CHAPTER-2

GENERAL TRANSFORMATION METHODS IN FLOW THEORY

2.0 INTRODUCTION:

An appropriate change of variables is probably one of the most useful methods available for solving the partial differential equations of mathematical physics. The most general criterion for a transformation is simply to change a given problem into a simpler problem in some sense or other; that is either to a form which will yield to more standard solution techniques or possible to a form which has been previously solved in connection with a related or similar problem. Hodograph transformations and conformal transformations are well-known methods for transforming given problems into forms which yield to classical techniques (such as separation of variables). Ames (1965) classifies the various transformation techniques into three groups: transformations only of the dependent variables, transformations only of the independent variables, and mixed transformations of both independent and dependent variables. However, all three groups have a common goal: to find a relation or more specifically, a basis of comparison between different physical (or mathematical) problems. This broad concept of comparison is the definition of the term “generalized similarity.”

Generalized similarity might be applied in fluid mechanics to attempt to answer such questions as “Is there any similarity (i.e. basis of comparison) between compressible and incompressible flow problems, axisymmetric and planar flows or in general, any more complicated and a less complicated flow?” By contrast, the usual term “similarity” as used by Birkhoff (1950), Morgan (1952), Hansen (1964), Abbott and Kline (1960) and others is defined in terms of independent variables of a problem. Thus it may ultimately refer to a physical
similarity within a given problem, such as similarity of velocity or temperature profiles.

Much of the previous work on similarity was motivated by the desire to develop simple methods for the reducing the number of independent variables. Out of this previous work came the realization that each of the proposed methods was based on an assumed class of transformations and the recognition that more general classes of transformations might lead to the solution of a wider class of problems. The first part of this chapter is concerned with seeking the most general class of transformations for particular types of problems. The two viewpoints discussed in chapter 1 will be examined. First, the mathematical theory of transformations will be reviewed and it will be shown possible to postulate the so-called primitive transformation as the most general form of a class of transformations (under a certain assumption). Second, a separate approach, based on postulating the “complete physical problem” is examined for the special case of laminar boundary-layer flows and it is shown that this approach also yields the primitive transformation.

The second part of the chapter deals with the development of a technique for employing the primitive transformation to find the form of the variables for a wide variety of generalized similarity problems.

2.1 TYPICAL TRANSFORMATION USED IN FLUID MECHANICS:

As a means of developing some insight into the question of “the most general class of similarity transformations” and the concept of “generalized similarity”, Table 2.1 was compiled of examples of as many different types of known conventional transformations as could be found to represent the field of fluid Mechanics.
TABLE 2.1 CONVENTIONAL TRANSFORMATIONS

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Transformed Independent variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Sun (1960)]</td>
<td></td>
</tr>
<tr>
<td>Similarity</td>
<td>$\xi(x) = x, \eta(x, y) = y, \gamma(x)$</td>
</tr>
<tr>
<td>Meksyn-Gortler</td>
<td>$\xi(x) = \int_0^x u_1(x) , dx, \eta(x, y) = u_1(x) \frac{y}{\sqrt{x}}$</td>
</tr>
<tr>
<td>Von Mises</td>
<td>$\xi(x) = x, \eta(x, y) = \psi(x, y)$</td>
</tr>
<tr>
<td>Crocetto</td>
<td>$\xi(x) = x, \eta(x, y) = u(x, y)$</td>
</tr>
<tr>
<td>Mangler</td>
<td>$\xi(x) = \int_0^x r_0^2(x) , dx, \eta(x, y) = r_0(x) , y$</td>
</tr>
<tr>
<td>Stewartson</td>
<td>$\xi(x) = \int_0^x a_1(x) , p_1(x) , dx,$ $\eta(x, y) = \int_0^x a_1(x) , \rho(x, y) , dy$</td>
</tr>
<tr>
<td>Dorodnitsyn</td>
<td>$\xi(x) = \int_0^x p_1(x) , dx, \eta(x, y) = \int_0^x \rho(x, y) , dy$</td>
</tr>
</tbody>
</table>

Similarity Rules of High Speed Flow (Refer. Liepmann et al (1965)):

1. Prandtl –Glauert
   $\xi(x) = x, \eta(y) = By$

2. Gothert
   $\xi(x) = x, \eta(y) = By$

3. Von Karman Transonic

   By comparing the various transformations in the table, an interesting conclusion can be made: all of the transformed independent variables are of the general form $\xi = \xi(x)$ and $\eta = \eta(x, y)$ (that is, one of the new variables is a function of only one of the original variables.) This realization raises the question “Is the general transformation $\xi(x), \eta(x, y)$ the most general class of transformations that need be considered for physical problems?” In an attempt to answer this question, it will first prove useful to review the mathematical theory of transformations.
2.2 THE PRIMITIVE TRANSFORMATION:

