CHAPTER X

COMMON FIXED POINT THEOREMS IN
COMPLETE METRIC AND PROBABILISTIC METRIC SPACES
CHAPTER X

COMMON FIXED POINT THEOREMS IN
COMPLETE METRIC AND PROBABILISTIC METRIC SPACES

10.1 In this chapter some common fixed point theorems for four continuous mappings in Menger and metric space have been established. These mappings are assume to satisfy some generalizations of the contraction condition.

Recently Stojakovic\cite{160} proved the following theorem.

THEOREM A : Let \((X, \mathcal{F}, t)\) be a complete Menger space with a continuous \(T\)-norm \(t\) and let \(h: X \rightarrow X, k: X \rightarrow X, f: X \rightarrow h(X)\) and \(g: X \rightarrow k(X)\) be continuous mappings such that \(f\) commutes with \(k\) and \(g\) commutes with \(h\). Further, suppose that for all \(x, y\) in \(X\) and for all \(\varepsilon > 0\) the following inequality holds

\[
(10.1.1) \quad F_{fx, gx}(\varepsilon) \geq F_{kx, hy}(\phi(\varepsilon)),
\]

where \(\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is an increasing function such that \(\lim_{n \to \infty} \phi^n(t) = \infty\) for all \(\varepsilon > 0\). If these exists \(x_1\) in \(X\) such that the sequence \(\{y_n\}_{n \in \mathbb{N}}\) formed by
\[(10.1.2) \quad y_{2n-1} = g x_{2n-1} = k x_{2n} \]

\[y_{2n} = f x_{2n} = h x_{2n+1}, \quad n \text{ in } \mathbb{N} \]

is probabilistically bounded, then there exists a unique common fixed point for the mappings \(f, g, h, \text{ and } k\).

Before going to state our generalizations of Theorem A we give the following definitions.

**DEFINITION (10.1.1)**: A mapping \(F: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is called a distribution function if it is non-decreasing, left continuous with \(\inf F = 0\) and \(\sup F = 1\), where \(\mathbb{R}\) denote the set of real numbers and \(\mathbb{R}^+=\{x \in \mathbb{R} : x \geq 0\}\).

We will denote by \(\Delta\) the set of all distribution functions.

**DEFINITION (10.1.2)**: A mapping \(t:[0,1] \times [0,1] \rightarrow [0,1]\) is a T-norm if it satisfies

1. \(t(a,1) = 1, \quad t(0,0) = 0\)
2. \(t(a,b) = t(b,a)\)
3. \(t(c,d) \geq t(a,b), \quad \text{for } c \geq a, \quad d \geq b,\)
4. \(t(t(a,b),c) = t(a,t(b,c))\)

that is a commutative, associative and non-decreasing mapping \(t:[0,1] \times [0,1] \rightarrow [0,1]\) is a T-norm if and only if \(t(a,1) = a\) for all \(a \in [0,1]\) and \(t(0,0) = 0\).
DEFINITION (10.1.3) : A Menger space is a triple \((X,F,t)\) where \(X\) is a set, \(F\) is a mapping from \(X \times X\) into \(\Delta\) and \(t\) is a \(T\)-norm. We shall denote the distribution function \(F_{x,y}(\varepsilon)\) by \(F_{x,y}(\varepsilon)\) and \(F_{x,y}(\varepsilon)\) will represent the value of \(F_{x,y}\) at \(\varepsilon\) in \(R\). The function \(F_{x,y}(x,y)\) in \(X\) are assumed to satisfy the following conditions:

1. \(F_{x,y}(\varepsilon)=1,\) for \(\varepsilon > 0\) if and only if \(x=y\).
2. \(F_{x,y}(0)=0,\) for all \(x,y\) in \(X\).
3. \(F_{x,y}(\varepsilon)=F_{y,x}(\varepsilon),\) for all \(x,y\) in \(X\).
4. \(F_{x,y}(\varepsilon+\delta) \geq t(F_{x,z}(\varepsilon),F_{z,y}(\delta)),\) for all \(x,y,z\) in \(X\).

DEFINITION (10.1.4) : A function \(H\) is said to be the specific distribution function if

\[
H(\varepsilon) = \begin{cases} 
0 & \varepsilon \leq 0, \\
1 & \varepsilon > 0.
\end{cases}
\]

The concept of neighbourhoods in Menger space was introduced by Schweizer and Sklar [135] in the following manner.

DEFINITION (10.1.5) : If \(x\) in \(X\), \(\varepsilon > 0\) and \(\lambda \in (0,1)\), then an \((\varepsilon,\lambda)\)-neighbourhood of \(x\), called \(U_x(\varepsilon,\lambda)\), is defined by

\[
U_x(\varepsilon,\lambda) = \{ y \in X : F_{x,y}(\varepsilon) > 1-\lambda \}.
\]

DEFINITION (10.1.6) : If \(t\) is continuous, then \((X,F,t)\)
is a Hausdorff space in the topology induced by the family

\[ \{ U_x(\varepsilon, \lambda) : x \in X, \varepsilon > 0, \lambda \in (0,1) \} \]

of neighbourhoods.

