CHAPTER V

PURE AND COPURE MODULES
The case of purity related to relative divisibility by elements of ring was first dealt with in the case of arbitrary ring by Hattori [90] in 1960. He proved that a sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) of \( R \)-modules is pure if it stays exact after tensoring with cyclic right modules of the type \( \frac{R}{rR} \oplus \text{re}R \).

i.e. \( 0 \rightarrow A \oplus \frac{R}{rR} \rightarrow B \oplus \frac{R}{rR} \rightarrow C \oplus \frac{R}{rR} \rightarrow 0 \)

is an exact sequence of \( R \)-modules.

In this chapter after the brief survey of purity developed by Walker, we study particular purities with respect to finitely generated modules and cocyclic modules and following results are proved:

(i) \( \mathfrak{F} \mathfrak{J} \)- pure module has been shown to be equivalent to local summand of Bergman [25] i.e. \( L \) is a direct summand of \( A+L \) whenever \( A \) is a finitely generated submodule of \( M \).

(ii) Flatness relative to \( \mathfrak{F} \mathfrak{J} \)- pure module has been shown to be equivalent to locally projective module.
(iii) Conditions on the ring are determined which are necessary and sufficient to render all short exact sequences copure.

The natural generalization of the characterization of purity (considered in Chapter 4) will be to replace a class of modules whose relative projectivity is being considered by an arbitrary class of modules $\mathcal{P}$.

**Definition 1:**

An exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is called $\mathcal{P}$-pure if objects of $\mathcal{P}$ are projective relative to $E$. Equivalently $E$ is $\mathcal{P}$-pure if given a commutative square

$$
\begin{array}{c}
\text{K} \\
\downarrow \\
\text{A}
\end{array} \quad \begin{array}{c}
\quad \rightarrow \quad F \\
\quad \downarrow \\
\quad \rightarrow \quad \text{B}
\end{array}
$$

there exists $F \rightarrow A$ making the upper triangle commutative where $F$ is free, $K \leq F$ and $F/K \in \mathcal{P}$.

Dulizing the above definition, we have an exact sequence $\mathcal{P}$-copure if objects of $\mathcal{P}$ are injective relative to $E$.

In $\mathcal{P}$-pure and $\mathcal{G}$-copure, $\mathcal{P}$ can be taken as a class of finitely generated groups and $\mathcal{G}$ can be taken as a class of cocyclic groups.
Definition 2:

Given a class of modules \( \mathcal{P}(\mathcal{A}) \) the class of exact sequences with respect to which objects of \( \mathcal{P}(\mathcal{A}) \) are relatively projective (injective) is called the class of \( \mathcal{P} \)-pure (\( \mathcal{A} \)-copure) sequences and is denoted by \( \Theta^{-1}(\mathcal{P}) \) (\( i^{-1}(\mathcal{A}) \))

i.e. \( \Theta^{-1}(\mathcal{P}) = \{ \mathcal{E} : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow O/P \text{ is } \mathcal{E} \text{-pure } \forall P \in \mathcal{P} \} \)

Given any class of exact sequence \( \mathcal{E} \), objects projective (injective) relative to \( \mathcal{E} \) are called \( \mathcal{E} \)-projective(\( \mathcal{E} \)-injective) and their class is denoted by \( \Theta(\mathcal{E})(i(\mathcal{E})) \) i.e. \( \Theta(\mathcal{E}) = \{ P \in \mathcal{P} \text{ is } \mathcal{E} \text{-pure} \} \).

A class of exact sequences \( \mathcal{E} \) is called projectively (injectively) closed if \( \mathcal{E} = \Theta^{-1}(\Theta(\mathcal{E})) \) [\( \mathcal{E} = i^{-1}(i(\mathcal{E})) \)] and this happens exactly when \( \mathcal{E} = \Theta^{-1}(\mathcal{P}) \) [\( \mathcal{E} = i^{-1}(\mathcal{A}) \)] for some class of modules \( \mathcal{P}(\mathcal{A}) \).

