I.1 L-embedded Banach space

Definition I.1. (L-Projection) [HWW93]: A linear projection $P$ on a Banach space $X$ is called an L-projection if $||x|| = ||Px|| + ||x - Px||$ for all $x \in X$.

The set of all L-projections is a complete Boolean algebra of projections, hence abstractly a spectral algebra. The notion of L-structure is useful in the theory of vector valued measures.

Proposition I.2. [HWW93]: Let $X$ be a Banach space. If $P$ is an L-projection on $X$ and $E$ is a contractive projection on $X$ with the same kernel, then $P = E$. 

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Definition I.3. [HWW’93]: A closed subspace $J \subset X$ is called an L-summand if it is the range of an L-projection.

Notation I.4. $X \oplus_1 Y$ denotes the direct sum of two Banach spaces equipped with $l^p$-norm.

Note I.5. [HWW’93]: If $J$ is an L-summand in $X$, then there is uniquely determined closed subspace $J'$ such that $X = J \oplus_1 J'$, and $J'$ is called complementary L-summand.

Note I.6. [HWW’93]: If $X$ is a Banach space and $J$ is a closed subspace of $X$ such that $X = J \oplus J'$, then there exists an L-projection on $X$ with range $J$. That is $J$ is L-summand in $X$.

Definition I.7. (L-embedded Banach space) [HWW’93]: An L-embedded Banach space is a Banach space which is L-summand on its bidual.

Note I.8. [HWW’93]: $X$ is an L-embedded Banach space if and only if there exists a closed subspace $X_s$ in the bidual $X^{**}$ such that $X^{**} = X \oplus_1 X_s$.

Examples of such L-embedded spaces include $l^1(\Gamma) = (c_0(\Gamma))^* \ (\Gamma \text{ any set}), N(H) = (K(H))^*(\text{the nuclear operators on a Hilbert space } H), (K(l^p,l^q))$
where $1 < p \leq q < \infty$, the Hardy space $H^0_1 = (C(T)/A)$, where $A$ is the disk algebra.

**Theorem I.9.** [HWW’93]: $L$-embedded Banach spaces are weakly sequentially complete.

**Lemma I.10.** [HWW’93]: For an $L$-embedded Banach space $X$ (with $L$-decomposition $X'' = X \oplus X_s$ and $L$-projection $P$ on $X''$ with range $X$) and a closed subspace $Y$ of $X$ the following assertions are equivalent.

(i) $Y$ is $L$-embedded.

(ii) $Y^{\perp\perp} = Y \oplus_1 (Y^{\perp\perp} \cap X_s)$.

(iii) $P\overline{B_Y^w} = B_Y$, where $B_Y$ is the closed unit ball of $Y$.

(iv) $PY^{\perp\perp} = Y$.

In particular if $Y$ is $L$-embedded and if one identifies $Y'' = Y \oplus_1 Y_s$ and $Y^{\perp\perp} \subset X''$ then $Y_s = Y^{\perp\perp} \cap X_s$.

**Definition I.11.** (Abstract measure topology) [Pft00a]: Let $X$ be a Banach space. A system $\tau_\mu$ of subsets of $X$ is called an abstract measure topology if it satisfies the following four conditions.

1. $(X, \tau_\mu)$ is a sequential space in which every convergent sequence has a
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unique limit.

2. $\tau_\mu$ is weaker than the norm topology.

3. $\tau_\mu$ is translation invariant for sequences more precisely, $x_n \xrightarrow{\tau_\mu} x$ if and only if $x_n - x \xrightarrow{\tau_\mu} 0$ for any sequence $(x_n)$ in X.

4. Each bounded sequence in X that spans $l^1$ asymptotically $\tau_\mu$-converges to 0, and each sequence in X that $\tau_\mu$-converges to zero is bounded and contains a subsequence which spans $l^1$ asymptotically or tends to 0 in norm.

Theorem I.12. [Pft00a]: Every L-embedded Banach space admits an abstract measure topology.

Theorem I.13. [Pft00a]: Let X be an L-embedded Banach space with abstract measure topology $\tau_\mu$. Then the following statements hold.