**DEFINITION (10.1.7):** A set \( M \subseteq X \) is called probabilistically bounded if and only if

\[
\sup_{\varepsilon > 0} \inf_{x, y \in M} F_{x, y}(\varepsilon) = 1.
\]

**10.2** Now we prove the following theorem.

**THEOREM 1:** Let \((X, F, t)\) be a complete Menger space with a continuous T-norm \( t \) and let \( h: X \to X \), \( k: X \to X \), \( f: X \to h(X) \) and \( g: X \to k(X) \) be continuous mappings such that \( f \) commutes with \( k \) and \( g \) commutes with \( h \). Further, suppose that for all \( x, y \) in \( X \) and for all \( \varepsilon > 0 \) the following inequality holds

\[
(10.2.1) \quad F_{fx, gy}(\varepsilon) \geq \min \{ F_{kx, hy}(\phi(\varepsilon)), F_{fx, kx}(\phi(\varepsilon)) \},
\]

\[
F_{gy, hy}(\phi(\varepsilon)), F_{fx, hy}(2\phi(\varepsilon)), F_{gy, kx}(2\phi(\varepsilon)) \}
\]

where \( \phi: \mathbb{R}^+ \to \mathbb{R}^+ \) is an increasing function such that

\[
\lim_{n \to \infty} \phi^n(t) = \infty \text{ for all } t > 0.
\]

If there exists \( x_1 \) in \( X \) such that the sequence \( \{ y_n \}_{n \in \mathbb{N}} \) formed by (10.1.2) is probabilistically bounded, then there exists a unique common fixed point for the mappings \( f, g, h \) and \( k \).

**Proof:** Since \( f: X \to h(X) \) and \( g: X \to k(X) \), we can form the
sequence as defined in (10.1.2). Let us prove that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence. In order to prove that, we shall show that

\[
\lim_{m \to \infty} F_{y_m, y_p} (\varepsilon) = H(\varepsilon), \text{ for every } \varepsilon \text{ in } \mathbb{R}
\]

\[
\lim_{n \to \infty} F_{y_n} (\varepsilon) = H(\varepsilon), \text{ for every } \varepsilon \text{ in } \mathbb{R}
\]

Now from (10.2.1), we have

\[
F_{y_{2n}, y_{2n-1}} (\varepsilon) = F_{f\chi_{2n}, g\chi_{2n-1}} (\varepsilon) \\
\geq \min \{ F_{k\chi_{2n}, h\chi_{2n-1}} (\phi(\varepsilon)), F_{k\chi_{2n}, h\chi_{2n-1}} (2\phi(\varepsilon)) \}
\]

\[
F_{g\chi_{2n-1}, k\chi_{2n}} (\phi(\varepsilon)) \}
\]

\[
= \min \{ F_{y_{2n-1}, y_{2n-2}} (\phi(\varepsilon)), F_{y_{2n-1}, y_{2n-2}} (2\phi(\varepsilon)) \}
\]

\[
F_{y_{2n-1}, y_{2n-2}} (2\phi(\varepsilon)) \}
\]

\[
F_{y_{2n}, y_{2n-1}} (\phi(\varepsilon)) \}
\]

giving

(10.2.2) \( F_{y_{2n}, y_{2n-1}} (\varepsilon) \geq F_{y_{2n-1}, y_{2n-2}} (\phi(\varepsilon)) \)

since, \( F_{y_{2n}, y_{2n-2}} (2\phi(\varepsilon)) \geq \min \{ F_{y_{2n}, y_{2n-1}} (\phi(\varepsilon)), \}

\[
F_{y_{2n-1}, y_{2n-2}} (\phi(\varepsilon)) \}
\]
and $F_{y_{2n} \cdot y_{2n-1}} (\varepsilon) \geq F_{y_{2n} \cdot y_{2n-1}} (\phi(\varepsilon))$ is impossible,

also $F_{y_{2n-1} \cdot y_{2n-1}} (2\phi(\varepsilon)) = H(\varepsilon)$.

Repeated use of (10.2.1) gives

$$F_{y_{2n} \cdot y_{2n-1}} (\varepsilon) \geq F_{y_{2n-1} \cdot y_{2n-2}} (\phi(\varepsilon)) \geq F_{y_{2n-2} \cdot y_{2n-3}} (\phi^2(\varepsilon))$$

$$\geq \ldots \ldots \geq F_{y_0 \cdot y_1} (\phi^{2n}(\varepsilon))$$

As $n \to \infty$, $F_{y_0 \cdot y_1} (\phi^{2n}(\varepsilon)) \to H(\varepsilon)$; thus for arbitrary positive reals there exists an integer $K=K(\phi(\varepsilon)-1, \lambda)$ such that

(10.2.3) $F_{y_{n-1} \cdot y_n} (\phi(\varepsilon)-1) > 1-\lambda$.

Now if $m=2i$ and $p=2j-1$ (let $j>i$) then from (10.2.1), we have

$$F_{y_{2i} \cdot y_{2j-1}} (\varepsilon) = F_{x_{2i} \cdot y_{2j-1}} (\varepsilon)$$

$$\geq \min\{F_{y_{2i-1} \cdot y_{2j-2}} (\phi(\varepsilon)), F_{y_{2i} \cdot y_{2i-1}} (\phi(\varepsilon)), F_{y_{2j-2} \cdot y_{2j-1}} (\phi(\varepsilon)), F_{y_{2j-1} \cdot y_{2j-1}} (2\phi(\varepsilon)), F_{y_{2j} \cdot y_{2j}} (2\phi(\varepsilon))\}$$
since \( F_{y_{2i-1}, y_{2j-1}} (2\phi(E)) \geq \min\{F_{y_{2i-1}, y_{2j-2}} (\phi(E)), F_{y_{2j-2}, y_{2j-1}} (\phi(E))\} \)

and \( F_{y_{2i}, y_{2j-2}} (2\phi(E)) \geq \min\{F_{y_{2i}, y_{2i-1}} (\phi(E)), F_{y_{2i-1}, y_{2j-2}} (\phi(E))\} \).

