In this chapter our main results are based on generalization of Banach contraction principle on cone rectangular metric spaces result given by Akbar Azam, Muhammad Arshad and Ismat Beg, [7] and the Kannans fixed point theorem in a cone rectangular metric [101]. Also C.T.Aage and J.N. Salunke, [12] proved fixed point theorem for expansion onto mappings on cone metric spaces and we have extended result for expansive onto mappings on cone rectangular metric.

5.2.2 Preliminary Notes

Definition 5.2.1. [101] Let $X$ be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X \setminus \{x, y\}$ [Rectangular Property].

Then $d$ is called a cone rectangular metric on $X$ and $d(X, d)$ is called cone rectangular
metric space.

**Example 5.2.1.** [101] Let \( X = \mathbb{N}, E = \mathbb{R}^2 \) and \( P = \{(x, y) \in E | x, y \geq 0\} \).

Define \( d : X \times X \to E \) as follow:

\[
 d(x, y) = \begin{cases} 
 (0, 0) & \text{if } x = y, \\
 (3, 9) & \text{if } x \text{ and } y \text{ are in } \{1, 2\}, \\
 (1, 3) & \text{if } x \text{ and } y \text{ can not both in } \{1, 2\}.
\end{cases}
\]

Now \((X, d)\) is a cone rectangular metric space but \((X, d)\) is not a cone metric space because it lacks the triangular property:

\[
 (3, 9) = d(1, 2) > d(1, 3) + d(3, 2) = (1, 3) + (1, 3) = (2, 6),
\]

as \((3, 9) - (2, 6) = (1, 3) \in P\).

**Example 5.2.2.** [101] Let \( E = \mathbb{R}^2 \) and \( P = \{(x, y) \in E | x, y \geq 0\} \) and \( X = \mathbb{R} \). Define \( d : X \times X \to E \) as follow:

\[
 d(x, y) = \begin{cases} 
 (0, 0) & \text{if } x = y, \\
 (3\alpha, 3) & \text{if } x \text{ and } y \text{ are in } \{1, 2\}, x \neq y, \\
 (\alpha, 1) & \text{if } x \text{ and } y \text{ can not both in } \{1, 2\}, x \neq y,
\end{cases}
\]

where \( \alpha > 0 \) is a constant. Then \((X, d)\) is a cone rectangular metric space but \((X, d)\) is not a cone metric space, since we have

\[
 (3\alpha, 3) = d(1, 2) > d(1, 3) + d(3, 2) = (2\alpha, 2).
\]

**Example 5.2.3.** Let \( E = \mathbb{R}^2 \) and \( P = \{(x, y) \in E | x, y \geq 0\} \) and \( X = \{a, b, c, e\} \). Define \( d : X \times X \to E \) such that

\[
 \begin{cases} 
 d(x, x) = (0, 0) & \text{for all } x \in X, \\
 d(x, y) = d(y, x) & \text{for all } x, y \in X, \\
 d(a, b) = (3, \alpha), \\
 d(a, c) = d(b, c) = (1, \alpha), \\
 d(a, e) = d(b, e) = d(c, e) = (2, \alpha),
\end{cases}
\]
where $\alpha > 0$ is a constant. Then $(X, d)$ is a cone rectangular metric space but it is not a cone metric space, since we have

$$d(a, b) = (3, \alpha) \text{ and } d(a, c) + d(c, b) = (2, 2\alpha),$$

but $(3, \alpha)$ and $(2, 2\alpha)$ cannot be compared with respect to $\leq$.

**Definition 5.2.2.** [101] Let $(X, d)$ be a cone rectangular metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in $X$. Then

(a) $\{x_n\}_{n \geq 1}$ is said to converge to $x$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

(b) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(c) $(X, d)$ is called a complete cone rectangular metric space if every Cauchy sequence is convergent in $X$.

**Lemma 5.2.4.** [101] Let $(X, d)$ be a cone rectangular metric space, $P$ be a normal cone. Let $\{x_n\}$ be a sequence in $X$. Then

$$x_n \to x \text{ as } n \to +\infty \iff ||d(x_n, x)|| \to 0 \text{ as } n \to +\infty.$$

Note that if $(X, d)$ is a cone metric space and $\{x_n\}$ is a convergent sequence in $X$, then the limit of $\{x_n\}$ is unique ([72]-Lemma 2). In our case uniqueness of the limit is not satisfied in general. We have an example to illustrate this remark.

