CHAPTER - I

INTRODUCTION

1.1 Fixed point theorems are mainly useful in the existence of differential equations, integral equations, partial differential equations, space mechanics and other related areas. Fixed point theory has a very fruitful application in eigenvalue problems, boundary value problems, nonlinear oscillations, fluid flows, topological dynamics, theory of games, probabilistic analysis.

A topological space is said to have a fixed point if every continuous function $T: X \rightarrow X$ has a point $Tx=x$, for some $x$ in $X$ i.e. a point which remains invariant under a transformation $T$ is called a fixed point. The first result on fixed points applied to cases in which $T$ is topological simple subset of $R^n$ and $T$ is continuous map of $E$ into itself.

In 1912, Brouwer proved the following useful theorem: Let $C$ be a unit ball in $E^n$ and $T:C \rightarrow C$ a continuous function, then $T$ has a fixed point in $C$, or $Tx = x$ has a solution. This fundamental result was extended by Schauder[134] to compact convex sets in Banach space in 1927 in the following manner: Every
continuous mapping of a convex compact subset $K$ of Banach space has a fixed point. Later in 1930, he[133] generalized his own result by relaxing the stringent condition of compactness on $K$ as follows: Let $X$ be a Banach space and $C$ be a compact convex subset of $X$ and $T:C \to C$ be a continuous map. Then $T$ has at least one fixed point in $C$. Meanwhile in 1922, Banach[8] proved an important result on fixed points known as the Banach contraction principle. The proof of this result does not involve much of topological machinery. The statement of his theorem is as follows:

If $T$ be a selfmap of a complete metric space $(X,d)$ satisfying

$$(1.1.1) \quad d(Tx,Ty) \leq kd(x,y)$$

for all $x,y$ in $X$ and for some $k$, $0 \leq k < 1$, then $T$ has a unique fixed point.

A mapping $T$ satisfying $(1.1.1)$ is known as contraction mapping. It is important to note that any such contraction $(1.1.1)$ of $X$ is obviously a continuous map of $X$ into itself. However a continuous mapping is not necessarily a contraction. During the last six decade, a lot of fixed point theorems have been established and we find that Banach contraction principle is at the base of most of these results.
established so far i.e. the Banach contraction principle has served as the main source containing the important idea of contraction, which has given much inspiration to a number of mathematicians and researchers to work over this field of fixed point spaces.

In 1965, Chu and Diaz[30] has shown that even if T is not a contraction it is possible that $T^n$ ($n^{th}$ iterate of T) is a contraction for some positive integer n. This result has been further generalized by Sehgal[136] in the following way:

Let $T : X \rightarrow X$ be a continuous mapping of a complete metric space $(X,d)$. If for each $x$ in $X$, there exists a positive integer $n(x)$ such that for all $x,y$ in $X$

\[(1.1.2) \ d(T^n x, T^n y) \leq kd(x,y)\]

for some constant $k$, $0 < k < 1$. Then $T$ has a unique fixed point.

A further generalization of (1.1.2) of Sehgal was obtained by Holmes[70] in 1969, where $n$ depends on both $x$ and $y$. For the further generalization about this type of work we refer to Khazanchi[89], Iseki[73], Guseman[59], Cirić[34], Sharma[143] and Ray and Rhoades[119]. Rakotch[116], Browder[20] and Boyd and
Wong[18] further generalized the Banach contraction principle by replacing the Lipschitz constant $k$ by some real valued functions, whose values are less than 1. Rakotch[116] defined a family of functions by

$$d(x,y) = k(d(x,y)), \quad 0 < k(d) < 1.$$  

for $d > 0$ and $k(d)$ is a monotone decreasing function of $d$. Bryant[22], Fukushima[53], Reich[121] and Wong[170] extended the work on the same line as that of Rakotch[116].