(a) A sequence converges in norm if and only if it converges both weakly and with respect to $\tau_\mu$, and all limits coincide.

(b) A norm closed subspace $Y \subset X$ is reflexive if and only if $\tau_\mu$ and the norm topology coincide on the unit ball of $Y$.

Theorem I.14. [Pft00a]: Let X be an L-embedded Banach space endowed with its abstract measure topology $\tau_\mu$. Then a norm closed subspace $Y \subset X$ is
L-embedded if and only if its unit ball $B_Y$ is $\tau_\mu$-closed.

**Lemma I.15.** [Pft00a]: If a Banach space $X$ admits an abstract measure topology $\tau_\mu$ then $\tau_\mu$ has the following properties.

(a) $(X, \tau_\mu)$ is a $T_1$-space.

(b) The relative topology of $\tau_\mu$ on a subspace of $X$ is again an abstract measure topology.

(c) Closedness and sequential closedness coincide for $\tau_\mu$. Sequentially continuous maps on $X$ are continuous.

(d) If $X$ does not contain a copy of $l_1$ the norm topology is an abstract measure topology and is the only one.

(e) $\tau_\mu$ is unique.

(f) If $X$ is the predual of a finite von Neumann algebra then $\tau_\mu$ coincides on bounded sets with the usual measure topology; on unbounded sets it does not in general.

(g) Multiplication by scalars is $\tau_\mu$-continuous.

(h) If $X$ is L-embedded (i.e. $X^* = X \oplus_1 X_s$) and if a net $(x_\gamma)$ in $X$ w*-converges to $x_s \in X_s$ such that $\|x_\gamma\| \leq \|x_s\|$ then $x_\gamma \xrightarrow{\tau_\mu} 0$
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(i) If $X$ is $L$-embedded then for any $x'' \in \overline{B_X}$ there exists a net $(x_\gamma)$ in $B_X$ such that both $x_\gamma \xrightarrow{\tau_\mu} 0$ and $x_\gamma \xrightarrow{w^*} x''$

Note I.16. [Pft00a]: Norm topology is finer than abstract measure topology.

Theorem I.17. [Pft00a]: Closedness and sequential closedness coincide for $\tau_\mu$.

Definition I.18. [Pft00a]: Every bounded sequence in an $L$-embedded Banach space admits a sequence of convex combinations which converges with respect to the measure topology.

Theorem I.19. [Pft00a]: $(X, \tau_\mu)$ is a topological vector space.

Theorem I.20. [Pft00b]: A Banach space that is isomorphic to a subspace of a separable $L$-embedded space is the unique predual of its dual.

Theorem I.21. [JP02]: Let $X$ be an $L$-embedded Banach space. If a bounded sequence $(x_n)$ converges to $0$ in abstract measure topology then

$$\limsup_{n \to \infty} \| x + x_n \| = \| x \| + \limsup_{n \to \infty} \| x_n \| \text{ for every } x \in X.$$ 

Theorem I.22. [Pft00b]: Almost isometric copies of $l_1$ which are subspaces of $L$-embedded Banach spaces are $L$-embedded.
Theorem I.23. \([Pft00b]\): Every non reflexive subspace of an \(L\)-embedded Banach space fails the fixed point property.

I.2 Generalized 2-normed Space

In 1963, S.Gahler introduced the concept of 2-metric spaces in his paper entitled “2-metric Spaces and their topological structures”. Later in 1965, he introduced the concept of 2-normed space as a special class of 2-metric spaces, which are linear and defined on them a 2-norm in his article entitled “Linear 2-normed spaces”. He investigated on the topological properties of the spaces and also proved that linear 2-normed spaces are normable and uniformizable provided the dimension of the space is greater than one. In fact, if \(X\) is a linear 2-normed space with \(a\) and \(b\) being linearly independent vectors in \(X\), then \(\|x\| = \|x, a\| + \|x, b\|\) defines a norm on \(X\). Conversely if the space is a linear normed space, then it is possible to define a 2-norm on it. The properties of 2-normed space were suggested by the area function for a triangle determined by a triple in Euclidean space. Every 2-normed space is a locally convex topological vector space. A white defined cauchy sequences and
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convergent sequences in such spaces, which lead to the theory of 2-Banach spaces, where he proved that every 2-dimensional linear 2-normed spaces is a 2-Banach space. He also considered linear 2-functionals (bilinear functionals) and their properties.