**Example 5.2.5.** [101] We take $E = \mathbb{R}$ and $P = \{x \in \mathbb{R} | x \geq 0\}$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Q}$ and $a, b \in \mathbb{R}/\mathbb{Q}$, $a \neq b$. We put $X = \{x_1, x_2, \ldots, x_n, \ldots\} \cup \{a, b\}$ and we
consider $d : X \times X \to \mathbb{R}$ defined by

$$
\begin{align*}
    d(x, x) &= 0 \text{ for all } x \in X, \\
    d(x, y) &= d(y, x) \text{ for all } x, y \in X, \\
    d(x_n, x_m) &= 1 \text{ for all } n, m \in \mathbb{N}, n \neq m, \\
    d(x_n, b) &= \frac{1}{n} \text{ for all } n \in \mathbb{N}, \\
    d(x_n, a) &= \frac{1}{n} \text{ for all } n \in \mathbb{N}, \\
    d(a, b) &= 1.
\end{align*}
$$

We remark $(X, d)$ is not a cone metric space because we have

$$
d(x_2, x_3) = 1 > d(x_2, a) + (a, x_3) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.
$$

However, $(X, d)$ is a cone rectangular metric space. Now, since $d(x_n, a) = \frac{1}{n} \to 0$ as $n \to +\infty$, we obtain that $x_n \to a$ as $n \to +\infty$. Also, we have $d(x_n, b) = \frac{1}{n} \to 0$ as $n \to +\infty$ and then $x_n \to b$ as $n \to +\infty$.

**Lemma 5.2.6.** [101] Let $(X, d)$ be a cone rectangular metric space, $P$ be a normal cone. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

**Lemma 5.2.7.** [101] Let $(X, d)$ be a complete cone rectangular metric space, $P$ be a normal cone with normal constant $\kappa$. Let $\{x_n\}$ be a Cauchy sequence in $X$ and suppose that there is positive number $N$ such that

(i) $x_n \neq x_m$ for all $n, m > N$.
(ii) $x_n, x$ are distinct points in $X$, for all $n > N$.
(iii) $x_n, y$ are distinct points in $X$, for all $n > N$.
(iv) $x_n \to x$ and $x_n \to y$, as $n \to +\infty$.

Then $x = y$. 
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5.2.3 On Banach Contraction Principle in a Complete Normal Cone Rectangular Metric Space

In this article we generalized the Banach contraction mapping principle in a complete normal cone rectangular metric space.

First we present two theorems whose proofs are similar to Huang and Zhang [[72], Lemmas 1 and 4].

**Theorem 5.2.8.** Let \((X, d)\) be a cone rectangular metric space and \(P\) be a normal cone with normal constant \(\kappa\). Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) if and only if \(\|d(x_n, x)\| \to 0\) as \(n \to \infty\).

**Theorem 5.2.9.** Let \((X, d)\) be a cone rectangular metric space, \(P\) be a normal cone with normal constant \(\kappa\). Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if \(\|d(x_n, x_{n+m})\| \to 0\) as \(n \to \infty\).

**Theorem 5.2.10.** Let \((X, d)\) be a cone rectangular metric space, \(P\) be a normal cone with normal constant \(\kappa\) and the mapping \(T : X \to X\) satisfies:

\[
d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty)
\]

for all \(x, y \in X\), where \(0 \leq \alpha + \beta < 1\). Then \(T\) has a unique fixed point.