Therefore,

\[
F_{y_{2i}, y_{2j-1}} (\phi(E)) \geq \min\{F_{y_{2i-1}, y_{2j-2}} (\phi(E)), F_{y_{2i}, y_{2i-1}} (\phi(E)), F_{y_{2j-2}, y_{2j-1}} (\phi(E))\}
\]

Thus in view of (10.2.3), we have

\[
F_{y_{2i}, y_{2j-1}} (\phi(E)) \geq F_{y_{2i-1}, y_{2j-2}} (\phi(E))
\]

Proceeding in the same way, we have

\[
F_{y_{2i}, y_{2j-1}} (\phi(E)) \geq F_{y_{2i-1}, y_{2j-2}} (\phi(E)) \geq F_{y_{2i-2}, y_{2j-3}} (\phi^{2}(E)) \\
\geq \ldots \geq F_{y_{0}, y_{2j-1-2i}} (\phi^{2i}(E)) \\
\geq \sup t < \phi^{2i}(E) \inf_{n \in \mathbb{N}} F_{y_{n}, y_{k}} (t) = \mathcal{D}\{y_{n}\}_{n=1} (\phi^{2i}(E))
\]

Since \( \{y_{n}\}_{n \in \mathbb{N}} \) is probabilistically bounded, letting \( i \to \infty \) and \( j \to \infty \), we get
\[
\lim_{i \to \infty} D\{y_n\}_{n=1}^{\infty} (4^{2i}(E)) = H(E).
\]

Repeating this procedure we can prove a similar result for \(m=2i-1\) and \(p=2j\).

If \(m\) and \(p\) are both even or both odd, we proceed as follows.

\[
F_{y_{2i}, y_{2j}} (E) \geq t(F_{y_{2i-1}, y_{2i+1}} (E/2), F_{y_{2j-1}, y_{2j}} (E/2))
\]

\[
\rightarrow t(H(E), H(E)) = H(E),
\]

\[
F_{y_{2i-1}, y_{2j-1}} (E) \geq t(F_{y_{2i-1}, y_{2i}} (E/2), F_{y_{2j-1}, y_{2j}} (E/2))
\]

\[
\rightarrow t(H(E), H(E)) = H(E)
\]

if \(i \to \infty\) and \(j \to \infty\), for all \(E > 0\).

Thus we have proved that \([y_n]_{n \in N}\) is a Cauchy sequence in \(X\) which means that there exists \(y^*\) in \(X\) such that \(\lim_{n \to \infty} y_n = y^*\).

To establish \(fy^* = gy^* = hy^* = ky^*\), we proceed as follows.

\[
fy^* = f \lim_{n \to \infty} kx_{2n} = \lim_{n \to \infty} fkx_{2n} = \lim_{n \to \infty} kfx_{2n}
\]

\[
= k \lim_{n \to \infty} fx_{2n} = ky^*
\]
\[ g y^* = \lim_{n \to \infty} g x_{2n+1} = \lim_{n \to \infty} g h x_{2n+1} = \lim_{n \to \infty} h g x_{2n+1} \]

\[ = h \lim_{n \to \infty} g x_{2n+1} = h y^* \]

since

\[
F_{f y^*} , g y^* (\varepsilon) \geq \min \{ F_{k y^*} , h y^* (\phi (\varepsilon)) , F_{f y^*} , k y^* (\phi (\varepsilon)) \}
\]

\[
F_{g y^*} , h y^* (\phi (\varepsilon)) , F_{f y^*} , h y^* (2\phi (\varepsilon)) , F_{g y^*} , k y^* (2\phi (\varepsilon)) \}
\]

\[ = \min \{ F_{f y^*} , g y^* (\phi (\varepsilon)) , F_{f y^*} , f y^* (\phi (\varepsilon)) , F_{g y^*} , g y^* (\phi (\varepsilon)) ,
F_{f y^*} , g y^* (2\phi (\varepsilon)) , F_{g y^*} , f y^* (2\phi (\varepsilon)) \}
\]

\[ = F_{f y^*} , g y^* (\phi (\varepsilon)). \]

Proceeding in the same way, we get

\[
F_{f y^*} , g y^* (\varepsilon) \geq F_{f y^*} , g y^* (\phi (\varepsilon)) \geq F_{f y^*} , g y^* (2\phi (\varepsilon)) \geq \ldots
\]

\[ \geq F_{f y^*} , g y^* (\phi^n (\varepsilon)) \rightarrow H (\varepsilon) \text{ for all } \varepsilon > 0,\]

we have proved that \( f y^* = g y^* = k y^* = h y^* \).

The point \( f y^* \) is a fixed point for the mappings \( f , g , h , k \). We shall show this for the mapping \( f \); the proof for the mappings \( g , h , k \) is analogous.

\[
F_{f f y^*} , f y^* (\varepsilon) = F_{f f y^*} , g y^* (\varepsilon)
\]

\[ \geq \min \{ F_{k f y^*} , h y^* (\phi (\varepsilon)) , F_{f f y^*} , k f y^* (\phi (\varepsilon)) \}
\]

(inequality continued on next page)
\[ F_{gy^*}, h_y^*(\frac{1}{2}(\epsilon)), F_{fy^*}, h_y^*(2\frac{1}{2}(\epsilon)), F_{gy^*}, k_fy^*(2\frac{1}{2}(\epsilon)) \]

\[ \geq \min \{ F_{fy^*}, gy^*(\frac{1}{2}(\epsilon)), H(\epsilon), H(\epsilon), F_{fy^*}, gy^*(2\frac{1}{2}(\epsilon)), \]

\[ F_{gy^*}, fy^*(\frac{1}{2}(\epsilon)) \} \]

\[ = F_{fy^*}, gy^*(\frac{1}{2}(\epsilon)) . \]

Proceeding in the same way, we get

\[ F_{fy^*}, y^*(\epsilon) \geq F_{fy^*}, gy^*(\frac{1}{2}(\epsilon)) \geq F_{fy^*}, gy^*(\frac{1}{2}^2(\epsilon)) \geq \ldots \]

\[ \geq F_{fy^*}, gy^*(\frac{1}{2}^n(\epsilon)) \rightarrow H(\epsilon) \text{ for } n \rightarrow \infty , \]

for all \( \epsilon > 0 \), which means that \( f_y^* \) is a common fixed point for the mappings \( f, g, h \) and \( k \).

