CHAPTER IV

RELATIVE PROJECTIVITY
The notion of purity plays a fundamental role in the theory of abelian groups. The connection of purity with tensor products was first explored by Cohn [40] in 1959. He proved that a module is flat if and only if it renders every short exact sequence pure in which it stays in the third position.

L. Fuch [68] in 1970 find that A is pure in B if and only if solvability in B of a finite consistent system of linear equations in A implies it's solvability in A.

In this chapter our aim is to study purity of modules by means of relative projectivity, cyclic modules and q*-rings. The main results established are following:

(i) Purity relative to a given column finite matrix \((r_{ij})\) of arbitrary sets has been characterized.

(ii) A submodule \(K\) of \(M\) is \(R/I\) - pure if and only if given \(m\in M\) such that \(\text{Im} \subseteq K\), there exists \(m'\in M\) such that \(\text{Im}' = 0\) and \(m - m' \in K\).

(iii) A ring \(R\) is a q*-ring if and only if given any \(r\in R\) and any left ideal \(I\) there exists \(i_r \in I\) such that \(I(r - i_r) \subseteq I\).
(iv) If $I$ is a two sided ideal of a left $q^*$-ring $R$ then cyclic left module $R/I$ is quasi-projective.

**Definition 1:**

An $R$-module $M$ is projective relative to an $R$-module $U$ ("$M$ is $U$-projective") in case the functor $\text{Hom}_R(M, \cdot)$ preserves the exactness of all short exact sequences

$$0 \longrightarrow A \longrightarrow U \longrightarrow B \longrightarrow 0$$

**Linear equations:**

The concept of purity arose in the theory of abelian groups in connection with solvability of linear equations with integer coefficient in groups. It comes out that arbitrary systems of linear equations $\sum n_{ij} x_j = y_i \ (i \in I, \ j \in J)$ which are consistent are solvable if and only if equations with one unknown $nx=y$ are solvable i.e. the group is divisible.

In the above system of equations, for each $i$ the sum is over a finite number of $j$'s only and therefore for any $i$, $n_{ij}$ must be equal to zero for almost all $j$.

It comes out that a subgroup $A$ of a group $B$, is a direct summand of $B$ if and only if solvability in $B$ of any system of linear equations in $A$ implies its solvability in $A$. 
Definition 2:

A subgroup A is called pure in B if solvability in B of the equation nx = y, y ∈ A implies its solvability in A i.e. if given nb = y, b ∈ B, y ∈ A there exists a ∈ A such that na = y. Clearly A is pure in B if and only if $A \cap nB = nA$ for all $n \in \mathbb{Z}$.

Purity can be defined in terms of relative solvability in the submodule of a finite consistent system of linear equations or just one equation. The concept of purity becomes clear when it comes out that an equation of the type $rx = a$ in A is a map $g : Rr \rightarrow A$ with $g(r) = a$ and its consistency is equivalent to $g$ being a homomorphism. This system is solvable in B if and only if there exists $h : R \rightarrow B$ such that the square

\[ \begin{array}{ccc}
Rr & \xrightarrow{g} & R \\
\downarrow & \searrow & \downarrow h \\
A & \xleftarrow{f} & B
\end{array} \]

commutes. And solvability of it in A values to the existence of $f : R \rightarrow A$ such that $f|_{Rr} = g$. Hence A is pure in B in the sense of solvability of single equation, i.e. relative divisibility if and only if given such a commutative square, $f : R \rightarrow A$ exists such that the upper triangle commutes.
For arbitrary systems of linear equations we have to replace $R$ by free module $F$ over the set of unknowns and $Rr$ by the submodule of $F$ generated by the left hand side expressions of the system of equations. Thus $\Sigma r_{ij}x_j = a_i$ $(i \in I, j \in J)$ is relatively solvable in $A$ if and only if given a commutative square

$$
\begin{array}{c}
R' \subseteq \Theta R \\
g \downarrow \quad f \quad \downarrow h \\
A \quad \leftarrow \quad R \\
\end{array}
$$

Where $R'$ is generated by $\{ \sum_{j} r_{ij}x_j \}_{i \in I}$ there exists $f : \Theta R \rightarrow A$ making the upper triangle commutative.