**Proof.** Let \(x_0\) be an arbitrary point in \(X\). Define a sequence of points in \(X\) as follows:

\[
x_{n+1} = Tx_n = T^{n+1}x_0, \quad n = 0, 1, 2, ...
\]

We can suppose that \(x_0\) is not a periodic point, in fact if \(x_n = x_0\), then,

\[
d(x_0, Tx_0) = d(x_n, Tx_n) = d(T^n x_0, T^{n+1} x_0) \leq \alpha d(T^{n-1} x_0, T^n x_0) + \beta d(T^n x_0, T^{n+1} x_0)
\]
5.2 Cone Rectangular Metric Space

\[ \leq h d(T^{n-1}x_0, T^n x_0) \]

\[ \vdots \]

\[ \leq h^n d(x_0, Tx_0), \]

where \( 0 \leq \alpha + \beta < 1 \) and \( h = \frac{\alpha}{1-\beta} \). It follows that \([h^n - 1]d(x_0, Tx_0) \in P\).

It further implies that \([\frac{h^n - 1}{1-h^n}]d(x_0, Tx_0) \in P\).

Hence \(-d(x_0, Tx_0) \in P\) and \(d(x_0, Tx_0) = 0\), this means \(x_0\) is a fixed point of \(T\). Thus in this sequel of proof we can suppose that \(x_m = x_n\), for all distinct \(m, n \in \mathbb{N}\). Now by using rectangular property for all \(y \in X\), we have,

\[ d(y, T^4y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^4y) \]

\[ \leq d(y, Ty) + hd(y, Ty) + h^2d(y, T^2y). \]

Similarly,

\[ d(y, T^6y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^6y) \]

\[ \leq d(y, Ty) + hd(y, Ty) + h^2d(y, Ty) + h^3d(y, Ty) + h^4d(y, T^2y) \]

\[ = \sum_{i=0}^{3} h^i d(y, Ty) + h^4 d(y, T^2y), \text{ for all } y \in X. \]

Now by induction, we obtain for each \(k = 2, 3, 4, \ldots, \)

\[ d(y, T^{2k}y) \leq \sum_{i=0}^{2k-3} h^i d(y, Ty) + h^{2k-2} d(y, T^2y). \quad (5.2.1) \]

Moreover, for all \(y \in X\),

\[ d(y, T^5y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^5y) \]

\[ = \sum_{i=0}^{4} h^i d(y, Ty). \]

By induction, for each \(k = 0, 1, 2, \ldots\) we have,

\[ d(y, T^{2k+1}y) \leq \sum_{i=0}^{2k} h^i d(y, Ty). \quad (5.2.2) \]
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Using Inequality (5.2.1), for \( k = 1, 2, 3, \ldots \) we have,

\[
d(T^n x_0, T^{n+2k} x_0) \leq h^n d(x_0, T^{2k} x_0)
\]

\[
\leq h^n \left[ \sum_{i=0}^{2k-3} h^i (d(x_0, T x_0) + d(x_0, T^2 x_0)) + h^{2k-2} d(x_0, T x_0) + d(x_0, T^2 x_0) \right]
\]

\[
\leq h^n \sum_{i=0}^{2k-2} h^i [d(x_0, T x_0) + d(x_0, T^2 x_0)]
\]

\[
\leq \frac{h^n(1 - h^{2k-1})}{1 - h} \left[ d(x_0, T x_0) + d(x_0, T^2 x_0) \right]
\]

\[
\leq \frac{h^n}{1 - h} \left[ d(x_0, T x_0) + d(x_0, T^2 x_0) \right].
\]

Similarly, for \( k = 0, 1, 2, \ldots \), Inequality (5.2.2) implies that

\[
d(T^n x_0, T^{n+2k+1} x_0) \leq h^n d(x_0, T^{2k+1} x_0)
\]

\[
\leq h^n \sum_{i=0}^{2k} h^i d(x_0, T x_0)
\]

\[
\leq \left( \frac{h^n}{1 - h} \right) \left[ d(x_0, T x_0) + d(x_0, T^2 x_0) \right].
\]

\[
d(T^n x_0, T^{n+m} x_0) \leq \left( \frac{h^n}{1 - h} \right) \left[ d(x_0, T x_0) + d(x_0, T^2 x_0) \right].
\]

Since \( P \) is a normal cone with normal constant \( \kappa \), therefore,

\[
\|d(T^n x_0, T^{n+m} x_0)\| \leq \left[ \frac{h^n}{1 - h} \right] \kappa \left[ \|d(x_0, T x_0) + d(x_0, T^2 x_0)\| \right].
\]