The concept of generalized 2-normed space was introduced by Zofia Lewandowska in the year 1999, as a generalization of the 2-normed space given by Gahler. There are appropriate connection between normed spaces and generalized 2-normed spaces - a generalized 2-normed space can be obtained from any normed space, each 2-normed space is a generalized 2-normed space and each generalized 2-normed space is a semi normed space. Z. Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces in 1999-2003. Zofia also proved that the space of all bounded 2-linear operators from a generalized 2-normed space into a generalized symmetric sequentially complete 2-normed space is a Banach space. Consequently the collection of bounded bilinear functionals on a generalized 2-normed space is a Banach space.

There are some works on characterization of 2-normed spaces, extension of bi-
linear functionals and approximation in 2-normed spaces. Banach-Steinhaus theorem for a bounded bilinear operator on a 2-normed space is also proved by Zofia. Sh.Rezapour and Mehmet Acikgoz have done some works in $\epsilon$-approximation theory and proximinal approximation in generalized 2-normed spaces. Rezapour studied 2-Proximinal subspace and 2-best approximation of a point in a subspace in Generalized 2-Normed Spaces. We begin with some basic definitions and facts.

**Definition I.24. (2-normed Space)** [FYJC01]: Let $X$ be a real linear space of dimension greater than one. Suppose $\|.,.\|$ is a non-negative real valued function on $X \times X$ satisfying the following conditions:

1. $\|x,y\| = 0$ if and only if $x$ and $y$ are linearly dependent,
2. $\|x,y\| = \|y,x\|$, for all $x,y \in X$,
3. $\|\lambda x,y\| = |\lambda|\|x,y\|$, for all $\lambda \in \mathbb{R}$ and for all $x,y \in X$,
4. $\|x + y,z\| \leq \|x,z\| + \|y,z\|$.

Then $\|.,.\|$ is called a 2-norm on $X$ and the pair $(X,\|.,.\|)$ is called a 2-normed space.

Some of the basic properties of 2-norms are that they are non-negative and
\[ \| x, y + \alpha x \| = \| x, y \|, \forall x, y \in X \text{ and } \forall \alpha \in \mathbb{R}. \]

**Example I.25.** For \( X = \mathbb{R}^3 \), define \( \| x, y \| = \max\{ |x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1|, |x_1y_2 - x_2y_1| + |x_2y_3 - x_3y_2| \} \), where \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \in \mathbb{R}^3 \). Then \( \|.,.\| \) is a 2-norm on \( \mathbb{R}^3 \).

**Example I.26.** Let \( P_n \) denotes the set of all real polynomials of degree \( \leq n \), on the interval \([0,1]\). By considering usual addition and scalar multiplication, \( P_n \) is a linear vector space over the reals. Let \( \{x_0, x_1, ..., x_{2n}\} \) be distinct fixed points in \([0,1]\). Let \( f, g \in P_n \).

Define \( \|f, g\| = 0 \), if \( f \) and \( g \) are linearly dependent.

\[
= \sum_{k=0}^{2n} |f(x_k)g(x_k)|, \text{if } f \text{ and } g \text{ are linearly independent.}
\]

Then \( (P_n, \|.,.\|) \) is a 2-normed space.

**Definition I.27.** (Generalized 2-normed Space) [Lew99]: Let \( X \) and \( Y \) be a real linear space of dimension greater than 1. Denote by \( D \), a non-empty subset of \( X \times Y \) such that for every \( x \in X, y \in Y \), the sets \( D_x = \{ y \in Y : (x, y) \in D \} \) and \( D_y = \{ x \in X : (x, y) \in D \} \) are linear subspaces of the spaces \( Y \) and \( X \) respectively. The function \( \|.,.\| : D \to [0, \infty) \) will be called a generalized 2-norm on \( D \) if it satisfies the following conditions:
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1. \( \|x, \alpha y\| = |\alpha| \|x, y\| \) for any real number \( \alpha \) and all \( (x, y) \in D \),

2. \( \|x, y + z\| \leq \|x, y\| + \|x, z\| \) for \( x \in X, y, z \in Y \)
   such that \( (x, y), (x, z) \in D \),

3. \( \|x + y, z\| \leq \|x, z\| + \|y, z\| \) for \( x, y \in X, z \in Y \)
   such that \( (x, z), (y, z) \in D \).