**Definition 3:**

If $I$ and $J$ are arbitrary sets, $I \times J$ matrix $(r_{ij})$ is called row finite if for any given $i$, $r_{ij} = 0$ for almost all $j$'s. Similarly the matrix is called column finite if for any given $j$, $r_{ij} = 0$ for almost all $i$'s.

Submatrix of the above matrix is defined as an $I' \times J'$ matrix $(s_{ij})$ such that there exists subset $I' \subseteq I$ and $J' \subseteq J$ and $s_{ij} = r_{ij}$ for $i \in I'$ and $j \in J'$ and $(s_{ij}) = 0$. 
Proposition 5.2:

Let \((r_{ij})\) be a row finite for arbitrary set \(I\) and \(J\). The system of equations \(\Sigma r_{ij}x_j = a_i\) (\(i \in I, \ j \in J\)) is relatively solvable in a submodule \(A\) of \(B\) if and only if all left modules \(M\) satisfies the exactness of the sequence

\[
\begin{align*}
\theta R \rightarrow & \\
\oplus & \\
\to & \\
I & \quad J \\
\oplus R & \rightarrow M \rightarrow O
\end{align*}
\]

with \(\mu\) given by \((r_{ij})\) are relatively projective with respect to \(O \rightarrow A \rightarrow B \rightarrow B/A \rightarrow O\).

Proof:

By the given condition on \(M\) is equivalent to saying that \(K = \text{Ker} \delta\) has system of generators. 
\(y_i = \Sigma r_{ij}x_j\) where \((x_j)_J\) are a system of free generators of \(\oplus R\).

By the projectivity of \(\oplus R\) and taking \(K = \text{Im} \mu\).

We draw the following diagram.
Suppose that \((x_j)_j\) be a free basis of \(\oplus R\) and let \(\{y_i\}_I\) generate \(K\). Therefore by the commutativity of the left hand square \(g(y_i) = f\mu_2(y_i) = \Sigma r_{ij}f(x_j)\).

Given that there exists \(a'_j \in A\) such that \(\Sigma r_{ij}a'_j = g(y_i)\). Mapping \(x_j\) to \(a'_j\) we have a homomorphism \(q: \oplus R \rightarrow A\) and \(q \circ \mu_2(y_i) = \Sigma r_{ij}q(x_j) = \Sigma r_{ij}a'_j = g(y_i)\) and thus \(q \circ \mu_2 = g\).

Thus by the above diagram existence of \(q\) is equivalent to existence of \(\mathbf{1}: M \rightarrow B\) such that \(\Theta \cdot \mathbf{1} = h\).

Conversely:

For a given system of equations. We generate a module \(M\) with generators \((x_j)_j\) and relations \(\Sigma r_{ij}x_j = 0\).

Then

\[0 \rightarrow K \rightarrow \oplus R \rightarrow M \rightarrow 0\]

is an exact sequence where \(y_i = \Sigma r_{ij}x_j\) generate \(K\) and then \(M\) satisfies the given conditions. If the equations are solvable in \(B\), there exist \(b_j \in B\) such that \(\Sigma r_{ij}b_j = a_i\). Mapping \(x_j\) to \(b_j\), \(y_j\) is mapped to \(a_i\) and we have the following commutative diagram.
Now existence of \( l \) satisfying \( \Theta l = h \) is equivalent to existence of \( q: \Theta R \rightarrow A \) such that \( q \cdot \mu_2 = q \). Now \( q(x_j) \cdot e_A \) and \( \Sigma r_{ij} q(x_j) = q \cdot \mu_2 (y_i) = q(y_i) a_i \). Thus the system is solvable in \( A \).

**Remark 1:**

A homomorphism \( \mu: \Theta R \rightarrow \Theta R \) of right \( R \)-modules induces a group homomorphism \( \mu': \Theta A \rightarrow \Theta A \) for any left module \( A \) where

\[
\mu': \Theta A \xrightarrow{\mu \Theta A} \Theta R \Theta A \xrightarrow{\Theta \mu A} \Theta A.
\]

And any such \( \mu \) is given by a column finite matrix \( (r_{ij})_{I \times J} \) where \( \mu(y_j) = \Sigma x_i r_{ij} \) and \( (y_j)_J \) and \( (x_i)_I \) are free bases of \( \Theta R \) and \( \Theta R \).