The following theorem can be found in the mathematical literature of general transformation theory (Refer. Courant (1956))

**THEOREM:** An arbitrary one-to-one continuously differentiable transformation

\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \]

of a region \( R \) in the \( x, y \) - plane onto a region \( R \) in the \( \xi, \eta \) - plane can be resolved in the neighborhood of any point interior to \( R \) into one or more continuously differentiable “primitive” transformations provided that throughout the region \( R \), the Jacobian

\[ J(\xi, \eta) = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x \eta_y - \xi_y \eta_x \]

differs from zero. A “primitive” transformation is of the form

\[ \xi = \xi(x), \quad \eta = \eta(x, y) \]

Now one-to-one transformations have an important interpretation and application in the representation of deformation or motions of continuously distributed systems, such as fluids. For example, if a fluid is spread out at a given time over a region \( R \) and then is deformed by motion, the motion of the fluid is described by the coordinates in the physical \( R \)-plane. If the fluid motion in \( R \) is characterized by the coordinates \( x, y \), then the corresponding motion in the transformed region \( R \) is characterized by coordinates \( \xi, \eta \). The one-to-one character of the transformation obtained by bringing every point \( x, y \) into correspondence with a single point \( \xi, \eta \) is simply the mathematical expression of the physically obvious fact that the fluid motion in the physical \( R \)-plane must remain recognizable after transformation to the transformed \( R \) -plane i.e. that the corresponding motions remain distinguishable.
Physically, since most “practical” transformations of interest for solving physical problems should be one-to-one, an exception to this rule; for example is the Von Mises transformation of Table 2.1, which is singular along the X-axis. However, this transformation is used for computational (not physical) reasons and hence its use is motivated by a completely different line of reasoning than that under consideration here. That is, have a unique inverse (except possibly at a finite number of singular points), it appears the primitive transformation or the resolution into primitive transformations should be considered in the search for “the most general class of transformations.”

For the sake of completeness, the following two properties of the primitive transformation are noted:

(1) If the primitive transformation
\[ \xi = \xi(x), \quad \eta = \eta(x, y) \]
is continuously differentiable and its Jacobian
\[ J(\xi, \eta) = \frac{\partial (\xi, \eta)}{\partial (x,y)} = \xi_x \eta_y - \xi_y \eta_x \]
differs from zero at a point \( P(x_0, y_0) \), then in the neighborhood of \( P \) the transformation has a unique inverse, and this inverse is also a primitive transformation of the same type.

(2) For primitive transformation, the sense of rotation in the \( x, y \)-plane is preserved or reversed in the \( \xi, \eta \)-plane according as the sign of the Jacobian is positive or negative, respectively.

In summary, the primitive transformation appears to be the most general class of transformations that need be considered, purely from a mathematical viewpoint, as long as it is required that a unique inverse of the transformation must exist. In the next section, the question of the most general transformation will be approached from a fundamentally viewpoint and it will be shown that again the
primitive transformation appears to be a requirement from conclusions based on uniqueness arguments.

2.3 REQUIREMENTS IMPOSED BY THE PHYSICAL DESCRIPTION:

The question of the most general class of transformations will now be examined from a more physical viewpoint. Some mathematics will be involved, of course, but the physical problem will be kept near at hand and frequent reference to it will be made throughout the analysis.

Because in what follows by definition involves a particular physical problem, it will be convenient to introduce such a problem. As an example, consider the following equations describing the two-dimensional flow of a laminar incompressible boundary layer:

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = -\frac{dp}{dx} \tag{2.1}$$

Or an equivalent single equation in terms of the stream functions;

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^3 \psi}{\partial y^3} = -\frac{dp}{dx} \tag{2.2}$$

For connivance in the development, it is assumed that all of the variables are non-dimensional. The objective is now to consider very general transformation of variables and see what conditions must be met by this transformation so as not to violate any known physical properties of the problem.

Let us begin by specifying a transformation of variables. For the independent variables, no restrictions at present will be placed on the assumed form; thus the form is specified simply as
\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \quad (2.3) \]

The dependent variable \( \psi \) (the stream function) is also transformed to a new function, say \( \Psi \). Previous work on fluid flow transformations has often assumed that the two stream functions, \( \psi \) and \( \Psi \), should be the same at corresponding points and hence that streamlines in one plane are transformed into streamlines in the other. However, this restriction is found to be unwarranted in many problems and will be avoided here. Instead, the relationship between \( \psi \) and \( \Psi \) will be specified in some weak sense by the form

\[ \Psi(\xi, \eta) = g(x, y), \psi(x, y) \quad (2.4) \]

It should be recognized that this assumption is not trivial and other forms could be considered. Nevertheless, Equation (2.4) represents a more general case than usually employed and attention will be restricted to this form in the present chapter.