Next, in 1969, Meir and Keeler[98] introduced a mapping known as weakly uniformly strict contraction. Let $T:X \rightarrow X$, if for given $\epsilon > 0$, there exists $\delta > 0$, such that

$$\epsilon < d(x,y) < \epsilon + \delta, \text{implies } d(Tx,Ty) < \epsilon,$$

then $T$ is called a weakly uniformly strict contraction mapping. Meir and Keeler[98] generalized the theorem of Boyd and Wong[18]. In 1978, Maiti and Pal[97] obtained the further generalization of Meir and Keeler[98] and of Boyd and Wong[18]. Further Park and Bae[111], Park and Rhoades[112], Rhoades[127], Ganguly[57], Maiti, Achari and Pal[95] obtained more general results along these lines.

In 1968, Kannan[83] considered a self mapping $T$ satisfying the condition
(1.1.5) $d(Tx, Ty) \leq k \left[ d(x, Tx) + d(y, Ty) \right]$

for all $x, y$ in $X$, $0 \leq k < \frac{1}{2}$ and obtained a unique fixed point.

In 1971 Reich[122] unified the theorem of Banach[8] and Kannan[83] and considered a selfmapping $T$ of a complete metric space $(X, d)$ satisfying the following condition

(1.1.6) $d(Tx, Ty) \leq a \cdot d(x, Tx) + b \cdot d(y, Ty) + c \cdot d(x, y)$

for all $x, y$ in $X$, where $0 \leq a + b + c < 1$. Then $T$ has a unique fixed point in $X$.

Rakotch[116] obtained the generalization of (1.1.1) by replacing the constant $k$ with the monotone decreasing function $k : [0, \infty) \rightarrow [0, 1]$ with this idea Reich[121] has proved fixed point theorems with the condition

(1.1.7) $d(Tx, Ty) \leq a(d(x, y))d(x, Tx) + b(d(x, y))d(y, Ty)$

$+ c(d(x, y))d(x, y)$

For the further work on the same lines, we refer to Wong[171], Bryant[22] and Fukushima[53].

Chatterjea[26] studied a mapping $T : X \rightarrow X$ of complete metric space $(X, d)$ satisfying

(1.1.8) $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$

for all $x, y$ in $X$, where $0 \leq k < \frac{1}{2}$, and obtained unique fixed point of $T$. 
In 1973, Hardy and Rogers [67] unified the mappings of Reich [122] and Chhaterjea [26] and studied a mapping \( T : X \rightarrow X \) of a complete metric space \((X,d)\) satisfying

\[
(1.1.9) \quad d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) \\
+ a_4 d(y, Tx) + a_5 d(x, y),
\]

for all \( x, y \) in \( X \) with \( a_1 > 0 \) and \( \sum_{i=1}^{5} a_i < 1 \), which is the generalization of (1.1.1), (1.1.5), (1.1.6) and (1.1.8).

The work of Hardy and Rogers [67] was further generalized by Wong [170] under the additional assumptions \( a_1 = a_2 \) and \( a_3 = a_4 \) and he obtained some results of Hardy and Rogers [67], Kannan [83], Rakotch [116], Reich [122], Gupta and Shrivastava [62] and Banach [8]. Hardy and Rogers [67] works was further generalized in various directions by Ciric [31], Sharma [144], Iseki [74], Singh and Meade [156], Hussain and Sehgal [142], [141] and many others.

Recently in 1984, Wong and others [166] obtained some fixed point theorems for expansion mappings on complete metric space \((X,d)\) in quite different way, which are as follows:

(i) If there is a constant \( a > 1 \) such that

\[
d(Tx, Ty) \geq a d(x, y)
\]

for all \( x, y \) in \( X \) and \( T \) is onto, then \( T \) has a unique fixed point.
(ii) If there exists non-negative real numbers $a, b, c$ with $a + b + c > 1$ and $a < 1$ such that

$$d(Tx, Ty) \geq a \ d(x, Tx) + b \ d(y, Ty) + c \ d(x, y)$$

for all $x, y$ in $X$ with $x \neq y$ and $T$ is onto, then $T$ has a fixed point.

(iii) If there exists a real number $a > 1$ such that

$$d(Tx, Ty) \geq \min \{d(x, Tx), d(y, Ty), d(x, y)\}$$

for all $x, y$ in $X$ and $T$ is onto and continuous, then $T$ has a fixed point.