**Theorem 1:**

Given a column finite \( I \times J \) matrix \( (r_{ij}) \) where \( I \) and \( J \) are arbitrary sets the following are equivalent for an exact sequence \( O \rightarrow A \rightarrow B \rightarrow C \rightarrow O \) of left \( R \)-modules.

(i) All such left \( A \)-modules \( M \), can be expressed in the form \( O \rightarrow K \xrightarrow{J} \Theta R \rightarrow M \rightarrow O \) such that \( K \) has a family of generator \( (y_i)_I \) which can be expressed as

\[
(y_i) = \Sigma r'_{ij} x_j (x_j)_J
\]

being a free basis of \( \Theta R \) where \( (r'_{ij}) \) is a row finite submatrix of the given matrix are
relatively projective with respect to the given sequence.

(ii) Any system of linear equations \( \Sigma r_{ij} x_j = a_i \), where \( a_i \in A \) and \( r_{ij} \) is a row finite submatrix of the given matrix is solvable in \( A \) whenever it is solvable in \( B \).

(iii) \( \mu_1^I(\Theta A) = \Theta A \cap \mu_1^J(\Theta B) \) as subgroups of \( \Theta B \) for all \( I \) \( J \), right \( R \)-homomorphism where \( \mu_1 \) is given by a submatrix of the given column finite matrix.

(iv) \( O \rightarrow N \Theta A \rightarrow N \Theta B \) is exact for all right modules \( N \) having a presentation \( \Theta R \xrightarrow{\mu_1} \Theta R \xrightarrow{\delta} N \rightarrow O \) where \( \mu_1 \) is given by a submatrix of the given matrix \( (r_{ij}) \).

Proof:

(i) \( \Rightarrow \) (ii)

By above proposition.

(ii) \( \Rightarrow \) (iii)

For a given \( \mu_1 \), if \( (a_i)_I \cap \Theta A \cap \mu_1^J(\Theta B) \) then there exists \( (b_j)_J \) such that \( \Sigma r_{ij} b_j = a_i \) where \( (r_{ij}) \) is the matrix provided by \( \mu_1 \) through \( \mu_1(v_j) = \Sigma u_i r_{ij} \) where \( (u_i)_I \) and \( (v_j)_J \) are the bases of \( \Theta R \) and \( \Theta R \) respectively. The \( r_{ij} \)'s occurring in the system of equations clearly form a row finite submatrix of \( \mu_1 \) and hence of the given matrix.
\( \mu \) and hence the system can be solved in \( A \) by hypothesis. Therefore

\[
(a_i) \in \mu_1'_{\{I\}} \text{ and } \mu_1'_{\{I\}} \subseteq \Theta A \cap \mu_1'_{\{J\}}.
\]

(iii) \( \Rightarrow \) (ii)

For a system of linear equations

\[ \Sigma r_{ij} x_j = a_i \]

if the system is solvable in \( B \), therefore \((a_i) \in \Theta A \cap \mu_1'_{\{I\}} \) where \( \mu_1 = (r_{ij}) \). It is given that \((a_i)_I \in \mu_1'_{\{I\}}\) and hence there exists \( a_i' \) such that \( a_i = \Sigma r_{ij} a_j' \) and thus solvability in \( A \).

(iii) \( \Rightarrow \) (iv)

\( 0 \rightarrow N A \rightarrow N B \) is an exact sequence for all right modules \( N \) having a presentation \( \Theta R \xrightarrow{\mu_1} \Theta R \xrightarrow{\delta} N \rightarrow 0 \), we draw the following commutative diagram