The next step is to carry out the transformation of Equation (2.3) and (2.4) on the left-hand side of Equation (2.2). Application of the transformations (2.3) and (2.4) to the boundary – layer equation is not new and was considered previously by Coles (1962). However, the generalized similarity technique was not under consideration by the previous authors.

The results of the transformation of Equation (2.2) are given in Table 2.2. For convenience in interpreting the physical terms after transformation, transformed velocities \( U, V \) have been defined by

\[ U = \frac{\partial \Psi}{\partial \eta}, \quad V = -\frac{\partial \Psi}{\partial \xi} \]

The transformed side of Equation (2.2) given in Table 2.2 is essentially unmanageable in the present form. While it is unclear at the present time what conditions are either necessary or sufficient to ensure any particular mathematical behavior, it is interesting to consider an argument proposed
by Coles (1962). Coles’ argument is based on the a priori requirement that the transformed flow outside of the shear layer (i.e., for large values of y) should conform both physically and formally to the original flow. Thus, he argues since the physical flow is bounded for large y and is, in fact; at most a function of x and then the transformed side of Equation (2.2) must behave in a similar fashion. His argument is then that the substantial derivative terms and the $U^2$ terms become at most functions of $\xi$ for large y, whereas the remaining terms behave either like $y$ or $y^2$. His conclusion is to require the $\eta$ and $\eta^2$ terms to vanish identically, which can be accomplished by requiring that $g = g(x)$, $\xi = \xi(x)$ and $\eta_y = \gamma(x)$. At the present time, about all that can be said is that this assumed form is one possible form of a transformation which will preserve a certain sense of physical correspondence between the given and transformed flow; this form of the transformation will be employed throughout the remainder of the present chapter.

In summary, the postulated general transformation, based on the physical mode is:

$$\Psi(\xi, \eta) = g(x) \psi(x, y)$$

$$\xi = \xi(x), \eta = y (\gamma(x))$$

**TABLE 2.2 (THE RESULTS OF EQUATION (2.2) ARE GIVEN.)**

| Transformation | $\Psi(\xi, \eta) = g(x, y) \psi(x, y)$; $\xi = \xi(x, y)$, $\eta = \eta(x, y)$, |
| $U = \frac{\partial \Psi}{\partial \eta}$, $V = -\frac{\partial \Psi}{\partial \xi}$ | $- \frac{dp}{dx} = \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^3 \psi}{\partial y^3} = \left( U \frac{\partial U}{\partial \xi} + V \frac{\partial U}{\partial \eta} \right) (\eta_y - \xi_y) \frac{j(\xi, \eta)}{g^2} $ |

$$- U \left[ \frac{\eta_y}{g^2} \right] \frac{\xi}{y} - 2 \left( \frac{g_y}{g^2} \right) \eta_y - \frac{g_y}{g^2} \eta_{yy}$$

$$+ V \left[ \frac{\xi_y}{g^2} \right] \frac{\xi}{y} - 2 \left( \frac{g_y}{g^2} \right) \xi_y - \frac{g_y}{g^2} \xi_{yy}$$
It is of interest to note that the resulting form of the independent variables is a primitive transformation as discussed in the last section, but it has been shown by considering a particular problem and relying on physical argument that it may
be possible to restrict the form of \( n \) to be a linear function of \( y \) for this particular class of problems (boundary-layer flows). Thus, the two approaches, mathematical and physical, have lead to the same conclusion regarding a postulated “most general class of transformation.” Of course it is necessary, as previously pointed out to emphasize that there may be some cases of practical interest which will lie outside the realm of the mathematical conclusions presented here. However, it has been found that all of the cases listed in Table 2.1 are satisfied by the transformation of Equation (2.5) (with the exception of the Von Mises transformation previously discussed).

It is now worthwhile to return to Table 2.1 for a moment. Recall that from Table 2.1, it was noted that a wide class of different transformations were all primitive transformation has been shown to yield to simple general analysis for its derivation for particular problems. In fact, at the present time there are two types of similarity analyses that are founded primarily on simple transformation theory: the free parameter method of Hansen (1964) and the separation of variables method of Abbott and Kline (1960). It is thus of interest to speculate on the following question: Would it be possible to derive all of these “generalized similarity” transformations by the same technique that is used to derive the similarity transformation? The Generalized Similarity Analysis was developed as an attempt to answer this question.