Therefore, \( \|d(T^n x_0, T^{n+m} x_0)\| \to 0 \) as \( n \to \infty \). Now Theorem 5.2.9 implies that \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists \( u \in X \) such that \( x_n \to u \). By Theorem 5.2.8, we have

\[
\|d(T^n x_0, u)\| \to 0, \text{ as } n \to \infty.
\]

By rectangular property, we have

\[
d(Tu, u) \leq d(u, T^{n-1} x_0) + d(T^{n-1} x_0, T^{n+1} x_0) + d(T^{n+1} x_0, u)
\]
\[ \leq d(u, T^{n-1}x_0) + h^{n-1}d(x_0, T^2x_0) + d(T^{n+1}x_0, u). \]

Thus
\[ \|d(Tu, u)\| \leq \kappa \left[ \|d(u, T^{n-1}x_0)\| + h^{n-1}\|d(x_0, T^2x_0)\| + \|d(T^{n+1}x_0, u)\| \right]. \quad (5.2.3) \]

Letting \( n \to \infty \), we have \( \|d(u, Tu)\| = 0 \). Hence \( u = Tu \). Now we show that \( T \) have a unique fixed point. For this, assume that there exists another point \( v \) in \( X \) such that \( v = Tv \). Now,
\[
\begin{align*}
d(v, u) &= d(Tv, Tu) \\
&\leq \alpha d(v, Tv) + \beta d(u, Tu) \\
&= 0.
\end{align*}
\]

Hence, \( u = v \). \( \square \)

5.2.4 On Kannans Fixed Point Theorem in a Cone Rectangular Metric

In this article our main result is On Kannans fixed point theorem in a cone rectangular metric space.

**Theorem 5.2.11.** Let \( (X, d) \) be a complete cone rectangular metric space, \( P \) be a normal cone with normal constant \( k \). Suppose a mapping \( T : X \to X \) satisfies the contractive condition
\[
d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(x, y) + \gamma d(y, Ty),
\]
for all \( x, y \in X, \alpha, \beta, \gamma > 0 \) and \( \alpha + 2\beta + \gamma < 1 \). Then,

(i) \( T \) has a unique fixed point in \( X \).

(ii) For any, the iterative sequence converges to the fixed point.
Proof. Let \( x_0 \in X \) and \( Tx_n = x_{n+1} \). Now

\[
d(Tx_0, T^2x_0) = d(Tx_0, T(T(x_0)) \leq \alpha d(x_0, Tx_0) + \beta d(x_0, Tx_0) + \gamma d(Tx_0, T^2x_0)
\]

\[
(1 - \gamma)d(Tx_0, T^2x_0) \leq (\alpha + \beta)d(x_0, Tx_0)
\]

\[
d(Tx_0, T^2x_0) \leq \left(\frac{\alpha + \beta}{1 - \gamma}\right)d(x_0, Tx_0), \tag{5.2.4}
\]

\[
d(T^2x_0, T^3x_0) \leq \left(\frac{\alpha + \beta}{1 - \gamma}\right)d(Tx_0, T^2x_0) \leq \left(\frac{\alpha + \beta}{1 - \gamma}\right)^2\left(\frac{\alpha + \beta}{1 - \gamma}\right)d(x_0, Tx_0) = \left(\frac{\alpha + \beta}{1 - \gamma}\right)^2d(x_0, Tx_0). \tag{5.2.5}
\]

Thus in general, if \( n \) is positive integer, then

\[
d(T^nx_0, T^{n+1}x_0) \leq h^n d(x_0, Tx_0),
\]

where \( h = \frac{\alpha + \beta}{1 - \gamma} \) and \( 0 < h < 1 \).

We divide the proof into two cases.

**Case I:** First assume that \( T^mx_0 = T^nx_0 \) for \( m, n \in \mathbb{N}, m \neq n \).

Let \( m > n \), then \( T^{m-n}(T^nx_0) = T^mx_0 \), i.e. \( T^py = y \), where \( p = m - n \), \( y = T^nx \). Now since \( p > 1 \), we have

\[
d(y, Ty) = d(T^py, T^{p+1}y) \leq h^p d(y, Ty),
\]

where \( 0 < h < 1 \). We obtain \(-d(y, Ty) \in P \) and \( d(y, Ty) \in P \) which implies that \( \|d(y, Ty)\| = 0 \), i.e. \( Ty = y \).