For the further generalization about this type of work we refer to Rhoades[129], Park[109], Fisher[47] and Park and Rhoades[113].

In 1962, Edelstein[43] introduced the contractive mapping of a metric space $X$, which satisfies the condition

$$(1.1.10) \quad d(Tx, Ty) < d(x, y)$$

for all $x, y$ in $X$ with $x \neq y$.

A contractive mapping is clearly continuous and it is more general than contraction mappings. Completeness of the space is not enough to ensure the existence of fixed points for such mappings and we require further extra condition on the space, on the mapping or on its range. Pertaining to contractive
mapping Bailey[6], Byrant, Guseman and Peters[23], Sehgal[138], Ciric[32], Jaggi and Sharma[81], Singh[155] and others obtained several fixed point theorems.

A mapping \( T : X \to X \) is said to be non-expansive mapping if

\[
(1.1.11) \quad d(Tx,Ty) \leq d(x,y)
\]

for all \( x, y \) in \( X \). It is remarked here that non-expansive mapping are more general than contractive mappings. This class of mappings include contraction and contractive mappings and doesn't assure the uniqueness of fixed point. Edelstein[42] gave the first result for non-expansive mappings in the year 1961. Later on, the result for a general type of non-expansive mappings in non-compact metric space was proved independently by Browder[19], Godhe[58] and Kirk[90] in 1965. Cheney and Goldstein[29] have proved a result which was further generalized by Singh and Zorzitto[158]. This class of mapping have further studied by Belluce, Kirk and Steiner[9], Browder and Petryshyn[21] and Kirk[91] and many others.

1.2 COMMON FIXED POINTS

(i) A point \( z \) in \( X \) is said to be a common fixed point of a pair of self-mapping \( S \) and \( T \) on \( X \) if \( Sz = z = Tz \).
(ii) If $A$, $S$ and $T$ are the three selfmappings of a metric space $(X,d)$ for any $z$ in $X$ if $Az = z = Sz = Tz$, then $z$ is said to be a common fixed point of $A$, $S$ and $T$.

In the same way, fixed point for any number of maps can be defined.

Kannan[84] obtained a sufficient condition for the existence of a unique fixed point for a pair of mappings, each defined on a complete metric space, for the first time in 1968. Later on, Kannan[83], Sehgal[137], Isoki[74], Rus[132], Yeh[174] and many others worked along these lines. In fact, a great deal of work on common fixed points has been done by various mathematicians and researchers in different directions. Ciric[31], obtained common fixed point for the most general type of selfmapping of a metric space $X$. The result of Ciric[31] were generalized by Maiti and Pal[96],[97], which were further generalized by Rhoades[128].

In a recent paper Fisher[50] obtained a unique fixed point theorem for a pair of selfmappings $S$, $T$ of the complete metric space $(X,d)$ satisfying a new contraction type condition:

(1.2.1) $[d(Sx,Ty)]^2 \leq a \ d(x,Ty)d(x,Sx) + b \ d(y,Sx)d(y,Ty)$
for all \(x, y\) in \(X\), where \(a, b > 0\) and
\[
[a + (a^2 + 4a)^\frac{1}{2}] [b + (b^2 + 4b)^\frac{1}{2}] < 4.
\]

Hc[50] also proved an analogous result for compact metric spaces. Pachpatte[108],[105] obtained some results for the similar type of mappings. Many other authors such as Wong[169], Ray[117], Singh[148],[147], Khan[88] and Yeh[173] have also studied some results on common fixed points in metric spaces.

1.3 FIXED POINT IN QUASI-GAUGE FUNCTION SPACES

Quasi-gauge space was first development by Reilly[123],[124]. The study of sets and spaces of functions were introduced by Ascoli[4] and Arzela[3] respectively. These paper marked the beginning of not only of functions space theory but also of the general topological space. Fixed point theorems in function space were tried by Frechet[52], Weyl[167], Hausdorff[69] and Reich[121]. Subrahmaniyam[161], Gupta and Bawa[60] have contributed fixed point theory in quasi-gauge spaces and quasi-gauge function spaces.