This new method is based on a single assumption concerning the admissible class of transformations of the independent variables, namely that the assumed transformation should be one-to-one (that is, have a unique inverse). On the previous pages it was shown that this requirement would lead to the primitive transformation, and further that a particular class of problems, namely \( \xi = \xi(x) \), \( \eta = y(\gamma(x)) \) for the boundary-layer flows. Of course the generality of the present ideas goes beyond a particular case, such as boundary-layer flow analysis and should be applicable to a wide class of problems.
2.4 THE GENERALIZED SIMILARITY ANALYSIS:

The development of the generalized similarity analysis was motivated as an extension of the method of finding similarity variables (i.e. the reduction of number of independent variables) to the problem of finding a transformation of variables which will convert a given physical problem into an alternate problem under certain prescribed conditions. For example, the prescribed condition for a similarity solution is that the number of independent variables must be reduced. By contrast, the prescribed condition for the Mangler transformation is that the axisymmetric boundary – layer equations are to be transformed into the planar form the equations and so forth with similar statements for the rest of the examples in table 2.1.

There are three distinct steps to the generalized similarity analysis. These steps are:

(a) The general mathematical theory of transformations states that any continuous one-to-one transformation can be resolved into one or more primitive transformations of the form

\[ \xi = \xi(x), \quad \eta = (x, y) \]

Thus, a primitive transformation form is assumed a priori, where it is recognized that the general analysis has the possibility of being repeated more than once, depending on the particular problem at hand.

(b) The given equation, transformed under a primitive transformation to the new independent variables \((\xi, \eta)\) is required to satisfy the state requirement; for example, that for a similarity analysis, the transformed equation should be a function only one of the new variables.

(c) Simultaneously the boundary conditions for the given equation are required to be satisfied when expressed in terms of the transformed variables.
These three steps will be shown to completely and uniquely determine the explicit form of the new variables $\xi(x)$ and $\eta(x, y)$ if, in fact, the original problem and associated boundary conditions do admit a generalized similarity solution.

As an example of the method, two problems will be examined in detail. First the problem of boundary layer flow of an incompressible visco-elastic second grade fluid along a stretching sheet will be solved to give a relatively simple motivation of the basic ideas. Second and more difficult Mangler transformation will be derived as an example of the broad applicability of the method.

2.4.1 EXAMPLE 1:

2.4.1.1 FORMULATION OF THE PROBLEM:

Coleman and Noll (1960) originally suggested a constitutive equation for the incompressible viscoelastic second grade fluid, based on the postulate of fading memory as

$$T = - p I + \mu A_1 + a_1 A_2 + a_2 A_1^2$$  \hspace{1cm} (2.6)

Where

- $T$: is the stress tensor
- $P$: is the pressure
- $\mu$: is the dynamic viscosity
- $a_1, a_2$: are the first and second normal stress coefficients
- $A_1, A_2$: are the kinematic tensors expressed as:
\[ A_1 = \text{grad}V + (\text{grad}V)^T \]

\[ A_2 = \frac{d}{dt} A_1 + A_1(\text{grad}V) + (\text{grad}V)^T A_1 \]

Where \( V \) is the velocity and \( \frac{d}{dt} \) is the material time derivative.

Let us consider the flow of second grade fluid, governed by (2.6) past a plane wall \( y = 0 \), the flow being confined to the region \( y > 0 \). The wall is stretched on both sides from a fixed origin along the x-axis, the origin being kept fixed by applying two equal and opposite forces given by Dandapat and Gupta (1989).

Beard and Walters (1964) derived the steady two-dimensional boundary layer equations for this fluid as:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2} - k \left[ \frac{\partial}{\partial x} \left( u \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^3 v}{\partial y^3} \right] \]

Where

\( v = \frac{\mu}{\rho} \) : is the kinematic viscosity

\( \mu \) : is the viscosity of the fluid

\( \rho \) : is the density and

\( k \): is a positive parameter associated with the viscoelastic fluid.

The relevant boundary condition for \( x \geq 0 \) is:

\[ u = \alpha x \quad \text{and} \quad v = 0 \quad \text{at} \quad y = 0 \]

\[ u \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty \]  
\[ \text{(2.9)} \]

Where \( \alpha \) is a constant.

We solve equations (2.7) and (2.8) by similarity method in the next section.
2.4.1.2 SIMILARITY APPROACH: SOLUTION TO THE PROBLEM:

We introduce the stream function \( \psi \) as:

\[
    u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}
\]  

(2.10)

Equation (2.7) is satisfied on substitution of (2.10) in it. Now, substituting (2.10) in (2.8), we obtain easily

\[
    \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} - \frac{k}{\partial x} \left[ \frac{\partial^2 \psi}{\partial x \partial y^2} \frac{\partial^3 \psi}{\partial y^2} + \frac{\partial \psi}{\partial y} \frac{\partial^4 \psi}{\partial x \partial y^3} - \frac{\partial^2 \psi}{\partial x \partial y^3} \frac{\partial^3 \psi}{\partial y \partial x \partial y^2} + \frac{\partial \psi}{\partial y} \frac{\partial^4 \psi}{\partial x^2} \right]
\]

(2.11)

(All variables are dimensional, where \( \nu \) is the kinematic viscosity) with the boundary conditions

\[
    y = 0: \frac{\partial \psi}{\partial y} = \alpha, \quad \frac{\partial \psi}{\partial x} = 0
\]  

(2.11a)

\[
    y \to \infty: \frac{\partial \psi}{\partial y} = 0
\]

To make the problem statement complete, it is necessary to state the given requirement for the transformation \( x, y \to \zeta, \eta \).