**Case II:** Assume that \( T^mx_0 \neq T^nx_0 \) for \( m, n \in \mathbb{N}, m \neq n \).

Clearly we have

\[
d(T^nx_0, T^{n+1}x_0) \leq h^n d(x_0, Tx_0),
\]
and

\[ d(T^n x_0, T^{n+2} x_0) = d(T(T^n x_0, T(T^{n+1} x_0)) \]

\[ \leq \alpha d(T^{n-1} x_0, T^n x_0) + \beta d(T^{n-1} x_0, T^{n+1} x_0) + \gamma d(T^{n+1} x_0, T^{n+2} x_0) \]

\[ \leq \alpha h^{n-1} d(x_0, T x_0) + \beta \left[ d(T^{n-1} x_0, T^n x_0) + d(T^n x_0, T^{n+2} x_0) + d(T^{n+2} x_0, T^{n+1} x_0) \right] + \gamma h^{n+1} d(x_0, T x_0) \]

\[ (1 - \beta) d(T^n x_0, T^{n+2} x_0) \leq \alpha h^{n-1} d(x_0, T x_0) + \beta h^{n-1} d(x_0, T x_0) + \beta h^{n+1} d(x_0, T x_0) + \gamma h^{n+1} d(x_0, T x_0) \]

\[ = h^{n-1}(\alpha + \beta) d(x_0, T x_0) + h^{n+1}(\beta + \gamma) d(x_0, T x_0) \]

\[ d(T^n x_0, T^{n+2} x_0) \leq h^{n-1} d(x_0, T x_0). \]

Now if \( m > 2 \) is odd then writing \( m = 2\ell + 1, \ell \geq 1 \) and using the fact that \( T^p x_0 \neq T^r x_0 \)

for \( p, r \in \mathbb{N}, p \neq r \), we get

\[ d(T^n x_0, T^{n+m} x_0) \leq d(T^n x_0, T^{n+1} x_0) + d(T^{n+1} x_0, T^{n+2} x_0) + \ldots + d(T^{n+2\ell} x_0, T^{n+2\ell+1} x_0) \]

\[ \leq h^n d(x_0, T x_0) + h^{n+1} d(x_0, T x_0) + \ldots + h^{n+2\ell} d(x_0, T x_0) \]

\[ < \frac{h^n}{1 - h} d(x_0, T x_0). \]

Again if \( m > 2 \) is even then writing \( m = 2\ell, \ell \geq 2 \) and using the same arguments as before, we get

\[ d(T^n x_0, T^{n+m} x_0) \leq d(T^n x_0, T^{n+2} x_0) + d(T^{n+2} x_0, T^{n+3} x_0) + \ldots + d(T^{n+2\ell-1} x_0, T^{n+2\ell} x_0) \]

\[ \leq h^{n-1} d(x_0, T x_0) + h^{n+2} d(x_0, T x_0) + \ldots + h^{n+2\ell-1} d(x_0, T x_0) \]

\[ \leq [h^{n-1} + h^n + h^{n+1} + \ldots] d(x_0, T x_0) \]

\[ < \frac{h^{n-1}}{1 - h} d(x_0, T x_0). \]

Thus combining all the cases we have

\[ d(T^n x_0, T^{n+m} x_0) < \frac{h^{n-1}}{1 - h} d(x_0, T x_0), \]
for all $m, n \in \mathbb{N}$.

Hence we get

$$\| d(T^n x_0, T^{n+m} x_0) \| \leq k \left( \frac{h^{n-1}}{1-h} \right) \| d(x_0, Tx_0) \|,$$

for all $m, n \in \mathbb{N}$.

Since $k \left( \frac{h^{n-1}}{1-h} \right) \| d(x_0, Tx_0) \| \to 0$, as $n \to +\infty$, $\{T^n x_0\}$ is a Cauchy sequence. By the completeness of $X$, there is $x^* \in X$ such that $T^n x_0 \to x^*$, as $n \to +\infty$.