1.4 FIXED POINTS IN BIMETRIC AND IN TWO METRIC SPACES

In 1968, Haia[94] proved some interesting results on bimetric space and thereby generalized the Banach
contraction principle by taking two metrics on a set $X$. He obtained a unique fixed point for a mapping $T:X \to X$, when $T$ is a contraction with respect to one metric and continuous for the other metric. Singh[154] generalized the continuity of $T$ over the whole space $X$ by the continuity at a point in $X$. Further Singh[154], Iseki[72], Rus[131], Rhoades[125], Pachpatte[107],[106], Singh and Pant[153], Mishra[101] and many others have generalized these results in different directions.

Recently Fisher[46],[49] obtained some fixed point theorems for two different metric spaces. He assumed quite different conditions from those, which has been considered by Maia[94] for obtained fixed points.

1.5 **FIXED POINT THEOREMS FOR ORBITALLY CONTINUOUS MAPPINGS**

Let $(X,d)$ be a metric space. A mapping $T$ on $X$ is orbitally continuous if $\lim_{i} T_{i}^{n} x = z$ implies $\lim_{i} T_{i}^{n} x = Tz$ for each $x$ in $X$. A space $X$ is said to be $T$-orbitally complete if every Cauchy sequence of the form $\{T_{i}^{n} x\}_{i=1}^{\infty}$, $x$ in $X$, converges in $X$. Achari[1],[2], Cirić[37], Pachpatte[107], Dhage[38], Pathak[114], Tasković[164] and many others obtained fixed point theorems for orbitally continuous mappings.
1.6 FIXED POINT THEOREMS FOR SEQUENCE OF MAPS

Several mathematicians have investigated the conditions under which the convergence of a sequence of mappings $T_n (n=1,2,\ldots)$ to a mapping $T$ of a metric $X$ into itself implies the convergence of their fixed points $z_n (n=1,2,\ldots)$ to a fixed point $z$ of $T$.

In 1962, Bonsall[17] gave some basic theorems for sequence of maps. Later on, Singh and Russel[157], Nadler Jr.[104], Chatterjea[27], Ray[118], Cirić[35], Park[110], Khan[86] and many other followed the ideas of Bonsall[17] and obtained various results for sequence of mappings. In general, all of these have been considered three types of fixed point theorems, which are as follows:

(a) If each pair $T_i, T_j$ satisfies the same contractive conditions, then $T_n$ gives the existence of a fixed point.

(b) If $T_n$ satisfies the same contractive conditions and $T_n$ tends pointwise to a limit function $T$, then $T$ has a fixed point $z$, which is the limit of each of fixed points $z_n$ of $T_n$ and

(c) If each $T_n$ has a fixed point $z_n$ and that $T_n$ converges uniformly to a function $T$ which satisfies a particular contractive condition with $z$ as the fixed
point of $T$, then $z_n$ converges to $z$.

1.7 FIXED POINTS OF CONTRACTIVE TYPE MULTIVALUED FUNCTIONS

Following the Banach contraction principle, Nadler Jr.[103] introduced the concept of multivalued contraction mappings and established that a multivalued contraction mapping possesses a fixed point in a complete metric space. Subsequently a number of fixed point theorems in metric spaces have been proved for multivalued mappings satisfying contraction type conditions by Dubey and Singh[41], Assad and Kirk[5], Smithson[159], Cirić[33], Iseki[75], Kaulgud and Pai[85], Rus[130], Kuhfthing[92], Itoh[78], Itoh and Takahashi[79], Yanagi[172], Hu[71], Singh and Kulshrestha[150] and many others. Also Mishra[100], [102], [99] extended some of these results to uniform spaces.

1.8 FIXED POINT THEOREMS IN PROBABILISTIC METRIC SPACES

Due to the applications of metric spaces in various disciplines of Mathematics and Mathematical sciences, the Banach contraction principle has been extensively studied and generalized on many settings and fixed point theorems have been established.
A Menger space is a space in which the concept of distance is considered to be probabilistic, rather than deterministic. The theory of Menger space is of fundamental importance in probabilistic functional analysis. Recently, some fixed point theorems for mappings in Menger spaces have been proved by several authors: Bocsan[12], Bharucha-Reid[10], Sehgal[139], Ciric[36], Chang and Kim[25], Hadzic[64],[65], Stojakovic[160] and others.