REQUIREMENT: Under the transformation, the given equation and boundary conditions (2.11) and (2.11a) are to reduce to a single independent variable (i.e. the similarity variable).

The steps of the generalized similarity analysis are carried out in order as follows:

At this stage we introduce transformations of the independent variables as

\[
    x = \zeta, \quad y = \frac{\eta}{g(\zeta)}
\]  

(2.12)
various derivatives appearing in (2.11) are transformed under (2.12) as:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \eta} \cdot \eta \cdot \frac{d \ln |g|}{d \zeta} + \frac{\partial \psi}{\partial \zeta}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot g(\zeta)$$

$$\frac{\partial^2 \psi}{\partial y^2} = g^2(\zeta) \cdot \frac{\partial^2 \psi}{\partial \eta^2}$$

$$\frac{\partial^3 \psi}{\partial y^3} = g^3(\zeta) \cdot \frac{\partial^3 \psi}{\partial \eta^3}$$

$$\frac{\partial^2 \psi}{\partial y \partial x} = \left[ g(\zeta) \cdot \frac{\partial^2 \psi}{\partial \eta \partial \zeta} + \frac{\partial \psi}{\partial \eta} \cdot g'(\zeta) \right] + \frac{\partial^2 \psi}{\partial \eta^2} \cdot g(\zeta) \cdot \eta \cdot \frac{d \ln |g|}{d \zeta}$$

$$\frac{\partial^2 \psi}{\partial x^2} \cdot \frac{\partial^3 \psi}{\partial y^2} = \left[ \frac{\partial^2 \psi}{\partial \eta^2} \cdot g^2(\zeta) \cdot \frac{\partial^3 \psi}{\partial \eta^3} \right]$$

Substituting the transformed derivative (2.13) into equation (2.11), we obtain

$$\nu g^3(\zeta) \frac{\partial^3 \psi}{\partial \eta^3} - k \left[ \left\{ g(\zeta) \cdot \frac{\partial^2 \psi}{\partial \eta \partial \zeta} + \frac{\partial \psi}{\partial \eta} \cdot g'(\zeta) \right\} g^2(\zeta) \cdot \frac{\partial^2 \psi}{\partial \eta^2} \right] =$$

$$\frac{\partial \psi}{\partial \eta} \cdot g(\zeta) \cdot \left\{ g(\zeta) \cdot \frac{\partial^2 \psi}{\partial \eta \partial \zeta} + \frac{\partial \psi}{\partial \eta} \cdot g'(\zeta) + \frac{\partial^2 \psi}{\partial \eta^2} \cdot g(\zeta) \cdot \eta \cdot \frac{d \ln |g|}{d \zeta} \right\}$$
The equation (2.14) is a complicated one. To proceed further, we adopt the separation of variables technique and accordingly, put

$$\psi = H(\varsigma) \cdot F(\eta)$$

(2.15)

Hansen (1964) has recommended that “Substitution of the product form of the dependent variable into the equation generally leads to an equation in which the functions of one variable cannot be isolated on the two sides of the equation unless certain parameters are specified”. Keeping this in view we proceed choosing simply $H(\varsigma) = \varsigma$, which reduces (2.15) to

$$\psi = \varsigma \cdot F(\eta)$$

(2.16)

Substituting (2.16) in (2.14), we obtain

$$\varsigma \cdot F'(\eta) \cdot g(\varsigma) \left[ g(\varsigma) \cdot F''(\eta) + g'(\varsigma) \cdot \varsigma \cdot F'(\eta) + g(\varsigma) \eta \frac{d}{d\varsigma} \cdot \varsigma \cdot F''(\eta) \right] -$$

$$- \left( F(\eta) + \eta \frac{d}{d\varsigma} \cdot \varsigma \cdot F'(\eta) \right) \cdot g^2(\varsigma) \cdot F''(\eta) \cdot \varsigma = \nu g^3(\varsigma) \cdot \varsigma \cdot F'''(\eta) -$$

$$k \left[ \left\{ [g(\varsigma) \cdot F'(\eta) + g'(\varsigma) \cdot \varsigma \cdot F'(\eta)] + g(\varsigma) \eta \frac{d}{d\varsigma} \cdot \varsigma \cdot F''(\eta) \right\} g^3(\varsigma) \cdot \varsigma \cdot F'''(\eta) +$$