The set \( D \) is called a 2-normed set.

In particular, if \( D = X \times Y \) the function \( \|\cdot,\cdot\| \) will be called a generalized
2-norm on \( X \times Y \) and the pair \( (X \times Y, \|\cdot,\cdot\|) \) generalized 2-normed space.

Moreover, if \( X = Y \) then the generalized 2-normed space will be denoted by
\( (X, \|\cdot,\cdot\|) \).

**Example I.28.** [Lew99]: Let \( X \) be a real linear space having two norms
(semi norms) \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \) and then \( X \times X \) is a generalized 2-normed Space
with the 2-norm defined by \( \|x, y\| = \|x\|_1 \|y\|_2 \).

In particular, every normed space (semi normed space) is a generalized
2-normed space.

**Example I.29.** [Aci09]: Let \( A \) be a Banach algebra and \( \|a, b\| = \|ab\| \) for all
\( a, b \in A \). In particular every Banach algebra is a generalized 2-normed space.
Example I.30. [Lew99]: Let \((X, \langle ., . \rangle)\) be a real inner product space.

Then \(X\) is a generalized 2-normed space with the 2-norm: \(\|x,y\| = |\langle x, y \rangle|\).

These examples show us that theory of generalized 2-normed spaces includes theory of normed spaces and theory of Banach algebras.

Example I.31. In the space \(\mathbb{R}^2\), \(\|(x_1, x_2), (y_1, y_2)\|_1 = |x_1y_1 + x_2y_2|\) and \(\|(x_1, x_2), (y_1, y_2)\|_2 = |x_1y_2 - x_2y_1|\) define two generalized 2-normed spaces.

Example I.32. [Lew99]: Let \(X\) and \(Y\) be two real linear spaces and let \(g: X \times Y \to \mathbb{R}\) be a functional satisfying the following properties.

1. the functional is linear with respect to the both variables;
2. if \(g(x, y) = 0\) for all \(y \in Y\), then \(x = \theta\);
3. if \(g(x, y) = 0\) for all \(x \in X\), then \(y = \theta\). Then the pair \((X \times Y, \|., .\|)\) is a generalized 2-normed space with the 2-norm defined by the formula \(\|x, y\| = |g(x, y)|\).

Assume that the generalized 2-norm satisfies, in addition, the symmetry condition. Then generalized symmetric 2-norm is defined as follows:

Definition I.33. Let \(X\) be a real linear space. Denote by \(X\) non-empty
subset $X \times X$ with the property $\chi = \chi^{-1}$ and such that the set $\chi_y = \{x \in \chi : (x, y) \in \chi\}$ is a linear subspace of $X$, for all $y \in X$. A function $\|.,.\| : \chi \to [0, \infty)$ satisfying the following conditions:

1. $\|x, y\| = \|y, x\|$ for all $(x, y) \in \chi$

2. $\|x, \alpha y\| = |\alpha|\|x, y\|$ for any real number $\alpha$ and all $(x, y) \in \chi$,

3. $\|x, y + z\| \leq \|x, z\| + \|y, z\|$ for $x, y, z \in X$ such that $(x, y), (x, z) \in \chi$

will be called a generalized symmetric 2-norm on $\chi$.

The set $\chi$ is called a symmetric 2-normed set.

In particular, if $\chi = X \times X$, the function $\|.,.\|$ will be called a generalized symmetric 2-norm on $X$ and the pair $(X; \|.,.\|)$ - a generalized symmetric 2-normed space.

**Example I.34.** If $\|.,\|_1 = \|.,\|_2$ ($= \|.,\|$) in Example [I.28], then $(X, \|.,\|)$ is a generalized symmetric 2-normed space with the symmetric 2-norm defined by the formula $\|x, y\| = \|x\|\|y\|$ for each $x, y \in X$.