We shall now show that $Tx^* = x^*$. Without any loss of generality, we can assume that $T^r x^* \neq x^*$, for any $r \in \mathbb{N}$. We have

$$d(x^*, Tx^*) \leq d(x^*, T^n x^*) + d(T^n x^*, T^{n+1} x^*) + d(T^{n+1} x^*, Tx^*)$$

$$\leq d(x^*, T^n x^*) + d(T^n x^*, T^{n+1} x^*) + \alpha d(T^n x^*, T^{n+1} x^*) + \beta d(T^n x^*, x^*) + \gamma d(x^*, Tx^*)$$

$$(1 - \gamma)d(x^*, Tx^*) \leq (1 + \alpha)d(T^{n+1} x^*, T^n x^*) + (1 + \beta)d(T^n x^*, x^*)$$

$$d(x^*, Tx^*) \leq \left( \frac{1 + \alpha}{1 - \gamma} \right) d(T^{n+1} x^*, T^n x^*) + \left( \frac{1 + \beta}{1 - \gamma} \right) d(T^n x^*, x^*)$$

$$\leq \left( \frac{1 + \alpha}{1 - \gamma} \right) \frac{h^n}{1-h} d(Tx^*, x^*) + \left( \frac{1 + \beta}{1 - \gamma} \right) \frac{h^{n-1}}{1-h} d(Tx^*, x^*).$$

Hence

$$\| d(x^*, Tx^*) \| \leq \left( \frac{1 + \alpha}{1 - \gamma} \right) \frac{h^n}{1-h} \| d(Tx^*, x^*) \| + \left( \frac{1 + \beta}{1 - \gamma} \right) \frac{h^{n-1}}{1-h} \| d(Tx^*, x^*) \|$$

$$\to 0, \text{ as } n \to +\infty.$$ 

So we obtain $d(x^*, Tx^*) = 0$, i.e., $Tx^* = x^*$.

Now, if $y^*$ is another fixed point of $T$, then

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

$$\leq \alpha d(Tx^*, x^*) + \beta d(x^*, y^*) + \gamma d(y^*, Ty^*)$$

$$< d(x^*, y^*), \text{ since } 0 < \beta < 1.$$

which implies that $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. \hfill \Box
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5.2.5 An Expansion Mappings on Cone Rectangular Metric Spaces

C.T. Aage and J.N. Salunke, [12] proved fixed point theorem for expansion onto mappings on cone metric spaces in this article we have proved fixed point theorems using same expansion onto mappings on cone rectangular metric spaces.

**Theorem 5.2.12.** Let \((X, d)\) be a complete cone rectangular metric space. Suppose the mapping \(T : X \to X\) is onto and satisfies the contractive condition

\[d(Tx, Ty) \geq Kd(x, y), \quad (5.2.6)\]

for all \(x, y \in X\), where \(K > 1\) is a constant. Then \(T\) has a unique fixed point in \(X\).

**Proof.** If \(Tx = Ty\), then

\[0 \geq Kd(x, y) \Rightarrow 0 = d(x, y) \Rightarrow x = y.\]

Thus, \(T\) is one to one. Define \(G = T^{-1}\)

\[d(x, y) = d(TT^{-1}x, TT^{-1}y) \geq Kd(T^{-1}x, T^{-1}y) = Kd(Gx, Gy),\]

So \(d(Gx, Gy) \leq Md(x, y)\), where \(M = \frac{1}{K} < 1\). By Theorem (2.3) in [123], \(G\) has unique fixed point \(u\) in \(X\) i.e. \(Gu = u \Rightarrow T^{-1}u = u \Rightarrow u = Tu\). Therefore, \(u\) is a unique fixed point of \(T\). \(\square\)

**Corollary 5.2.13.** Let \((X, d)\) be a complete cone rectangular metric space. Suppose a mapping \(T : X \to X\) is onto and satisfies for some positive integer \(n\)

\[d(T^n x, T^n y) \geq Kd(x, y),\]

for all \(x, y \in X\), where \(K > 1\) is a constant. Then \(T\) has a unique fixed point in \(X\).