\[
\begin{array}{cccccc}
\Theta A & \xrightarrow{J} & \Theta A & \xrightarrow{I} & N A & \xrightarrow{0} \\
\downarrow{} & & \downarrow{} & & \downarrow{} & \\
\Theta RA & \xrightarrow{J} & \Theta RA & \xrightarrow{I} & N A & \xrightarrow{0} \\
\downarrow{} & & \downarrow{} & & \downarrow{} & \\
\Theta RO & \xrightarrow{J} & \Theta RO & \xrightarrow{I} & N \Theta i & \xrightarrow{0} \\
\downarrow{} & & \downarrow{} & & \downarrow{} & \\
\Theta RB & \xrightarrow{J} & \Theta RB & \xrightarrow{I} & N \Theta B & \xrightarrow{0} \\
\downarrow{} & & \downarrow{} & & \downarrow{} & \\
\Theta B & \xrightarrow{J} & \Theta B & \xrightarrow{I} & N \Theta B & \xrightarrow{0} \\
\end{array}
\]
Since $\mathbf{R}$ and $\mathbf{R}$ are flat, $\mathbf{R}\mathbf{R}^I_I$ and $\mathbf{R}\mathbf{R}^J_J$ are monomorphism.

By chasing the diagram we get that $N\mathbf{R}^I_I$ is a monomorphism iff

$$u_1^J(\mathbf{R}^B)^I_I = u_1^J(\mathbf{R}^A)^I_I$$

by proposition 2.2

because the equality holds iff the left hand square is a pullback.

**Remark 1:**

We sum up the theorem over all such column finite matrices $(r_{ij})$.

**Theorem 2:**

The following are equivalent for the exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and for any sets $I$ and $J$.

(i) Any module $M$ with generators $(x_j)_J$ and relations $\Sigma r_{ij}x_j = 0$ given by a row and column finite matrix $(r_{ij})$ is projective relative to the sequence.
(ii) Any consistent system of linear equations $\Sigma r_{ij}x_j = a_i$, $a_i \in A$ is solvable in $A$ whenever it is solvable in $B$ where $(r_{ij})$ is a row and column finite $I \times J$ matrix.

(iii) $O \rightarrow N \otimes A \rightarrow N \otimes B$ is exact for all right $R$-modules $N$ which has a presentation $\bigoplus_R \bigoplus_J \bigoplus_I M \rightarrow N \rightarrow O$. Taking $I$ and $J$ to be any finite set, we get Cohn's purity [40].

**Proposition 5.3:**

The following conditions are equivalent:

(i) All finitely presented left modules are projective relative to $E : O \rightarrow A \rightarrow B \rightarrow C \rightarrow O$ i.e. $[M, B] \rightarrow [M, C] \rightarrow O$ is epic.

(ii) A finite consistent system of linear equations in $A$ is solvable in $A$ whenever it is solvable in $B$.

(iii) $N \otimes A \rightarrow N \otimes B$ is monic for all finitely presented right modules $N$.

(iv) $N \otimes A \rightarrow N \otimes B$ is monic for all right modules $N$.

(v) Every row and column finite system of equations is relatively solvable in $A$.

(vi) Every module $M$ generated by a set of generators $(r_j)_J$ and relations $\Sigma r_{ij}x_j = 0$ with a (row and)
column finite matrix \((r_{ij})\) is projective relative to the sequence \(E\).

(vii) Any left module which can be written as a factor of a free module by a finitely generated module projective relative to it.

Proof:

(i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii)
by the theorem 2.

(iv) \(\Rightarrow\) (v) \(\Rightarrow\) (vi) \(\Rightarrow\) (vii) \(\Rightarrow\) (i)
obviously.

(iii) \(\Rightarrow\) (iv)
Any module is a direct limit of finitely presented module (Lazard [108]) and direct limit is an exact functor \(\Rightarrow\) \([O \longrightarrow N@A \longrightarrow N@B\) for all finitely presented module \(N = N@A_1 \longrightarrow N@A_2\) is monic for all right modules \(N\).

Now, Relative projectivity will be considered by arbitrary class of modules \(\Phi\)

Definition 4:
An exact sequence

\[E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\]
is called \(\Phi\)-pure if objects of \(\Phi\) are projective relative \(E\).

Equivalently \(E\) is \(\Phi\)-pure if given a commutative square.
there exists $F \rightarrow A$ making the upper triangle commutative, where $P$ is free $K \leq F$ and $F/K \in \mathcal{P}$.

**Definition 5 [104]:**

A ring is called $q^*$-ring if all cyclic modules are quasi-projective.