**Theorem I.35.** [Lew99]: Let $(X \times Y; \|.,\|)$ be generalized 2-normed space. Then a family $\mathcal{B}$ of all sets defined by $\bigcap_{i=1}^{n} \{x \in X; \|x, y_i\| < \epsilon\}$, where $y_1, y_2, \ldots, y_n \in Y, n \in \mathbb{N}$ and $\epsilon > 0$, forms a complete system of neighbourhoods
of zero for a locally convex topology in $X$.

**Theorem I.36.** [Lew99]: Let $(X \times Y, \|., .\|)$ be generalized 2-normed space. Then the family $B$ of all sets defined by $\cap_{i=1}^{n} \{ y \in Y; \|x_i, y\| < \epsilon \}$ where $x_1, x_2, x_3, \ldots, x_n \in X, n \in N$ and $\epsilon > 0$ forms a complete system of neighbourhoods of zero for a locally convex topology in $Y$.

**Notation I.37.** We will denote the above topologies by the symbols $\tau(X,Y)$ and $\tau(Y,X)$ respectively.

In the case when $X = Y$ we will write: $\tau_1(X) = \tau(X;Y)$ and $\tau_2(X) = \tau(Y;X)$.

**Note I.38.** In $(X \times Y, \|., .\|)$ we can consider the topology $\tau(X,Y) \times \tau(Y,X)$.

**Note I.39.** Topologies generated by the norms in [I.24] coincide with the topologies $\tau_1(X)$ and $\tau_2(X)$ respectively.

**Note I.40.** [Lew01b]: If $B$ is a complete system of neighbourhoods of zero, then $B(a) = \{ a + u; u \in B \}$ is the complete system of neighbourhoods of $a$.

**Theorem I.41.** [Lew01b]: Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space.

1. The space $(X, \tau(X,Y))$ is a Hausdorff space if and only if
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\[ \forall y \in Y \| x, y \| = 0 \iff x = 0 \]

2. The space \((Y, \tau(Y, X))\) is a Hausdorff space if and only if

\[ \forall x \in X \| x, y \| = 0 \iff y = 0. \]

**Theorem I.42.** [Lew01b]: Let \((X \times Y, \| ., . \| )\) be a generalized 2-normed space.

Let \( \sum \) be a directed set.

(1) A net \( \{ x_\sigma; \sigma \in \sum \} \) is convergent to \( x_0 \in X \) in \((X, \tau(X, Y))\) if and only if for all \( y \in Y \) and \( \epsilon > 0 \) there exists \( \sigma_0 \in \sum \), such that \( \| x_\sigma - x_0, y \| < \epsilon \) for all \( \sigma \geq \sigma_0 \).

(2) A net \( \{ y_\sigma; \sigma \in \sum \} \) is convergent to \( y_0 \in Y \) in \((Y, \tau(Y, X))\) if and only if for all \( x \in X \) and \( \epsilon > 0 \) there exists \( \sigma_0 \in \sum \), such that \( \| x, y_\sigma - y_0 \| < \epsilon \) for all \( \sigma \geq \sigma_0 \).

**Definition I.43.** [Lew99]: Let \((X \times Y, \| ., . \| )\) be a generalized 2-normed space. A sequence \( \{ x_n; n \in N \} \subset X \) is called a Cauchy sequence if for every \( y \in Y \) and \( \epsilon > 0 \) there exists \( n_0 \in N \) such that the inequality \( n, m > n_0 \) implies \( \| x_n - x_m, y \| < \epsilon \).

**Definition I.44.** [Lew99]: Let \((X \times Y, \| ., . \| )\) be a generalized 2-normed space. A space \((X, \tau(X, Y))\) is called sequentially complete if every Cauchy
sequence in $X$ is convergent in this space.

By analogy we obtain definitions of Cauchy sequence in $Y$ and sequentially complete space $(Y, \tau(Y, X))$.

**Definition I.45.** [Lew99]: A generalized 2-normed space $(X \times Y, \|., .\|)$ is called sequentially complete space if spaces $(X, \tau(X, Y))$ and $(Y, \tau(Y, X))$ are sequentially complete spaces.