Any class of the form \( \Theta^{-1}(\mathcal{P}) \) (\( i^{-1}(\mathcal{A}) \)) of exact sequences is a proper class \( \mathcal{E} \) of exact sequences.

Definition 3 – Walker [147]:

Given a class of modules \( \mathcal{I}(\mathcal{J}) \), a sequence is called \( \mathcal{I} \)-pure (\( \mathcal{J} \)-copure) if \( A \) is a direct summand of \( D \) whenever \( A \leq D \leq B \) and \( D/A \in \mathcal{I} \) (\( A/S \) is a direct summand of \( B/S \) whenever \( S \leq A \) and \( A/S \in \mathcal{J} \)).
Remark 1:

Walker proved that the class of $\mathcal{I}$-pure ($\mathcal{I}$-copure) sequences form a proper class whenever $\mathcal{I}$ ($\mathcal{I}$) is closed under homomorphic images (submodules). In fact for any $\mathcal{I}$ ($\mathcal{I}$) if $E \in \Theta^{-1}(\mathcal{I})$ ($i^{-1}(\mathcal{I})$) then $E$ is $\mathcal{I}$-pure ($\mathcal{I}$-copure) in the above sense. And if $\mathcal{I}$ ($\mathcal{I}$) is closed under factors (submodules) then any $\mathcal{I}$-pure ($\mathcal{I}$-copure) sequence $E \in \Theta^{-1}(\mathcal{I})$ ($i^{-1}(\mathcal{I})$) and hence in this case Walker's $\mathcal{I}$-purity ($\mathcal{I}$-copurity) coincides with the earlier notion.

In view of Proposition 2.7* we have -

Proposition 5.9:

If $\mathcal{I}$ ($\mathcal{I}$) is closed under factors (submodules) then a sequence $E: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is $\mathcal{I}$-pure ($\mathcal{I}$-copure) if and only if given $C' (\subseteq C) \in \mathcal{I}$, there exist $B' \subseteq B$ such that $B' \cap C'$ and $A \cap B' = 0$. (Given $\frac{A}{A'} \in \mathcal{I}$, there exists $B' \subseteq B$ such that $B' + A = B$ and $\frac{A \cap B}{A'}$.)

In case of abelian groups Walker's theory of $\mathcal{I}$-purity and $\mathcal{I}$-copurity applies because the classes of finitely generated groups are closed under factors and class of cocyclic groups is closed under submodules.

Definition 4:

$L$ be any submodule of left $R$-modules $M$, $L$ is said to be pure submodule of $M$ if for all $m \in M$, $rM \subseteq M$ implies there exists $l \in L$ such that $r = rl$. 
The sequence \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) will be called as a pure exact sequence if \( L \) is pure in \( M \).

An \( R \)-module \( M \) will be called absolutely pure (respectively regular, flat) if every short exact sequence with \( M \) in the first (respectively second, third) place is pure exact.

i.e. A module \( M \) will be called absolutely pure if it is pure in every overmodule.

\( \mathcal{F}_g \) denotes the class of all finitely generated modules, and this class of modules is closed under factors.

Proposition 6.1:

The following are equivalent for an exact sequence \( E: 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \).

(i) The sequence \( E \) is \( \mathcal{F}_g \)-pure.

(ii) \( L \) is a local Summand of Bergman [25] i.e. \( L \) is a direct summand of \( A+L \) whenever \( A \) is a finitely generated submodule of \( M \).

(iii) Given any finitely generated submodule \( M' \) of \( M \), there exists \( g:M' \rightarrow L \) such that \( g|_{L\cap M'} = 1_{L\cap M'} \).

Proof:

(i) \( \Rightarrow \) (ii)
Given that $A$ is finitely generated, this implies that $rac{A+L}{L} \cong \frac{A}{A \cap L}$ is finitely generated $\Rightarrow$ there exists $f$ such that $\Theta.f = h$ $\Rightarrow$ there exists $f'$ such that $f'.i = 1_L$.

Therefore $L$ is a direct summand of $A+L$.