If we suppose that there exists another common fixed point \( z \) in \( X \), we get

\[ F_{fy^*}, z(\epsilon) = F_{fy^*}, gz(\epsilon) \]

\[ \geq \min \{ F_{kfy^*}, hz(\frac{1}{2}(\epsilon)), F_{fy^*}, kfy^*(\frac{1}{2}(\epsilon)), F_{gz}, hz(\frac{1}{2}(\epsilon)) \}

\[ F_{kfy^*}, hz(2\frac{1}{2}(\epsilon)), F_{gz}, kfy^*(2\frac{1}{2}(\epsilon)) \} \]

\[ = \min \{ F_{fy^*}, gz(\frac{1}{2}(\epsilon)), F_{fy^*}, f_y^*(\frac{1}{2}(\epsilon)), F_{gz}, gz(\frac{1}{2}(\epsilon)) \}, \]

\[ F_{fy^*}, gz(2\frac{1}{2}(\epsilon)), F_{gz}, fy^*(2\frac{1}{2}(\epsilon)) \]

\[ = \min \{ F_{fy^*}, gz(\frac{1}{2}(\epsilon)), H(\epsilon), H(\epsilon), F_{fy^*}, gz(2\frac{1}{2}(\epsilon)) \} \]

\[ = F_{fy^*}, gz(\frac{1}{2}(\epsilon)) = F_{fy^*}, z(\frac{1}{2}(\epsilon)). \]
Proceeding in the same way, we get

\[ F_{f(y)\ast,z}(\varepsilon) \geq F_{f(y)\ast,z}(\psi(\varepsilon)) \geq F_{f(y)\ast,z}(\psi^2(\varepsilon)) \geq \ldots \]

\[ \geq F_{f(y)\ast,z}(\psi^n(\varepsilon)) \rightarrow H(\varepsilon) \text{ for } n \rightarrow \infty, \]

for all \( \varepsilon > 0 \) which means that \( f(y) \) is a unique common fixed point for the mapping \( f,g,h \) and \( k \).

This completes the proof of the theorem.

**Theorem 2**: If in Theorem 1 the \( T \)-norm \( t \) satisfies \( t(a,b) \geq t_m(a,b) = \max\{a+b-1,0\} \), for all \( a,b \in [0,1] \), then the condition that \( f,g,h \) and \( k \) are continuous can be replaced (weakened) by the condition that at least one of the mappings \( f,g,h \) and \( k \) is continuous.

**Proof**: The continuity of the mappings \( f,g,h \) and \( k \) was necessary only to prove that \( f(y) = g(y) = h(y) \).

Now, we assume that at least one of the mappings, say \( k \), is continuous. By the commutativity of \( f \) and \( k \) we have

\[ \lim_{n \to \infty} k^{2n+1} \xrightarrow{n \to \infty} ky = ky \]

Then

\[ F_{f(k^{2n+1},h^{2n+1})(\varepsilon)} \geq \min \left\{ F_{k^{2n+1},h^{2n+1}}(\psi(\varepsilon)), F_{f(k^{2n+1},h^{2n+1})}(\psi(\varepsilon)) \right\}, \]

(inequality continued on next page)
\[ F_{g \times 2n+1 \cdot h \times 2n+1} \phi(\varepsilon), F_{f \times 2n \cdot h \times 2n+1} (2\phi(\varepsilon)), \]
\[ F_{g \times 2n+1 \cdot k^2 \times 2n} (2\phi(\varepsilon)) \}

Letting \( n \to \infty \), we get (only if \( t \geq t_m \), see [135])

\[ F_{k_y^*, y^*(\varepsilon)} \geq \min \{ F_{k_y^*, y^*(\phi(\varepsilon))}, F_{k_y^*, y^*(\phi(\varepsilon))} \}
\]

\[ F_{y^*, y^*(2\phi(\varepsilon))}, F_{y^*, k_y^*(2\phi(\varepsilon))} \]

\[ = F_{k_y^*, y^*(\phi(\varepsilon))}, \text{ for all } \varepsilon > 0, \]

and so \( k_y^* = y^* \).

Further

\[ F_{f_y^*, g \times 2n+1} (\varepsilon) \geq \min \{ F_{k_y^*, h \times 2n+1} (\phi(\varepsilon)), F_{f_y^*, k_y^*(\phi(\varepsilon))} \}
\]

\[ F_{g \times 2n+1 \cdot h \times 2n+1} (\phi(\varepsilon)), F_{f_y^*, h \times 2n+1} (2\phi(\varepsilon)), \]

\[ F_{g \times 2n+1 \cdot k_y^*(2\phi(\varepsilon))} \}

for all \( \varepsilon > 0 \). When \( n \to \infty \) we have,

\[ F_{f_y^*, y^*(\varepsilon)} \geq \min \{ F_{y^*, y^*(\phi(\varepsilon))}, F_{f_y^*, y^*(\phi(\varepsilon))}, F_{y^*, y^*(\phi(\varepsilon))} \}
\]

\[ F_{f_y^*, y^*(2\phi(\varepsilon))}, F_{y^*, y^*(2\phi(\varepsilon))} \}

\[ = F_{f_y^*, y^*(\phi(\varepsilon))} \]

