CHAPTER 4

SOME APPLICATIONS OF HYPERGEOMETRIC FUNCTIONS AND FRACTIONAL CALCULUS OPERATOR TO UNIVALENT AND MULTIVALENT FUNCTIONS
CHAPTER 4

Some Applications of Hypergeometric Functions and Fractional Calculus Operator to Univalent and Multivalent Functions

4.1 Introduction

We breakup this Chapter into two main Sections. The first Section dealing with the study of hypergeometric functions and Fox-Wright generalized hypergeometric functions applied to both univalent and multivalent functions. H. M. Srivastava [24], H. M. Srivastava and S. Owa [23], S. Ponnusamy [17], J. M. Jahangiri [11] have put on remarkable contributions in the study of hypergeometric functions and their applications to univalent starlike, convex and other functions. The hypergeometric integral is any integral which has hypergeometric function in its integral. Because of the complicated look of hypergeometric functions, the study was overlooked. The functions in these integrals range from simple Bessel and Whittaker functions upto generalized hypergeometric functions of several variables with many parameters. There are many types of hypergeometric functions in literature, Gauss series $\,_{2}F_{1}(a, b; c; z)$ with one variable and three parameters. All of these quantities may be any numbers, real or complex. This may be further extended to give rise to the multiple hypergeometric functions of many variables. The term “hypergeometric” was first used by Wallis in his work Arithmetica Infinito-
rum published in the year 1655, when he was a Professor at the University of Oxford.

Here we obtain coefficient estimates, distortion and growth bounds, closure results, neighbourhoods and integral operators for various subclasses, we also obtain some properties of subclass of holomorphic function for some convolution operator on Hilbert space by using the Fox-Wright generalized hypergeometric functions.

The Section 2 is interwoven with study of fractional calculus operator including both fractional derivative and fractional integral. The eminent mathematicians like P. K. Banerji, L. Debnath and G. M. Shenan [5], Y. C. Kim et. al. [14], H. M. Srivastava and S. Owa [23] have contributed a lot of results to study of univalent and multivalent functions using fractional calculus technique.

Fractional calculus is the field of mathematical analysis which deals with the investigation and application of integrals and derivatives of arbitrary order. The fractional calculus is an old subject and yet novel topic, it is an old topic starting from some speculations of G. W. Leibniz (1695) and L. Euler (1730), it has been developed upto nowadays. N number of mathematicians have provided important contributions up to middle of our century, includes P. S. Laplace (1812), J. B. J. Fourier (1822), N. H. Abel (1823), J. Liouville (1832), B. Riemann (1847), D. V. Widder (1941), M. Riesz (1949). In recent years considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering.
Our motivation stems from the results obtained earlier by these above authors.

Firstly, we have tried to obtain the application of fractional calculus for univalent functions with negative coefficients. Here we have generalized the results obtained by R. Aghalary and S. R. Kulkarni [1], by introducing the generalization, we have obtained the results which are routine, we have also extended the study of fractional calculus in particular fractional derivative to these subclasses of \( n \)-uniformly multivalent functions. We conclude this section by obtaining the varieties of the results having geometrical importance.

SECTION 1

4.2 Some Properties on Hypergeometric Functions

Let \( \mathcal{A}(p) \) denote the class of all holomorphic functions \( f(z) \) in the unit disk \( \mathcal{U} = \{ z : |z| < 1 \} \) defined by

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (a_k \geq 0, p \in \mathbb{N}) \tag{4.1}
\]

and let

\[
g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (b_k \geq 0)
\]

be fixed multivalent holomorphic function in \( \mathcal{U} \). Define the class \( MNR(r, s, \delta) \) by

\[
MNR(r, s, \delta) = \left\{ f \in \mathcal{A}(p) : Re \left( \frac{z(H_r^s[\alpha_1](f \ast g)(z))'}{H_r^s[\alpha_1](f \ast g)(z)} \right) < p\delta, (1 < \delta < 1 + \frac{1}{2p}, z \in \mathcal{U}) \right\}. \tag{4.2}
\]
where

\[ H_r^*[\alpha_1](f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} a_k b_k z^k, \quad [8] \]  

with \( H_r^*[\alpha_1]f(z) \) defined in (0.16).

The generalized hypergeometric function has been studied by many mathematicians in univalent function theory like de Branges [8] Merkes-Scott [15], Carlson-Shaffer [7] and Ruscheweyh - Sing [21]. Ahuja and Jahangiri [2] studied generalized hypergeometric functions under the above operator on univalent function on the certain classes.

Here we find the necessary and sufficient condition for function to be in the class \( MNR(r, s, \delta) \).

**Theorem 4.2.1** : A function \( f(z) \in MNR(r, s, \delta) \) if and only if

\[ \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} (k - p\delta) a_k b_k z^k < p(\delta - 1), \]  

where \( 1 < \delta < 1 + \frac{1}{2p} \).