$$+ g(\varsigma) \cdot \varsigma \cdot F'(\eta) [3 g^2(\varsigma) \cdot g'(\varsigma) \cdot \varsigma \cdot F'''(\eta) + g^3(\varsigma) \cdot F'''(\eta) + g^2(\varsigma) \cdot g'(\varsigma) \cdot \eta \cdot \varsigma \cdot F''(\eta)] -$$

$$- g^2(\varsigma) \cdot \varsigma \cdot F''(\eta) \left[ 2 g(\varsigma) \cdot g'(\varsigma) \cdot F''(\eta) + g^2(\varsigma) \cdot F''(\eta) + g^2(\varsigma) \eta \frac{d}{d\varsigma} \cdot \varsigma \cdot F''(\eta) \right] -$$

$$\frac{d}{d\varsigma} \cdot \varsigma \cdot F'(\eta) \right\} g^4(\varsigma) \cdot \varsigma \cdot F''(\eta) -$$

$$+ \left[ F(\eta) + \eta \frac{d}{d\varsigma} \cdot \varsigma \cdot F'(\eta) \right] \cdot [g^4(\varsigma) \cdot \varsigma \cdot F''(\eta)] \right]$$

(2.17)
On inspection, we can see that equation (2.17) is not yet easily separable. In view of the relations (2.10) and the transformations (2.12), it can be easily shown that the continuity equation (2.7) is satisfied if \( g(\zeta) = 1 \). Substituting \( g(\zeta) = 1 \) in equation (2.17), we obtain

\[
g(\zeta) \cdot F'(\eta) \cdot F'(\eta) - F''(\eta) \cdot F(\eta) \cdot \zeta = v \cdot \zeta \cdot F''(\eta) - k \left[ g(\zeta) \cdot F'(\eta) \cdot F'(\eta) + g(\zeta) \cdot F''(\eta) \right]
\]

Or

\[
v \cdot F''(\eta) - F''(\eta) + F(\eta) \cdot F''(\eta) = k \left[ 2F'(\eta) \cdot F''(\eta) - F''(\eta) + F(\eta) \cdot F''(\eta) \right]
\]

(2.18)

The boundary conditions (2.9) are transformed to

\[
F'(\eta) = \alpha, \quad F(\eta) = 0 \quad \text{as} \quad \eta = 0
\]

\[
F'(\eta) = 0 \quad \text{as} \quad \eta \to \infty
\]

(2.19)

Equation (2.18) with the boundary conditions (2.19) is an ordinary differential equation, so called similarity equation. From the present analysis it is also clear that the free parameter method and the separation of variables techniques are alike. Further it is worth to note that, an exact solution of equation (2.18) satisfying the boundary conditions (2.19) is given by

\[
F = \frac{1}{\gamma} (1 - e^{-\alpha \gamma \eta})
\]

(2.20)

Where

\[
\gamma = \alpha^{-\frac{1}{2}} (v - \alpha k)^{-\frac{1}{2}}, \quad 0 \leq \frac{\alpha k}{v} < 1
\]

Now taking (2.16), (2.20) and (2.12) with \( g(\zeta) = 1 \) into account in (2.10), we determine \( u \) and \( v \) as
\[ u = \alpha x e^{\frac{1}{2}(v - \alpha k)^{-\frac{1}{2}} y} \quad (2.21) \]

\[ v = \alpha^{\frac{1}{2}}(v - \alpha k)^{\frac{1}{2}} [1 - e^{\alpha^{\frac{1}{2}(v - \alpha k)^{-\frac{1}{2}} y}}] \quad (2.22) \]

Thus, generalized similarity technique results are exact solution of most complicated problem.

### 2.4.1.3 DISCUSSION:

In the applications of the generalized similarity method to two-dimensional boundary layer flows, stream function \( \psi \) is introduced and subsequently a non-linear partial differential equation in \( \psi \) is derived. Transformations of the independent variables are then sought. The resulting equation \( \psi \) is next subjected to the method of ‘separation of variables’. Equation (2.18) is derived utilizing the above concept and imposing some necessary restrictions namely, \( H(\zeta) = \zeta \) and \( g(\zeta) = 1 \). It is noticed that equation (2.18) is an ordinary differential equation in the similarity parameter \( \eta \). Further, equation (2.18) is a fourth-order differential equation but we have three boundary conditions in (2.19). In the viscous Newtonian case \( (k = 0) \), (2.18) however reduces to a third order equation. The suitability of present approach has been recommended by Hansen (1964) for the present type of problems. Some authors have solved the problem numerically.