**Theorem 5.2.14.** Let \((X, d)\) be a complete cone rectangular metric space. Suppose the mapping \(T : X \to X\) is continuous, onto and satisfies the contractive condition

\[d(Tx, Ty) \geq K(d(Tx, x) + d(Ty, y)),\]

for all \(x, y \in X\), where \(\frac{1}{2} < K \leq 1\) is a constant. Then \(T\) has a fixed point in \(X\).
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Proof. For each \( x_0 \in X \), since \( T \) is onto, there exist \( x_1 \in X \) such that \( Tx_1 = x_0 \). Similarly, for each \( n \geq 1 \) there exist \( x_{n+1} \in X \) such that \( x_n = Tx_{n+1} \). If \( x_{n-1} = x_n \), then \( x_n \) is a fixed point of \( T \). Thus, Suppose that \( x_{n-1} \neq x_n \) for all \( n \geq 1 \). Then

\[
d(x_n, x_{n-1}) = d(Tx_{n+1}, Tx_n) \\
\geq K(d(Tx_{n+1}, x_{n+1}) + d(Tx_n, x_n)) \\
= K(d(x_n, x_{n+1}) + d(x_{n-1}, x_n)),
\]

so,

\[
d(x_n, x_{n+1}) \leq \left( \frac{1 - K}{K} \right) d(x_{n-1}, x_n) \\
= hd(x_{n-1}, x_n).
\]

From this we get \( d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \), where \( h = \frac{1-K}{K} \) and \( 0 \leq h < 1 \). Now for \( n < m \) we have

\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
\leq (h^n + h^{n+1} + \cdots + h^{m-1})d(x_0, x_1) \\
\leq \frac{h^n}{1-h} d(x_0, x_1).
\]

Let \( 0 \leq c \) be given. Choose a natural number \( N_1 \) such that \( \frac{h^n}{1-h} d(x_0, x_1) \leq c \), for all \( n \geq N_1 \). Thus \( d(x_n, x_m) \leq c \), for \( n < m \). Therefore, \( \{x_n\}_{n \geq 1} \) is Cauchy sequence in \((X, d)\). Since \((X, d)\) is a complete cone rectangular metric space, there exist \( x^* \in X \) such that \( x_n \to x^* \), as \( n \to \infty \).

If \( T \) is continuous, then

\[
d(Tx^*, x^*) \leq d(Tx^*, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x^*) \to 0, \quad \text{as } n \to \infty,
\]

since \( x_n \to x^* \) and \( Tx_n \to Tx^* \) as \( n \to \infty \). Therefore \( d(Tx^*, x^*) = 0 \) and so \( Tx^* = x^* \).

Thus \( T \) having fixed point in \( X \). \( \square \)
Theorem 5.2.15. Let \((X, d)\) be a complete cone rectangular metric space. Suppose the mapping \(T : X \to X\) is continuous, onto and satisfies the contractive condition
\[
d(Tx, Ty) \geq kd(x, y) + ld(Tx, y),
\]
for all \(x, y \in X\), where \(k > 1, l \geq 0\) are constants. Then \(T\) has a fixed point in \(X\).