**Proof** : Let \( f \in MNR(r, s, \delta) \). Then by (4.2) we have

\[ \text{Re} \left( p z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} k a_k b_k z^k \right) < p\delta. \]

By letting \( z \to 1^- \) through real values, we obtain

\[ p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} k a_k b_k \]

\[ 1 + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} a_k b_k \]

then

\[ \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} (k - p\delta) a_k b_k < p(\delta - 1). \]
Conversely, assume that (4.4) holds true, then we have

\[
\left| \frac{z(H'_s[a_1](f \ast g)(z)') - pH'_s[a_1](f \ast g)(z)}{z(H'_s[a_1](f \ast g)(z)') - (2\delta - 1)pH'_s[a_1](f \ast g)(z)} \right| \leq \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p+1})_{k-p}} (k - p)a_kb_k
\]

or equivalently \( f(z) \in MNR(r, s, \delta) \). \( \square \)

**Theorem 4.2.2** : Let \( f(z) \in MNR(r, s, \delta) \). Then

\[
a_k \leq \frac{p(\delta - 1)}{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}(k - p\delta)b_k
\]

The result is sharp for function of the form

\[
f_k(z) = z^p + \frac{p(\delta - 1)}{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}(k - p\delta)b_k \]

\( k = p + 1, p + 2, \ldots \). \( (4.6) \)

**Proof** : Since \( f(z) \in MNR(r, s, \delta) \) by assumption, then we have by (4.4)

\[
\frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p+1})_{k-p}} (k - p\delta)a_kb_k \\
\leq \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p+1})_{k-p}} (k - p\delta)a_kb_k \leq p(\delta - 1),
\]

or

\[
a_k \leq \frac{p(\delta - 1)}{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}(k - p\delta)b_k(\beta_1)_{k-p} \cdots (\beta_s)_{k-p+1})_{k-p}.
\]

It is clear that for

\[
f_k(z) = z^p + \frac{p(\delta - 1)}{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}(k - p\delta)b_k(\beta_1)_{k-p} \cdots (\beta_s)_{k-p+1})_{k-p}z^k \in MNR(r, s, \delta)
\]

for \( k = p + 1, p + 2, \ldots \), we have

\[
a_k = \frac{p(\delta - 1)}{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}(k - p\delta)b_k(\beta_1)_{k-p} \cdots (\beta_s)_{k-p+1})_{k-p}.
\] \( \square \)
Now we obtain the bounds and growth theorems for \( f(z) \in MNR(r, s, \delta). \)

**Theorem 4.2.3** : Let \( f(z) \in MNR(r, s, \delta). \) Then

\[
r^p \frac{p(\delta - 1)}{\alpha_1 \alpha_2 \cdots \alpha_r (p(1 - \delta) + 1)b_{p+1}} r^{p+1} \leq |f(z)| \leq r^p + \frac{p(\delta - 1)}{\alpha_1 \alpha_2 \cdots \alpha_r (p(1 - \delta) + 1)b_{p+1}} r^{p+1},
\]

for \( |z| = r < 1, \) provided \( b_k \geq b_{p+1} \geq 1. \)

The result is sharp for

\[
f(z) = z^p + \frac{p(\delta - 1)}{\alpha_1 \alpha_2 \cdots \alpha_r (p(1 - \delta) + 1)b_{p+1}} z^{p+1}.
\]

**Proof** : Since \( f(z) \in MNR(r, s, \delta) \) by assumption, then by using (4.4) with

\[
\left( \frac{\alpha_1 \alpha_2 \cdots \alpha_r}{\beta_1 \beta_2 \cdots \beta_s} (p(1 - \delta) + 1)b_{p+1} \right) \leq \left( \frac{\alpha_1}{\beta_1} k-p \alpha_2 k-p \cdots (\alpha_r k-p (k-p \delta) b_k, \right.
\]

we obtain

\[
\frac{\alpha_1 \alpha_2 \cdots \alpha_r}{\beta_1 \beta_2 \cdots \beta_s} (p(1 - \delta) + 1)b_{p+1} \sum_{k=p+1}^{\infty} a_k \leq p(\delta - 1),
\]

or

\[
\sum_{k=p+1}^{\infty} a_k \leq \frac{p(\delta - 1)}{\alpha_1 \alpha_2 \cdots \alpha_r (p(1 - \delta) + 1)b_{p+1}}. \tag{4.9}
\]

For \( |z| = r \) and by using (4.9) for the function \( f(z) \) defined by (4.1), we have

\[
|f(z)| \leq r^p + \sum_{k=p+1}^{\infty} a_k r^k \leq r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k \leq r^p + \frac{p(\delta - 1)}{\alpha_1 \alpha_2 \cdots \alpha_r (p(1 - \delta) + 1)b_{p+1}} r^{p+1},
\]

and similarly,

\[
|f(z)| \geq r^p - \frac{p(\delta - 1)}{\alpha_1 \alpha_2 \cdots \alpha_r (p(1 - \delta) + 1)b_{p+1}} r^{p+1}.
\]