and so \( f_y^* = y^* \).
From the condition that \( f(x) \in h(x) \), it follows that there exists \( z^* \in X \) such that \( h z^* = y^* \). Then

\[
F_{y^*}, gz^*(\varepsilon) = F_{f y^*}, g z^*(\varepsilon)
\]

\[
\geq \min \{ F_{k y^*}, h z^*(\phi(\varepsilon)), F_{f y^*}, k y^*(\phi(\varepsilon)), F_{g z^*}, h z^*(\phi(\varepsilon)), F_{f y^*}, h z^*(2\phi(\varepsilon)), F_{g z^*}, k y^*(2\phi(\varepsilon)) \}
\]

\[
= \min \{ F_{y^*}, y^*(\phi(\varepsilon)), F_{y^*}, y^*(\phi(\varepsilon)), F_{g z^*}, y^*(\phi(\varepsilon)), F_{g z^*}, y^*(2\phi(\varepsilon)) \}
\]

\[
= F_{y^*}, g z^*(\phi(\varepsilon))
\]

for all \( \varepsilon > 0 \), which means that \( g z^* = y^* \). Since \( g \) and \( h \) commute, the equality \( g h z^* = g y^* = h g z^* \) implies that \( g y^* = h y^* \). Thus

\[
F_{y^*}, g y^*(\varepsilon) = F_{f y^*}, g y^*(\varepsilon)
\]

\[
\geq \min \{ F_{k y^*}, h y^*(\phi(\varepsilon)), F_{f y^*}, k y^*(\phi(\varepsilon)), F_{g y^*}, h y^*(\phi(\varepsilon)), F_{f y^*}, h y^*(2\phi(\varepsilon)), F_{g y^*}, k y^*(2\phi(\varepsilon)) \}
\]

\[
= \min \{ F_{y^*}, g y^*(\phi(\varepsilon)), F_{y^*}, y^*(\phi(\varepsilon)), F_{g y^*}, g y^*(\phi(\varepsilon)), F_{g y^*}, y^*(2\phi(\varepsilon)) \}
\]

\[
= F_{y^*}, g y^*(\phi(\varepsilon)).
\]

for all \( \varepsilon > 0 \), that is \( y^* = g y^* \).
So, we have proved that \( y^* \) is a fixed point for \( f, g, h \) and \( k \), i.e. \( y^* = fy^* = gy^* = ky^* = hy^* \).

If we suppose that \( f, g, h \) is continuous, the proof is similar, so it is omitted.

The other parts of the proof of this Theorem are the same as for Theorem 1.

10.3 If \((M, d)\) is a metric space, then the metric \( d \) induces a mapping \( F: \mathbb{M} \times \mathbb{M} \to \mathcal{A} \), where \( F(x, y), x, y \) in \( M \), is defined by \( F(x, y)(\varepsilon) = F_x, y(\varepsilon) = H(\varepsilon - d(x, y)), \varepsilon \in \mathbb{R}^+ \). Further if \( t: [0, 1] \times [0, 1] \to [0, 1] \) is defined by \( t(a, b) = \min \{a, b\} \), then \((M, F, t)\) is a Menger space. It is complete if it is complete with respect to metric \( d \). The space \((M, F, \min)\) so obtained will be called the induced Menger space.

Using some results stated in the preceding section (i.e. 10.2) and the next two lemmas we shall study the existence and uniqueness of common fixed points in metric spaces.

Throughout this section let \((M, d)\) be a complete metric space, \( h: M \to M, k: M \to M, f: M \to h(M) \) and \( g: M \to k(M) \) continuous mappings, such that \( f \) commutes with \( k \) and \( g \) commutes with \( h \). Further let the following inequality hold

\[
(10.3.1) \quad d(fx, gy) \leq \max \{\psi(d(kx, hy)), \psi(d(fx, kx)), \psi(d(gy, hy)), \frac{1}{2}\psi(d(fx, hy)), \frac{1}{2}\psi(d(gy, kx))\}
\]
for all \(x, y\) in \(M\), where \(\Psi\) is a function, \(\Psi: \mathbb{R}^+ \to \mathbb{R}^+\).

Let us introduce the following conditions on the mapping \(\Psi\):

(10.3.2) \(\Psi\) is increasing and \(\Psi^{-1}\) is right upper semicontinuous, \(\Psi(t) < t\) for all \(t\) in \(\mathbb{R}^+\)

\[
\lim_{t \to \infty} (t - \Psi(t)) = \infty.
\]

(10.3.3) \(\Psi\) is increasing, right upper semicontinuous, for any real number \(q \in (0, \infty)\) there exists a real number \(t(q) \in (0, \infty)\) such that \(t(q)\) is an upper bound of the set \(\{t \in (0, \infty) : t - \Psi^2(t) < q\}\), and \(\lim_{n \to \infty} \Psi^n (t(q)) = 0\).

Since \(f: M \to h(M)\) and \(g: M \to k(M)\), it is obvious that for every \(x_1\) in \(M\) we can form the sequence \(\{y_n\}_{n \in \mathbb{N}}\) by

(10.3.4) \(y_{2n} = f x_{2n} = h x_{2n+1}, y_{2n+1} = g x_{2n+1} = k x_{2n+2}\).

Now we prove the following lemmas.

**Lemma 1**: If \(\Psi\) satisfies condition (10.3.2), then the sequence \(\{y_n\}_{n \in \mathbb{N}}\) defined by (10.3.4) is bounded.