The exact solution (2.20) obtained here is in terms of dimensional quantities and agrees with that of Rajagopal et al. (1984) and Siddappa and Abel (1985). Correct expressions for \( u \) and \( v \), as given by (2.21) and (2.22) respectively have been obtained as they satisfy the boundary conditions and the continuity equation (2.7). It is to mention in this context that the expressions derived by Dandapat and
Gupta (1989) for \(u\) and \(v\) from the exact solution do not satisfy the continuity equation.

### 2.4.2 EXAMPLE 2:

The well-known Mangler Transformations [Refer. Liepmann et al (1956)] serves as a good example of the broad meaning of the idea of generalized similarity because the transformation answers the question “Under what conditions is an axisymmetric boundary-layer flow similar to a two-dimensional planar flow?”. The answer lies in finding a transformation between the variables describing the two types of flow. The generalized similarity analysis is formulated in the same as the simple similarity analysis of the preceding example; that is, by specifying the equations (and boundary conditions) and the stated requirement to be satisfied by the transformation.

The governing equations for a thin laminar axisymmetric boundary layer are:

\[
\frac{\partial (r_0 u)}{\partial x} + \frac{\partial (r_0 v)}{\partial y} = 0
\]  

(2.23)

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_1 \frac{\partial u_1}{\partial x} + \nu \frac{\partial^2 u}{\partial^2 y}
\]  

(2.24)

Where \(r_0 = r_0(x)\) is a given quantity (\(r_0\) specifies the body shape relative to the axis of symmetric) and \(\nu\) the kinematic viscosity is a constant. The general boundary conditions are:

\[
y = 0: \quad u, v = 0
\]

\[
y \to \infty: \quad u \to u_1(x).
\]  

(2.25)

The problem statement is completed by writing down the required mathematical form of the desired equations, namely:
REQUIREMENT: Under the transformation \( u, v \rightarrow U, V \) and \( x, y \rightarrow \xi, \eta \) the transformed equations are to be a planar boundary-layer flow; that is, of the form

\[
\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0 \tag{2.26}
\]

\[
U \frac{\partial U}{\partial \xi} + V \frac{\partial V}{\partial \eta} = U_1 \frac{dU_1}{d\xi} + v \frac{\partial^2 U}{\partial \eta^2} \tag{2.27}
\]

The three parts of the analysis are as follows:

(a) Assume a primitive transformation. Again, since the problem involves the boundary-layer equations, it is sufficient to assume the same general form of the transformation as given in section (2.3) that is

\[
\Psi(\xi, \eta) = g(x) \psi(x, y)
\]

\[
\xi = \xi(x), \quad \eta = y \gamma(x) \tag{2.28}
\]

Where for the present case, the stream function \( \psi(x, y) \) is defined by the equations

\[
u = \frac{1}{r_0} \frac{\partial \psi}{\partial y}, \quad \nu = -\frac{1}{r_0} \frac{\partial \psi}{\partial x}
\]

so as to satisfy Equation (2.23). For the transformed flow to be in the form of planar equations, the transformed stream function \( \Psi(\xi, \eta) \) is defined by

\[
U = \frac{\partial \Psi}{\partial \eta}, \quad V = -\frac{\partial \Psi}{\partial \xi}
\]

Using these definitions and the transformation (2.28), the velocities and their derivatives become

\[
u(x, y) = \frac{1}{r_0g} \Psi(\xi, \eta)
\]

\[
v(x, y) = \frac{1}{r_0g} g' \psi(x, y) + \frac{\xi_x}{r_0g} V(\xi, \eta) - \frac{\eta_x}{r_0g} U(\xi, \eta)
\]
Substituting these expressions into Equations (2.23) and (2.24) yields, after rearrangement:

\[
\frac{\partial u}{\partial y} = \frac{\gamma}{r_0 g} \frac{\partial U}{\partial \eta_1} \\
\frac{\partial^2 u}{\partial^2 y} = \frac{\gamma}{r_0 g} \gamma^2 \frac{\partial^2 U}{\partial \eta^2} \\
\frac{\partial u}{\partial x} = \left( \frac{\gamma}{r_0 g} \right) \gamma \frac{\partial U}{\partial \xi} + \frac{\gamma}{r_0 g} \left( \frac{\partial U}{\partial \xi} \xi_x + \frac{\partial U}{\partial \eta} \eta_x \right)
\]

Substituting these expressions into Equations (2.23) and (2.24) yields, after rearrangement:

\[
J (\xi, \eta) \left( \frac{\partial u}{\partial \xi} + \frac{\partial U}{\partial \eta} \right) = 0 \tag{2.29}
\]

\[
\left( \frac{\gamma}{r_0 g} \right) \gamma \frac{\partial U}{\partial \xi} + V \frac{\partial V}{\partial \eta} - U_1 \frac{dU_1}{d\xi} = (\gamma r_0 g) v \frac{\partial^2 U}{\partial \eta^2}
\]

\[
+ \left[ \left( \frac{\gamma}{r_0 g} \right) \gamma \left( U_1^2 - U^2 \right) - \frac{g'}{g} \Psi \frac{\partial U}{\partial \eta} \right]
\]

\[
\tag{2.30}
\]