**Definition I.46.** (Bilinear mapping) [Lew01a]: Let $D \subset X \times X$ be a 2-normed set, $Y$ a normed space. A map $f : D \to Y$ is called a bilinear if it satisfies the conditions

1. $f(a + c, b + d) = f(a, b) + f(a, d) + f(c, b) + f(c, d)$ for $a, b, c, d \in X$ such that $a, c \in D^b \cap D^d$,

2. $f(\alpha a, \beta b) = \alpha \beta f(a, b)$ for $\alpha, \beta \in C$ and $(a, b) \in D$.

**Definition I.47.** [Lew01a]: A bilinear map $f$ is said to be bounded if there exists a non negative real number $M$ such that $\|f(a, b)\| \leq M\|a, b\|$ for all $a, b \in D$. The norm of the bilinear map $f$ is defined by

$$\|f\| = \inf \{k > 0; \|f(a, b)\| \leq k\|a, b\| \text{ for } (a, b) \in D\}.$$

**Example I.48.** [Lew01a]: Consider a real inner product space $(X, <.,.>)$
with the 2-norm $\|x, y\| = |<x, y>|$. An operator $f : X \times Y \to R$ defined
by $f(a, b) = <x, y>$. Then $f$ is bilinear and bounded. Moreover $\|f\| = 1$.

**Notation I.49.** Denote by $L_2(D; Y)$ the set of all bounded 2-linear maps
from $D$ into $Y$.

In particular, we will write $L_2(X; Y)$, if $X$ is a generalized 2-normed space.

**Theorem I.50.** [Lew01a]: Let $D \subset X \times X$ be a 2-normed set, $Y$ a normed
space, the set of all bounded bilinear operators from $D$ into $Y$, $L_2(D; Y)$ is a
Banach space with respect to the norm

$||f|| = \inf\{k > 0; ||f(a, b)|| \leq k\|a, b\|,(a,b) \in D\}$.

**Corollary I.51.** [Lew01a]: If $X$ is a symmetric 2-normed set, $Y$ is a Banach
space, then $L_2(X, Y)$ is symmetric sequentially complete 2-normed space with
the 2-norm $\|f, g\| = \|f\| \cdot \|g\|$ for $f, g \in L_2(X, Y)$.

**Theorem I.52.** [LMSMASM06]: Let $(X, \|., .\|)$ be a generalized 2-normed
space and $M$ be a linear subspace of $(X, \|., .\|)$. If $F_0$ is a real bounded bilinear
functional on $M$, then there exists a real bounded bilinear extension $F$ on $X$
such that $F(a, b) = F_0(a, b)$ for all $a, b \in M$, and $\|F\| = \|F_0\|$.
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Theorem I.53. \cite{LMSMASM06}: Let \( x_0, y_0 \) be vectors in the generalized 2-normed space \((X, \| . \|)\) such that \( \| x_0, y_0 \| \neq 0 \). Then there exists a real bounded 2-linear functional \( F \), defined on the whole space, such that \( F(x_0, y_0) = \| x_0, y_0 \| \) and \( \| F \| = 1 \).

Theorem I.54. Every generalized 2 normed space is semi-normed space.