**Proof**: We shall show that

\[
\sup_{n \in \mathbb{N}} \sup_{1 \leq m, p < \ell} d(y_m, y_p) < \infty.
\]
which means that \( \{y_n\}_{n \in \mathbb{N}} \) is a bounded set. If \( m=2i \) and \( p=2j-1 \) we have

\[
d(y_{2i}, y_{2j-1}) = d(fy_{2i}, gx_{2j-1})
\]

\[
\leq \max \{\psi(d(y_{2i-1}, y_{2j-2})), \psi(d(y_{2i}, y_{2i-1})), \psi(d(y_{2j-1}, y_{2j-2})), \frac{1}{2}\psi(d(y_{2i}, y_{2j-2})), \frac{1}{2}\psi(d(y_{2j-1}, y_{2i-1}))\}
\]

since \( \psi(d(y_{2i}, y_{2j-2})) \leq \{\psi(d(y_{2i}, y_{2i-1}))+\psi(d(y_{2i-1}, y_{2j-2}))\} \)

\[
\psi(d(y_{2j-1}, y_{2i-1})) \leq \psi(d(y_{2j-1}, y_{2j-2}))+\psi(d(y_{2j-2}, y_{2i-1}))
\]

and

\[
\max \{\psi(d(y_{2i}, y_{2i-1})), \psi(d(y_{2j-1}, y_{2i-2}))\}
\]

\[
< \max \{d(y_{2i}, y_{2i-1}), d(y_{2j-1}, y_{2j-2})\}
\]

\[
< d(y_{2i}, y_{2j-1}) \quad \text{for all } j > i.
\]

Thus, we have

\[
d(y_{2i}, y_{2j-1}) \leq \psi(d(y_{2i-1}, y_{2j-2}))
\]

similarly, we have

\[
\psi(d(y_{2i-1}, y_{2j-2})) \leq \psi^2 d(y_{2i-2}, y_{2j-3}) \leq \psi(d(y_{2i-2}, y_{2j-3}))
\]

as \( \psi(t) < t \).

and so,

\[
d(y_{2i}, y_{2j-1}) \leq \psi d(y_{2j-1}, y_{2j-2}) \leq \psi d(y_{2i-2}, y_{2j-3})
\]
that is

\[(10.3.5) \sup_{r \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq \psi \sup_{r-1 \leq i, j \leq n-1} d(y_{2i}, y_{2j-1}),\]

\[\leq \psi \sup_{r-1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}).\]

We know that

\[(10.3.6) \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq d(y_1, y_2) + \sup_{2 \leq i, j \leq n} d(y_{2i}, y_{2j-1}).\]

Putting \(r=2\) in (10.3.5) we get

\[\sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq \psi \sup_{1 \leq i, j \leq n-1} d(y_{2i}, y_{2j-1})\]

\[\leq \psi \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}).\]

Using the last inequality and (10.3.6) we have

\[\sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq d(y_1, y_2) + \psi \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}).\]

If we suppose that

\[\sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) = \infty \text{ for } n \to \infty,\]

then from the condition (10.3.2) and the last inequality we get

\[\infty = \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) - \psi \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq d(y_1, y_2),\]

which is a contradiction. So, we have proved that
\[
\sup_{n \in \mathbb{N}} \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) < \infty.
\]

Since
\[
\sup_{n \in \mathbb{N}} \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j}) \leq \sup_{n \in \mathbb{N}} \sup_{1 \leq i \leq n} d(y_{2i}, y_{2i+1}) + \sup_{n \in \mathbb{N}} \sup_{1 \leq i \leq n} d(y_{2i+1}, y_{2j}) < \infty + \infty = \infty,
\]
and
\[
\sup_{n \in \mathbb{N}} \sup_{1 \leq i, j \leq n} d(y_{2i-1}, y_{2j-1}) \leq \sup_{n \in \mathbb{N}} \sup_{1 \leq i \leq n} d(y_{2i-1}, y_{2i}) + \sup_{n \in \mathbb{N}} \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) < \infty + \infty = \infty,
\]
we have proved that \( \{y_n\}_{n \in \mathbb{N}} \) is bounded.

**Lemma 2:** Let \((M,d)\) be a metric space and \((M,F,\min)\) be an induced probabilistic metric space. If the inequality (10.3.1) holds in \((M,d)\), then the inequality (10.2.1) holds in \((M,F,\min)\) (\(\psi\) is an increasing and \(\hat{\psi} = \psi^{-1}\)).

**Proof:** The mapping \(\psi\) increasing and this implies the existence of the increasing mapping \(\psi^{-1} = \hat{\psi}\). To establish that condition (10.3.1) implies condition (10.2.1) we prove the next implication:

\[
d(fx, gy) \leq \max\{\psi(d(kx, hy)), \psi(d(fx, kx)), \psi(d(gy, hy)),
\]
\[
\quad \frac{1}{2} \psi(d(fx, hy)), \frac{1}{2} \psi(d(gy, kx))\}.
\]
\[ F_{fx,gy}(\varepsilon) \geq \min \{ F_{kx,hy}(\psi^{-1}(\varepsilon)), F_{fx,kx}(\psi^{-1}(\varepsilon)), F_{gy,hy}(\psi^{-1}(\varepsilon)), F_{kx,hy}(2\psi^{-1}(\varepsilon)), F_{gy,kx}(2\psi^{-1}(\varepsilon)) \} \]