(ii) $\Rightarrow$ (i)

Given $h:F \rightarrow M/L$. Now take $F_1 = \text{Im}h$
By taking $K = \Theta^{-1}(F_1)$. If $x_1, \ldots, x_n$ generates $F$, then $K = L + \sum_{i=1}^{n} Rx_i$ and hence by given condition $L$ is a direct summand of $K$ and therefore there exists $f : F_1 \to M$ such that $\Theta(f.L) = g.L = h$ and hence sequence is $\mathcal{F}_Q$-pure.

(i) $\Rightarrow$ (iii)

Consider the following diagram

\[
\begin{array}{ccccc}
0 & \to & M' \cap L & \to & M' & \to & M'/M'\cap L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & L & \to & M & \to & M/L & \to & 0 \\
\end{array}
\]

If $M'$ is finitely generated then $M'/M'\cap L$ is also finitely generated, and there exists $f$ such that $\Theta.f = h$ and therefore there exists $g : M' \to L$ such that $g|_{M'\cap L} = 1_{M'\cap L}$.

Hence the condition (iii) holds.

(iii) $\Rightarrow$ (i)

Given $h : F \to M/L$. We take the inverse image.
Now $P = \varnothing^{-1}(F_1) = M' + L$ where $M'$ is a finitely generated module and $F_1 \triangleleft \frac{M' + L}{L}$. Given that there exists $g : M' \to L$ such that $g|_{M' \cap L} = 1_{M' \cap L}$ Define $g' : P \to L$ by $g'(m' + l) = g(m') + l$. If $m' + l = 0$, then $m' \in M' \cap L$ and $g(m') = m'$ and therefore $g(m' + l) = 0$. Thus $g'$ is well defined and $g'(l) = 0 + l = l$ and hence there exists $f : F_1 \to M$ such that $\varnothing.f = h$ and hence the sequence is $\varnothing$-pure.

**Definition 5 [155]**

A module $M$ is called locally projective if given $g : M \to B$ and a finitely generated submodule $F$ of $M$, there exists $g' : M \to A$ such that $\varnothing.g'|F = g|F$. 
All locally projective modules are flat and the class lies strictly between flat and projective modules [155].

Examples of locally projective module [155].

(i) A pure submodule of a locally projective module is locally projective.

(ii) If $R$ is commutative and $M_R$, $N_R$ are locally projective, then $M \otimes_R N$ is locally projective (as an $R$-module) and each localization $M_p$ with respect to a prime ideal $p$ of $R$ is a locally projective $R_p$-module.

Theorem 1:

The following conditions are equivalent for any module $M$:

(i) $M$ is $\delta_f$ - pure flat.

(ii) $M$ is locally projective.

(iii) Given any sequence $O \rightarrow L \rightarrow N \overset{\delta}{\rightarrow} M \rightarrow O$ and any finitely generated submodule $F$ of $M$, there exists $k:M \rightarrow N$ such that $\delta_k | F = 1_F$. 
Proof:

(i) $\Rightarrow$ (ii)

Given the finitely generated submodule $F$ of $M$ and $g:M \rightarrow B$, $f:A \rightarrow B$ be $R$-homomorphisms. Now, consider the following diagram.

![Diagram](image)

The above diagram can be extended by pullback. By hypothesis the sequence $K \rightarrow M \rightarrow 0$ is $\phi_{g}$-pure and therefore exists $h_{F}:F \rightarrow K$ such that

$$f_{1}h_{F} = i_{F}$$

There exists a unique $h_{F}$ because $K$ is a pullback by proposition 2.2.

Now, take the directed set of all finitely generated submodules of $M$, then $\{(h_{F},K)\}$ form a directed compatible family by uniqueness of $h_{F}$ and $h_{F}$ induces a unique $h:M \rightarrow K$ the direct limit map such that $h.i_{F} = h_{F}$.
for all finitely generated submodule $F$ of $M$. Now, $f.g_1.h.i_F = g.f_1.h.i_F = g.f_1.h_F = g.i_F$ for all $F$.

Therefore $M$ is locally projective.

(ii) $\Rightarrow$ (iii)

Given a finitely generated submodule $F$ of $M$, there exists $g':M \rightarrow N$ such that $\delta.g'.i_F = i_F$ and thus the (iii) condition holds.