I.3 Ordered Normed Space

Ordered vector spaces were developed in the beginning of twentieth century in parallel with functional analysis and operator theory. F.Riesz, H.Freudenthal, L.V.Kantorovitch, Kakutanic, Krein brothers, H Nakanao, H H Schafer and Zannen initiated the study of ordered linear spaces in late 1930’s. This theory was developed into a mathematical discipline around 1950’s and now it is one of the most important branches of functional analysis being effectively used to solve such problems, which are posed in more general setting. It is an indispensable tool for studying a variety of problems in engineering and economics. Several authors-Leonard Asimow, Kung-Fu Ng, Jan A.Van Casteren, Bonsall, Jameson, Dalen, Cristescu, Martin, Karim and Hong have
studied ordered linear spaces and have made significant contributions in this direction. In every ordered vector space, the set of vectors that dominates zero is a cone called the positive cone. Conversely every cone induces a vector ordering on the vector space in which it is the positive cone. Let \( H \) be a Hilbert space. The totality of bounded self-adjoint operators on \( H \) is denoted by \( S(H) \). Let \( P \) be the set of all positive-definite operators on \( H \). Then \( P \) is a positive cone in \( S(H) \). Furthermore, \( P - P = S(H) \). Hence, \( S(H) \) is a partially ordered linear space. If \( E \) is a linear space, \( E^* \) is the dual space and if \( C \) is any cone in \( E \), \( C^* = \{ \lambda : \lambda \in E^*, \lambda(c) \geq 0, \text{ for any } c \in C \} \) is called the dual cone. When a real vector space \( Z \) is endowed with a partial ordering, induced by a cone that makes \( Z \) a lattice, we call it a vector lattice. Some useful concepts in this area are listed below.

**Definition I.55.** [Con01]: Let \( X \) be a vector space. A cone of \( X \) is a nonempty subset of \( X \) for which \( \alpha A \subset A \), whenever \( \alpha \in R_+ \).

**Definition I.56.** [Con01]: The cone \( A \) is called sharp if \( A \cap -A = 0 \).

**Definition I.57.** [Con01]: The cone \( A \) is convex if and only if \( A + A \subset A \).

**Example I.58.** The empty set, the space \( V \), and any linear subspace of \( V \)
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(including the trivial subspace 0) are convex cones. Other examples are the set of all positive multiples of an arbitrary vector $v$ of $V$.

**Definition I.59.** [ART07] (Lexicographic Order): Given two posets $(A_1, R_1)$ and $(A_2, R_2)$ we construct an induced partial order $R$ on $A_1 \times A_2$ as $< x_1, y_1 > R < x_2, y_2 >$ if and only if $x_1 R_1 x_2$ or if $x_1 = x_2$ and $y_1 R_2 y_2$.

**Definition I.60.** [Con01]: An ordered vector space is a vector space $X$ with an order relation $\leq$ such that given $x, y, z \in X$ and $\alpha \in R_+$, $x + z \leq y + z$ and $\alpha x \leq \alpha y$ whenever $x \leq y$.

**Note I.61.** Defining $X_+ = \{x \in X / x \geq 0\}$, we call the elements of $X_+$ positive and elements of $X_-$ negative.

**Example I.62.** The set of all real numbers with the usual order is an ordered vector space.

**Example I.63.** $R^2$ with lexicographic order is an ordered vector space.

**Example I.64.** $R^2$ is an ordered vector space with product order (ie, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$)
Theorem I.65. [Con01]: If $X$ is an ordered vector space then $X^+$ is a sharp convex cone in $X$.

Theorem I.66. [Con01]: Let $X$ be a vector space and $C$ a sharp convex cone in $X$. Then there is a unique order relation $\leq$ on $X$, with respect to which $X$ is an ordered vector space with $X_+ = C$ and the associated partial ordering of $Z$ is $x \leq y$ or $y \geq x$ whenever $y-x \in C$, and $x < y$ or $y > x$ whenever $0 \neq y-x \in C$.

Definition I.67. [Con01]: An ordered normed space is an ordered vector space $X$ endowed with a norm with respect to which $X_+$ is closed.

Definition I.68. [Con01]: An ordered Banach space is an ordered normed space with a complete norm.

Example I.69. : If $T$ is a topological (measurable)space, then $C(T) = \{x \in l^\infty(T)/x$ is continuous\}$ endowed with usual order is an ordered Banach space.

Theorem I.70. [ART07] If $X$ is an ordered vector space then $X \times X$ is an ordered vector space with Lexicographic order(or with product order).
Definition I.71. [ART07]: Let $X$ be an ordered vector space and $K$ a cone in $X$. Then $u \in K$ is called an order unit for $K$ if for each $x \in X$ there is some $\lambda > 0$ such that $x \leq \lambda u$.

Notation I.72. An ordered vector space $X$ with an order unit $u$ is denoted by $(X, u)$.

Note I.73. If $u$ is the order unit in $X$, then $(u, u)$ is the order unit in $X \times X$. 