Now, we proceed as follows:

\[ d(fx,gy) \leq \max \{ \psi(d(kx,hy)), \psi(d(fx,kx)), \psi(d(gy,hy)), \psi(d(fx,gy)), \psi(d(gy,kx)) \} \]

\[ \Rightarrow \varepsilon - d(fx,gy) \geq \max \{ \psi(d(kx,hy)), \psi(d(fx,kx)), \psi(d(gy,hy)), \psi(d(fx,gy)), \psi(d(gy,kx)) \} \]

\[ \Rightarrow H(\varepsilon - d(fx,gy)) \geq H(\max \{ \psi(d(kx,hy)), \psi(d(fx,kx)), \psi(d(gy,hy)), \psi(d(fx,gy)), \psi(d(gy,kx)) \}) \]

\[ \Rightarrow H(\varepsilon - d(fx,gy)) \geq H(\psi^{-1}(\varepsilon)) \max \{ \psi(d(kx,hy)), \psi(d(fx,kx)), \psi(d(gy,hy)), \psi(d(fx,gy)), \psi(d(gy,kx)) \} \]

\[ \Rightarrow F_{fx,gy}(\varepsilon) \geq \max \{ F_{kx,hy}(\psi^{-1}(\varepsilon)), F_{fx,kx}(\psi^{-1}(\varepsilon)), F_{gy,hy}(\psi^{-1}(\varepsilon)), F_{kx,hy}(2\psi^{-1}(\varepsilon)), F_{gy,kx}(2\psi^{-1}(\varepsilon)) \} \]

Thus we have obtained (10.3.1) implies (10.2.1).

This completes the proof of Lemma 2.

We shall use Theorem 1 and the last two lemmas for the proof of the next Theorem in which we consider the existence and uniqueness of common fixed points for the selfmappings of the metric space \((H,d)\).
THEOREM 3: Let \((M,d)\) be a complete metric space \(h:M \to M\), \(k:M \to M\), \(f:M \to h(M)\) and \(g:M \to k(M)\), where one of these mappings is continuous, \(f\) commutes with \(k\) and \(g\) commutes with \(h\). If inequality (10.3.1) holds, where \(\psi: \mathbb{R}^+ \to \mathbb{R}^+\) is a function for which condition (10.3.2) holds, then there exists a unique common fixed point for the mappings \(f,g,h\) and \(k\).

Proof: If \(\phi = \psi^{-1}\) then the mapping \(\phi\) satisfies the condition of Theorem 1 of Ding[40]. Lemma 1 and Lemma 2 provide the other conditions of Theorem 1, so this theorem is proved.

THEOREM 4: If the mapping \(\psi: [0, \infty) \to [0, \infty)\) satisfies condition (10.3.2), then the conclusion of the last theorem still holds.

Proof: The proof has four steps. We shall prove that

1. \(\{y_n\}\) is a Cauchy sequence and \(\lim_{n \to \infty} y_n = y^*\),
2. \(fy^* = gy^* = hy^* = ky^* = u\),
3. \(u\) is a common fixed point for \(f, g, h\) and \(k\),
4. \(u\) is a unique common fixed point.

1. We shall show that

\[
\lim_{\substack{m \to \infty \\ p \to \infty}} d(y_m, y_p) = 0.
\]
If \( m=2 \) and \( p=2j-1 \) we have

\[
(10.3.7) \quad d(y_{2i}, y_{2j-1}) \leq \psi d(y_{2i-1}, y_{2j-2}) \leq \psi^2 d(y_{2i-2}, y_{2j-3})
\]

That is

\[
(10.3.8) \quad \sup_{r \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq \psi^2 \sup_{r-1 \leq i, j \leq n-1} d(y_{2i}, y_{2j-1}) \leq \psi^2 \sup_{r-1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}).
\]

From (10.3.8) and (10.3.6) (letting \( r=2 \)) we get

\[
\sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq d(y_{1}, y_{2}) + \psi^2 \sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}).
\]

Letting the real number \( q=d(y_{1}, y_{2}) \) and using condition (10.3.2) it follows, from the last inequality, that there exists a real number \( t(q) \in (0, \infty) \) such that

\[
\sup_{1 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq t(q)
\]

Taking \( r=2 \) in (10.3.7) again, and considering the inequality we have

\[
\sup_{2 \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq \psi^2(t(q)).
\]

Repeating this procedure, for any positive integers \( r, n \) we obtain

\[
(10.3.9) \quad \sup_{r \leq i, j \leq n} d(y_{2i}, y_{2j-1}) \leq \psi^{2r-2}(t(q)).
\]
Letting $r \to \infty$ (hence $n \to \infty$) in (10.3.9), from condition (10.3.3) we get

$$0 \leq \lim_{r \to \infty} \sup_{r < i, j \leq n} d(y_{2i}, y_{2j-1}) \leq \lim_{r \to \infty} \frac{2r-2}{t(q)} = 0.$$

If $m$ and $p$ are both even or both are odd, we proceed as follows

$$\lim_{r \to \infty} \sup_{r < i, j \leq n} d(y_{2i}, y_{2j}) \leq \lim_{r \to \infty} \sup_{r < i, j \leq n} d(y_{2i}, y_{2i+1}) + \lim_{r \to \infty} \sup_{r < i, j \leq n} d(y_{2i+1}, y_{2j}) = 0 + 0 = 0.$$ 

and

$$\lim_{r \to \infty} \sup_{r < i, j \leq n} d(y_{2i-1}, y_{2j-1}) \leq \lim_{r \to \infty} \sup_{r < i, j \leq n} d(y_{2i-1}, y_{2i}) + \lim_{r \to \infty} \sup_{r < i, j \leq n} d(y_{2i}, y_{2j-1}) = 0 + 0 = 0.$$ 

This shows that $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space $(M, d)$ and so there exists $y^*$ in $X$ such that $\lim_{n \to \infty} y_n = y^*$.