(iii) $\Rightarrow$ (i)
Suppose \( 0 \to L \to N \to M \to O \) is an exact sequence and \( h: F \to M \), take \( F_1 = \text{Im} h \).

Given that for any finitely generated submodule \( F \) of \( M \), there exists \( k:M \to N \) such that \( \delta . k. i_{F_1} = i_{F_1} \) and therefore

\[
\delta (k . i_{F_1} . \eta) = i_{F_1} . \eta = h
\]

and thus the sequence is \( F \)-pure and \( M \) is \( F \)-pure flat.

**Proposition 6.2:**

The following conditions are equivalent for the sequence \( 0 \to A \to B \to C \to O \).

(i) The sequence \( 0 \to A \to B \to C \to O \) is copure relative to all cocyclic modules.

(ii) Given that \( O \neq a \in A \) and \( A' \leq A \) maximal such that \( a \in A' \) there exists \( B' \leq B \) maximal among submodules of \( B \) excluding \( a \) and for which \( A' = B' \cap A \) and \( B = B' + A \).

**Proof:**

(i) \( \Rightarrow \) (ii)

Every submodule of \( A/A' \) is of the form \( L/A' \) with \( A' \leq L \) and therefore \( a \in L \). Hence \( A/A' \) contains the submodule \( (Ra+A')/A' = R(a+M') \) and thus \( A/A' \) is cocyclic. Hence there is \( g:B \to A'/A \) such that \( g|A = \delta \).
Now take $B' = \ker g$. Then $\ker g|_A = \ker \delta = A'$ and therefore

$$A' = B' \cap A$$

Since $\delta(a) = g(a)$ generates the simple socle of $A/A'$.

Take any $a \neq 0$ such that

$$0 \neq f(a) \in \operatorname{Soc}(A/A')$$

Then $a \not\in \ker g$ and if $\ker g \subseteq L$ then $\operatorname{Soc}(A/A') = \operatorname{rg}(a) = (Ra + \ker g)/\ker g$ and $a \in L$. Thus $\ker g$ maximally excludes $a$ or $B'$ is maximal such that

$$a \in \ker A'$$

since

$$\frac{A}{A'} = \frac{A}{B' \cap A} \cap \frac{A + B'}{B'} \subseteq \frac{B}{B'}$$

and

$$\frac{B}{B'} = \operatorname{img} \subseteq \frac{A}{A'}$$

Therefore,
\[
\frac{B' + A}{B'} = \frac{B}{B'}
\]

and thus \( B = B' + A \).

(ii) \( \Rightarrow \) (i)

Given the \( \delta : A \to I \), a cocyclic module, now take any element \( a(\neq 0) \) such that \( 0 \neq \delta(a) \in \text{Soc} \, I \).

The \( I \uplus A/A' \) where \( A' \) is maximal with respect to excluding \( a \). It is given that there exists \( B' \) such that

\[
A' = B' \cap A \quad \text{and} \quad A + B' = B.
\]

Now,

\[
\frac{A}{A'} = \frac{A}{B' \cap A} \uplus \frac{A + B'}{B'} = \frac{B}{B'}
\]

Hence \( B \rightarrow \frac{B}{B'} \uplus \frac{A}{A'} = I \)

is the required map.

**Definition 6:**

A ring \( R \) is said to be (left) right \( V \)-ring if every simple \( R \)-module is injective.

**Proposition 6.3:**

The following conditions are equivalent for a ring \( R \):

1. (i) Every short exact sequence is copure relative to cocyclics.
2. (ii) \( R \) is \( V \)-ring.
Proof:

(i) \Rightarrow (ii)

By hypothesis every sequence is copure, then for given simple module $S$,

\[ 0 \rightarrow S \rightarrow E(S) \rightarrow \frac{E(S)}{S} \rightarrow 0 \]

is copure and thus the sequence splits as $S$ is cocyclic. Therefore every simple module is injective and hence $R$ is a $V$-ring.

(ii) \Rightarrow (i)

If $R$ is a $V$-ring, thus every cocyclic module is simple injective module and therefore every sequence is copure relative to them.