Now, we consider purity with respect to a fixed cyclic left module $R/I$. $\mathcal{P}$ can be taken as a class cyclic groups.

**Proposition 5.4:**

The following conditions are equivalent:

(i) A submodule $K$ of $M$ is $R/I$ - pure.

(ii) Given $m \in M$ such that $\text{Im} \subseteq K$, there exists $m' \in M$ such that $\text{Im}' = 0$ and $m - m' \in K$.

**Proof:**

(i) $\Rightarrow$ (ii)
Given \( m \in M \) such that \( \text{Im} \subseteq K \) define \( h \) by \( h(\bar{i}) \mapsto m+K \).
Take \( m' = f(\bar{i}) \). \( h(\bar{i}) = f(\bar{i}) + K \) this implies that \( m-m' \in K \).

(ii) \( \Rightarrow \) (i)

By the given condition and \( h : R/I \to M/K \), take
\( h(\bar{i}) = m+K \). Then \( Ih(\bar{i}) = 0 \).

\( \Rightarrow \text{Im} \subseteq K \).

Map \( f(\bar{i}) \) to \( m' \). Then \( f(\bar{i}) = im' = 0 \) for all \( i \in I \)
and hence \( f \) is an \( R \)-homomorphism. And \( f(\bar{r}) = rm' + K = rm + K \)
= \( h(\bar{r}) \) and hence a submodules \( K \) of \( M \) is \( R/I \) - pure in \( M \).

**Proposition 5.5:**

A submodule \( K/J \) of the cyclic module \( R/J \), is
\( R/I \) - pure in \( R/J \) if and only if given \( r \in R \) such that
\( Ir \subseteq K \), there exists \( r' \in R \) such that \( Ir' \subseteq J \) and \( r-r' \in K \).

**Proposition 5.6:**

The following conditions are equivalent:

(i) A cyclic module \( R/J \) is \( R/I \) - regular i.e. all
its submodules are \( R/I \) - pure in it.

(ii) Given \( r \in R \), there exists \( r' \in R \) such that \( Ir' \subseteq J \)
and \( r-r' \in J + Ir \).

**Proof:**

(i) \( \Rightarrow \) (ii)

By the given condition, if \( K/J \subseteq R/J \), then for
\( r \in R \) such that \( Ir \subseteq K \), there exists \( r' \in R \) such that \( Ir' \subseteq J \)
and \( r-r' \in J + Ir \subseteq K \).
(ii) $\Rightarrow$ (i)

Given that $r \in R$, $Ir \subseteq J + Ir$ and as $(J + Ir)/J$ is $R/I$-pure in $R/J$, there exists $r' \in R$ such that $Ir' \subseteq J$ and $r-r' \in J + Ir$.

Proposition 5.7:

A cyclic module $R/I$ is quasiprojective if and only if given $r \in R$, there exists $i_r \in I$ such that $I(r - i_r) \subseteq I$.

Proof:

By the above proposition, $R/I$ is quasiprojective if and only if given $r \in R$ there exists $r' \in R$ such that $Ir' \subseteq I$ and $r-r' \in I + Ir$.

If $R/I$ is quasi-projective and $r \in R$, then take $r'$ such that $Ir' \subseteq I$ and $r-r' \in I + Ir$. Now if $r-r' = i_1 + i_2 r$, then $I(r - i_2 r) = I(r' + i_1) \subseteq I$ and therefore there exists $i_r \in I$ such that $I(r - i_r) \subseteq I$.

Conversely:

Given that $r \in R$, we take $r' = r - i_r r$. Then $r-r' = i_r r \in J \subseteq I + Ir$ and $Ir' = I(r - i_r r) \subseteq I$, and so $R/I$ is quasiprojective.

Proposition 5.8:

If $I$ is a two sided ideal of a left $q^*$-ring $R$, then $R/I$ is a left $q^*$-ring.
Proof:

Given that \( r \in R \) and a left ideal \( J \supseteq I \), there exists \( i_r \in J \) such that \( J(r - i_r) \subseteq J \). Then \( J/I \ [(r + I) - (i_r + I)(r + I)] \subseteq J/I \). Hence the proposition.