Then the proof is complete. \( \square \)
**Theorem 4.2.4**: Let \( f(z) \in MNR(r, s, \delta) \). Then
\[
pr^{p-1} \frac{p(p+1)(\delta-1)}{\frac{1}{\beta_1} \frac{1}{\beta_2} \cdots \frac{1}{\beta_s} (p(1-\delta)+1) b_{p+1}} r^p \leq |f'(z)| \leq pr^{p-1} + \frac{p(p+1)(\delta-1)}{\frac{1}{\beta_1} \frac{1}{\beta_2} \cdots \frac{1}{\beta_s} (p(1-\delta)+1) b_{p+1}} r^p,
\]
(4.10)

\(|z| = r < 1\), provided \( b_k \geq b_{p+1} \). The result is sharp for the function given by (4.8).

**Proof**: The proof is simple and we avoid the details.

**Theorem 4.2.5**: Let \( F_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k \) \((j = 1, \cdots, n)\) be in \( MNR(r, s, \delta) \) and let \( \gamma_j \geq 0 \) for \( j = 1, \cdots, n \) and \( \sum_{j=1}^{n} \gamma_j \leq 1 \). Then
\[
f(z) = z^p + \sum_{k=p+1}^{\infty} \left( \sum_{j=1}^{n} \gamma_j a_{k,j} \right) z^k
\]
also in \( MNR(r, s, \delta) \).

**Proof**: By assumption we have \( F_j(z) \in MNR(r, s, \delta) \), then from Theorem 4.2.1, we obtain
\[
\sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} (k - \delta p) a_{k,j} b_k \leq p(\delta - 1)
\]
for every \( j = 1, \cdots, n \).

Hence
\[
\sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} (k - \delta p) b_k \left( \sum_{j=1}^{n} \gamma_j a_{k,j} \right)
\]
\[
= \sum_{j=1}^{n} \gamma_j \left( \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} (k - \delta p) a_{k,j} b_k \right) \leq p(\delta - 1).
\]

Therefore by Theorem 4.2.1, we conclude that \( f(z) \in MNR(r, s, \delta) \). \( \square \)

**Theorem 4.2.6**: Let \( G_p(z) = z^p \) and
\[
G_k(z) = z^p + \frac{p(\delta - 1)}{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}} \frac{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}}{(k - \delta p) b_k} z^k, \quad k = p+1, p+2, \cdots.
\]
Then \( f(z) \in MNR(r, s, \delta) \) if and only if \( f(z) \) can be expressed in the form
\[
f(z) = \gamma_p z^p + \sum_{k=p+1}^{\infty} \gamma_k G_k(z)
\]
such that \( \sum_{k=p}^{\infty} \gamma_k = 1 \) and \( \gamma_k \geq 0 \).

**Proof**: Let \( f(z) = \gamma_p z^p + \sum_{k=p+1}^{\infty} \gamma_k G_k(z) \). Then
\[
f(z) = \gamma_p z^p + \sum_{k=p+1}^{\infty} \gamma_k \left( z^p + \frac{p(\delta - 1)}{\beta_1 k-p-\cdots(\beta_s k-p(1) k-p)} (k-p\delta) b_k \right) z^k
\]
\[
= z^p + \sum_{k=p+1}^{\infty} \frac{\gamma_k p(\delta - 1)}{\beta_1 k-p-\cdots(\beta_s k-p(1) k-p)} (k-p\delta) b_k z^k,
\]
therefore
\[
\sum_{k=p+1}^{\infty} \frac{\gamma_k p(\delta - 1)}{\beta_1 k-p-\cdots(\beta_s k-p(1) k-p)} (k-p\delta) b_k z^k
\]
\[
= \sum_{k=p+1}^{\infty} \gamma_k = 1 - \gamma_p \leq 1.
\]

So by Theorem 4.2.1, we have \( f(z) \in MNR(r, s, \delta) \).

Conversely, let \( f(z) \in MNR(r, s, \delta) \). Then
\[
a_k \leq \frac{p(\delta - 1)}{\beta_1 k-p-\cdots(\beta_s k-p(1) k-p)} (k-p\delta) b_k \quad (k = p + 1, p + 2, \cdots).
\]

Now, if we put
\[
\gamma_k = \frac{b_k (k-p\delta)}{p(\delta - 1)}
\]
for \( k = p + 1, p + 2, \cdots \)
and \( \gamma_p = 1 - \sum_{k=p+1}^{\infty} \gamma_k \), we obtain
\[
f(z) = \gamma_p z^p + \sum_{k=p+1}^{\infty} \gamma_k G_k(z).
\]

This completes the proof of theorem. \( \square \)
**Theorem 4.2.7**: Let \( q(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k \) be holomorphic in \( U \) and \( 0 \leq c_k \leq 1 \). If \( f(z) \in MNR(r, s, \delta) \), then \((f * q)(z)\) is also in \( MNR(r, s, \delta) \).