Since every metric space is a Menger space, all results in Menger space, with some modifications, can be used in metric spaces.

Bharucha-Reid[10] underlined the need of the research regarding the study of fixed points of mappings on probabilistic metric space. Sehgal[139],[140] initiated the study of fixed points of contraction mappings on a probabilistic metric space. He introduced the notion of contraction mappings on a probabilistic metric space as a generalization of the Banach contraction principle and proved some fixed point theorems for such mappings. These results were first published in 1972 by Sehgal and Bharucha-Reid[140]. Subsequently several generalization of Sehgal's results were obtained on probabilistic metric spaces, see for
instance Bocsan[12], Cain and Kasriel[24], Chang and Kim[25], Ciric[36], Hadzic[63], [66], [65], Istratescu[77], Singh and Pant[151], [152] and several others.

1.9 Chapter II is devoted to the study of fixed points in sequentially complete quasi-gauge function space and it consists of three sections. In section I, we obtain a unique fixed point theorem for a pair of selfmaps. In section II, we generalize the result of section I for sequence of densifying and weakly contractive mappings. In the last section, we obtain some results having two different quasi-gauge function spaces.

In chapter III, we obtain some results on fixed points for a selfmap of a metric space \((X,d)\), which is not necessarily continuous. The mappings used in this section were introduced by Sharma and Yule[146] and Iseki, Rajput and Sharma[76].

In chapter IV, we improve the results of Ciric[37] and Dhage[38], on fixed points in orbitally complete metric spaces and also obtain the localized version of our results. In the last section of this chapter, we discuss the validity of the hypothesis and
degree of generality of our results in view of some examples.

In chapter V, some results on fixed point and common fixed point have been obtained, which in turn, generalize the results of Fisher[48], Harinath[68] on compact metric spaces and of Jain and Dixit[82] on pseudo-compact metric spaces.

In chapter VI, some fixed point and common fixed point theorems in Bimetric spaces and in two metric spaces have been established. Our results in Bimetric spaces, which extend and unify the results of chapters III and IV, are initiated by a result of Maia[94].

In chapter VII, we obtain some common fixed point theorems in complete 2-metric spaces. In the beginning, we obtain some results for three selfmaps and than generalize these results for sequence of selfmappings.

In chapter VIII, section I, some coincidence and fixed point theorems in a metric space are proved for a multivalued mapping that commutes with a single-valued mapping and satisfies a multi-valued contraction type condition. In section II, we have generalized the results of section I in uniform space.
In chapter IX, section I, we prove some common fixed point theorems for expansion mappings which generalize the results of Rhoades[129], Wong and others[166]. In section II, we obtain some common fixed point theorems for continuous selfmaps.

In chapter X, we confine ourselves to obtain some results on common fixed point for four maps of a probabilistic metric and of complete metric spaces. In section I, we study some problems related to common fixed point of four mappings $h : X \to X$, $k : X \to X$, $f : X \to h(X)$, $g : X \to k(X)$ satisfying the inequality

$$F_{fx,gy}(\varepsilon) \geq \min\{F_{kx,hy}(\phi(\varepsilon)), F_{fx,kx}(\phi(\varepsilon)), F_{gy,hy}(\phi(\varepsilon)), F_{fx,hy}(2\phi(\varepsilon)), F_{gy,kx}(2\phi(\varepsilon))\},$$

for all $\varepsilon > 0$, $x,y$ in $X$, where $\phi : R^+ \to R^+$ is an increasing function such that $\lim_{n \to \infty} \phi^n(t) = \infty$, for all $t > 0$, and obtain a unique common fixed point of $f,g,h$ and $k$.

In section II, we generalize the results of section I in complete metric spaces and in the last we give some examples to verify that, whether the conditions of our theorems are all necessary?

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