2. From the continuity of the mapping $f, g, h$ and $k$ in the same way as in Theorem 1, we get that $fy^* = ky^*$ and $gy^* = hy^*$. To prove $fy^* = gy^*$ we proceed as follows:

$$d(fy^*, gy^*) \leq \max\{\Psi(d(ky^*, hy^*)), \Psi(d(fy^*, ky^*)), \Psi(d(gy^*, hy^*))\},$$

$$\frac{1}{2}\Psi(d(fy^*, hy^*)), \frac{1}{2}\Psi(d(gy^*, ky^*))\}$$
\[ = \max\{\psi(d(fy^*, gy^*)), \psi(d(fy^*, fy^*)), \psi(d(gy^*, gy^*)), \]
\[ \frac{1}{2} \psi(d(fy^*, gy^*)), \frac{1}{2} \psi(d(gy^*, fy^*))\} \]
\[ = \psi(d(fy^*, gy^*)) \]

Similarly, we have
\[ d(fy^*, gy^*) \leq \psi(d(fy^*, gy^*)) \leq \psi^2(d(fy^*, gy^*)) \]

From (10.3.3) if \( q=0 \) there exists \( t(q) \in [0, \infty) \) such that
\[ d(fy^*, gy^*) \leq t(0) \Rightarrow \psi^n d(fy^*, gy^*) \leq \psi^n(t(0)) \]

Since
\[ d(fy^*, gy^*) \leq \psi^n d(fy^*, gy^*) \leq \psi^n(t(0)) \rightarrow 0 \text{ for } n \rightarrow \infty \]

we have that \( fy^* = gy^* \), that \( fy^* = gy^* = hy^* = ky^* = u \).

3. We shall prove that \( fu = u \) (similarly, we can prove the same for the mappings \( g, h \) and \( k \)).

\[ d(fu, u) = d(fu, gy^*) \leq \max\{\psi(d(ku, hy^*)), \psi(d(fu, ku)), \]
\[ \psi(d(gy^*, hy^*)), \frac{1}{2} \psi(d(ku, hy^*)), \]
\[ \frac{1}{2} \psi(d(gy^*, ku))\} \]
\[ = \max\{\psi(d(fu, gy^*)), \psi(d(fu, fu)), \]
\[ \psi(d(gy^*, gy^*)), \frac{1}{2} \psi(d(fu, gy^*)), \]
\[ \frac{1}{2} \psi(d(gy^*, fu))\} \]
\[ = \psi(d(fu, gy^*)) = \psi(d(fu, u)) \]
Proceeding in the same way, we obtain
\[ d(fu, u) \leq \psi(d(fu, u)) \leq \psi^2(d(fu, u)) \leq \ldots \leq \psi^n(d(fu, u)). \]

Putting \( q = 0 \) from (10.3.3) there exists \( t(0) \in [0, \infty) \) such that
\[ d(fu, u) \leq t(0) \Rightarrow \psi^n(d(fu, u)) \leq \psi^n(t(0)). \]

Then \( d(fu, u) \leq \psi^n(d(fu, u)) \leq \psi^n(t(0)) \to 0, \) as \( n \to \infty. \)

Therefore \( fu = u. \)

4. If we suppose that the point \( v \) in \( X \) is also a common fixed point for the mappings \( f, g, h \) and \( k \) we get
\[
\begin{align*}
d(u, v) &= d(fu, gv) \\
&\leq \max\{\psi(d(ku, hv)), \psi(d(fu, ku)), \psi(d(gv, hv)), \\
&\quad \frac{1}{2} \psi(d(fu, hv)), \frac{1}{2} \psi(d(gv, ku))\}
\end{align*}
\]

\[
= \max\{\psi(d(u, v)), \psi(d(u, u)), \psi(d(v, v)), \\
&\quad \frac{1}{2} \psi(d(u, v)), \frac{1}{2} \psi(d(v, u))\}
\]

\[ = \psi(d(u, v)) \]

Proceeding in the same way, we obtain
\[ d(u, v) \leq \psi(d(u, v)) \leq \psi^2(d(u, v)) \leq \ldots \leq \psi^n(d(u, v)) \]
so that if \( q = 0 \), then there exists \( t(0) \) such that
\[ d(u, v) \leq t(0) \Rightarrow d(u, v) \leq \psi^n(d(u, v)) \leq \psi^n(t(0)) \to 0, \) as \( n \to \infty. \)

implies \( u = v. \)
This completes the proof of the theorem.

In the following examples we shall show that we can not omit any of the conditions of Theorem 3 (and Theorem 2).

Example 1: Let $M=[0,1]$, $f(x)=g(x)=h(x)=0$, $k(x)=1-x$, $\psi=(3/4)t$. All the conditions of Theorem 3 are satisfied except for the commutativity of $f$ and $k$, and $f,g,h$ and $k$ have no common fixed point.

Example 2: If $M=[0,1]$, $f(x)=(2/5)x$, $g(x)=$ \[ \begin{cases} 2/5 & x=0 \\ (2/5)x & x \neq 0 \end{cases} \]

$h(x) = \begin{cases} 1 & x=0 \\ x & x \neq 0 \end{cases}$, $k(x)=x$, $\psi(t) = (4/9)t$, then all the conditions of Theorem 3 are satisfied except $f(M) \subset h(M)$ ($f(M) = [0,2/5]$, $h(M) = (0,1]$) and $f,g,h$ and $k$ have no common fixed point.

Example 3: If $M=[0,1]$, $f(x)=g(x)=$ \[ \begin{cases} 2/5 & x=0 \\ 2/5x & x \neq 0 \end{cases} \]

$k(x)=h(x)=$ \[ \begin{cases} 4/5 & x=0 \\ (4/5)x & x \neq 0 \end{cases} \]

$\psi(t) = (2/5)t$, then all the conditions of Theorem 3 are satisfied except that $f,g,h$ and $k$ are all discontinuous, and $f,g,h$ and $k$ have no common fixed point.

* * *