Proof. For each \(x_0 \in X\), since \(T\) is onto, there exist \(x_1 \in X\) such that \(Tx_1 = x_0\). Similarly, for each \(n \geq 1\) there exist \(x_{n+1} \in X\) such that \(x_n = Tx_{n+1}\). If \(x_{n-1} = x_n\), then \(x_n\) is a fixed point of \(T\). Thus, Suppose that \(x_{n-1} \neq x_n\) for all \(n \geq 1\). Then
\[
d(x_n, x_{n-1}) = d(Tx_{n+1}, Tx_n)
\geq kd(x_{n+1}, x_n) + ld(Tx_{n+1}, x_n)
= kd(x_{n+1}, x_n) + ld(x_n, x_n)
= kd(x_{n+1}, x_n),
\]
so,
\[
d(x_n, x_{n+1}) \leq \frac{1}{k}d(x_{n-1}, x_n)
= hd(x_{n-1}, x_n),
\]
where \(h = \frac{1}{k}\). Now for \(n < m\) we have
\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)
\leq (h^n + h^{n+1} + \cdots + h^{m-1})d(x_0, x_1)
\leq \frac{h^n}{1-h}d(x_0, x_1).
\]
Let \(0 \leq c\) be given. Choose a natural number \(N_1\) such that \(\frac{h^n}{1-h}d(x_0, x_1) \leq c\), for all \(n \geq N_1\). Thus \(d(x_n, x_m) \leq c\), for \(N_1 < n < m\). Therefore, \(\{x_n\}_{n \geq 1}\) is Cauchy sequence in \((X, d)\). Since \((X, d)\) is a complete cone rectangular metric space, there exist \(x^* \in X\) such that \(x_n \to x^*\), as \(n \to \infty\).
If $T$ is continuous, then
\[ d(Tx^*, x^*) \leq d(Tx^*, Tx) + d(Tx, Tx_{n+1}) + d(Tx_{n+1}, x^*) \to 0, \quad \text{as } n \to \infty, \]
since $x_n \to x^*$ and $Tx_n \to Tx^*$ as $n \to \infty$. Therefore $d(Tx^*, x^*) = 0$ and so $Tx^* = x^*$. Thus $T$ having fixed point in $X$.

**Theorem 5.2.16.** Let $(X, d)$ be a complete cone rectangular metric space. Suppose the mapping $T : X \to X$ is continuous, onto and satisfies the contractive condition
\[ d(Tx, Ty) \geq kd(x, y) + ld(x, Tx) + pd(y, Ty), \]
for all $x, y \in X$, where $k \geq -1$, $l > 1$ and $p < 1$ are constants with $k + l + p > 1$. Then $T$ has a fixed point in $X$.

**Proof.** For each $x_0 \in X$, since $T$ is onto, there exist $x_1 \in X$ such that $Tx_1 = x_0$. Similarly, for each $n \geq 1$ there exist $x_{n+1} \in X$ such that $x_n = Tx_{n+1}$. If $x_{n-1} = x_n$, then $x_n$ is a fixed point of $T$. Thus, Suppose that $x_{n-1} \neq x_n$ for all $n \geq 1$. Then
\[ d(x_n, x_{n-1}) = d(Tx_{n+1}, Tx_n) \]
\[ \geq kd(x_{n+1}, x_n) + ld(x_{n+1}, Tx_{n+1}) + pd(x_n, Tx_n) \]
\[ = kd(x_{n+1}, x_n) + ld(x_{n+1}, x_n) + pd(x_n, x_{n-1}), \]
so,
\[ d(x_n, x_{n+1}) \leq \left(\frac{1-p}{k+l}\right)d(x_n, x_{n-1}) \]
\[ = hd(x_n, x_{n-1}), \]
where $h = \frac{1-p}{k+l}$ with $0 < h < 1$. Now for $n < m$ we have
\[ d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \]
\[ \leq (h^n + h^{n+1} + \cdots + h^{m-1})d(x_0, x_1) \]
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\[ \leq \frac{h^n}{1-h} d(x_0, x_1). \]

Let \( 0 \leq c \) be given. Choose a natural number \( N_1 \) such that \( \frac{h^n}{1-h} d(x_0, x_1) \leq c \), for all \( n \geq N_1 \). Thus \( d(x_n, x_m) \leq c \), for \( N_1 < n < m \). Therefore, \( \{x_n\}_{n \geq 1} \) is Cauchy sequence in \((X, d)\). Since \((X, d)\) is a complete cone rectangular metric space, there exist \( x^* \in X \) such that \( x_n \to x^* \), as \( n \to \infty \).

If \( T \) is continuous, then

\[ d(Tx^*, x^*) \leq d(Tx^*, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x^*) \to 0, \text{ as } n \to \infty, \]

since \( x_n \to x^* \) and \( Tx_n \to Tx^* \) as \( n \to \infty \). Therefore \( d(Tx^*, x^*) = 0 \) and so \( Tx^* = x^* \). Thus \( T \) having fixed point in \( X \). \qed