It is found that the continuity equations (2.29) is invariant under the transformation (2.28) as long as the Jacobian \( J (\xi, \eta) = \gamma \xi_x \) differs from zero. Thus equation (2.29) does not force any requirements on the transformation. However, in comparing equations (2.27) and (2.30), it is seen that for the latter equation to fulfill the given requirement of being identical to the planar form given by Equation (2.27), it is necessary that

\[
\xi_x = \gamma r_0 g \tag{2.31}
\]

And

\[
\left( \frac{\gamma}{r_0 g} \right) \gamma \left( U_1^2 - U^2 \right) - \frac{g'}{g} \Psi \frac{\partial U}{\partial \eta} = 0 \tag{2.32}
\]
A sufficient condition for Equation (2.32) to be satisfied is for

\[ \frac{\nu}{\rho g} = \text{constant} = c_1 \]  

(2.33)

And

\[ g = \text{constant} = c_2 \]  

(2.34)

Although Equations (2.33) and (2.34) may not be necessary conditions for the satisfaction of (2.32), they at least provide one solution for the given requirement and this is usually satisfactory from an engineering viewpoint. Since there are no further requirements to restrict a choice for the constants \( c_1 \) and \( c_2 \), they are normally chosen for dimensional reasons to assume the values \( c_1 = 1 \) and \( g = 1/D \) where \( D \) is an arbitrary reference length. Hence, the final form for the transformation becomes:

\[ \Psi(\xi, \eta) = \frac{1}{D} \psi(x, y) \]  

(2.35)

\[ \xi = \int_0^x \frac{r_0^2(x)}{D^2} \, dx \]  

(2.36)

\[ \eta = y \frac{r_0(x)}{D} \]  

(2.37)

(b) It has already been shown that for the present case, the unique form of the transformation, equations (2.35), (2.36) and (2.37) are determined by requirements on the differential equation alone. Thus, the boundary conditions do not provide any additional information and in fact, they are found to carry over directly as follows:

\[ \eta = 0 : \quad U, V = 0 \]

\[ \eta \to \infty : \quad U \to U_1(\xi) \]  

(2.38)

The analysis is thus complete, Equations (2.35), (2.36) and (2.37) being known as the Mangler transformation.
2.5 CONCLUDING REMARKS:

It has been shown that the primitive transformation appears to be the most general class of transformations necessary to provide a transformation with a unique inverse. This result was obtained from the general mathematical theory of transformations, but was also supported by a physical argument for such cases as boundary-layer problems which show that the following transformations are possibly sufficient to ensure proper behavior of the equations. 

\[
\Psi(\xi, \eta) = g(x) \psi(x, y); \quad \xi = \xi(x), \quad \eta = y \gamma(x)
\]

The generalized similarity analysis was then introduced to solve a wide class of problems which could be formulated as a comparison between two given sub problems. Two examples were given, the Blasius similarity problem and a derivation of the Mangler transformation, however all of the cases given in Table - 2.1 can be derived in similar fashion.

A few comments can be made concerning the role of the function \( g(x) \) appearing in the transformation of the dependent variable for the boundary-layer equations. In certain cases, the value of \( g(x) \) will be uniquely determined by the problem. For example, \( g(x) \) is proportional to \( \sqrt{x} \) and \( \sqrt{\xi(x)} \) for the similarity and Meksyn-Görtler transformations, respectively and \( g \) is found to be constant for the Mangler transformation. However, in some cases, the choice for \( g \) is arbitrary; for example, \( g \) may take on any value for the von Mises transformation. Further, for the case of comparing compressible and in-compressible forms of the boundary-layer equations, different choices for \( g \) lead to different, but nevertheless useful, results: Stewartson chooses \( g = \text{constant} \) and Dorodnitsyn chooses \( g = \eta_y = \gamma(x) \). Coles discusses the significance of \( g \) for the difficult problem of compressible-incompressible transformations of the boundary-layer equations for turbulent flow. (Refer. Coles (1962)).
In summary, the ideas presented in this chapter are based on, or more appropriately, motivated by a physical description of a problem which is in some sense complete. Depending on the particular case under consideration, completeness may imply knowledge of all necessary boundary conditions or possibly only a statement of a particular requirement of the transformation. In any case, a fairly specific problem formulation is implied.

In the next chapter, a different technique will be examined which focuses attention on a more narrow applications; the simple similarity problem (in the sense of reducing the number of independent variables). This technique, the group theory method being less encumbered by statements of broad generality will prove to yield very elegant and powerful mathematical results for finding similarity solutions for a wide range of applications.