**Proof**: By Theorem 4.2.1, we have

\[
\sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} (k - p\delta) a_k b_k < p(\delta - 1).
\]

Thus from definition of convolution and since \( f \in MNR(r, s, \delta) \) then, we get

\[
\sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(1)_{k-p}} (k - p\delta) a_k b_k c_k \leq p(\delta - 1),
\]

so by Theorem 4.2.1, we obtain the required result. \( \square \)

**Definition 4.2.1**: A function \( f \in A(p) \) is said to be in the class \( MNR(\xi)(r, s, \delta) \) if there exists a function \( g(z) \in MNR(r, s, \delta) \) such that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| < p - \xi, \quad (z \in U, 0 \leq \xi < 1).
\]

(4.11)

Now we define the \( \lambda \)-neighbourhood of a function \( f \in A(p) \) by

\[
N_\lambda(f) = \left\{ g \in A(p) : g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=p+1}^{\infty} k|a_k - b_k| \leq \lambda \right\}.
\]

(4.12)

**Theorem 4.2.8**: If \( g \in MNR(r, s, \delta) \) and

\[
\xi = p - \frac{\lambda(\alpha_1 \alpha_2 \cdots \alpha_r)(p(1 - \delta) + 1)b_{p+1}}{(p+1)(\alpha_1 \alpha_2 \cdots \alpha_r)(p(1 - \delta) + 1)b_{p+1} - p(\delta - 1)(\beta_1 \cdots \beta_s)},
\]

(4.13)

then \( N_\lambda(g) \subset MNR(\xi)(r, s, \delta) \).

**Proof**: Let \( f \in N_\lambda(g) \). Then from (4.12), we get

\[
\sum_{k=p+1}^{\infty} k|a_k - b_k| < \lambda
\]

which implies that

\[
\sum_{k=p+1}^{\infty} |a_k - b_k| \leq \frac{\lambda}{p+1}.
\]
Since \( g \in MNR(r, s, \delta) \), then we have from (4.4)
\[
\sum_{k=p+1}^{\infty} a_k \leq \frac{p(\delta - 1)(\beta_1 \beta_2 \cdots \beta_s)}{(\alpha_1 \alpha_2 \cdots \alpha_r)(p(1 - \delta) + 1)b_{p+1}},
\]
so that
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < \left| \sum_{k=p+1}^{\infty} (a_k - b_k)z^k \right| < \sum_{k=p+1}^{\infty} |a_k - b_k| \frac{z^p + \sum_{k=p+1}^{\infty} b_kz^k}{1 - \sum_{k=p+1}^{\infty} b_k} \leq \frac{\lambda}{p + 1} \cdot \frac{(\alpha_1 \alpha_2 \cdots \alpha_r)(p(1 - \delta) + 1)b_{p+1}}{(\alpha_1 \alpha_2 \cdots \alpha_r)(p(1 - \delta) + 1)b_{p+1} - p(\delta - 1)(\beta_1 \beta_2 \cdots \beta_s)} = p - \xi,
\]
provided \( \xi \) is given by (4.13).

Therefore, by Definition 4.2.1, we obtain \( f \in MNR(\xi)(r, s, \delta) \). This completes the proof of theorem. \( \square \)

**Theorem 4.2.9**: Let \( f(z) \in MNR(r, s, \delta) \). Then the generalized Bernardi integral operator defined by [6]
\[
F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1}f(t)dt \quad (c > -1, z \in U),
\]
is also in \( MNR(r, s, \delta) \).

**Proof**: Since
\[
F(z) = f(z) \ast \left( z^p + \sum_{k=p+1}^{\infty} \frac{c + p}{c + k}z^k \right) = z^p + \sum_{k=p+1}^{\infty} \frac{c + p}{c + k}a_kz^k
\]
and \( \frac{c + p}{c + k} \leq 1 \), then in view of Theorem 4.2.1, we have the required result. \( \square \)

Next, we study some properties of subclasses of holomorphic functions for some convolution operator on Hilbert space by using the Fox-Wright generalized hypergeometric functions.

Let \( \mathcal{A}_1 \) denote the class of functions \( f \) of the form
\[
f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad (a_1 > 0),
\]
(4.14)
which are holomorphic in the unit disk $\mathcal{U} = \{ z : |z| < 1 \}$.

For complex parameters

$$\alpha_1, \ldots, \alpha_q \left( \frac{\alpha_j}{A_j} \neq 0, -1, -2, \ldots; j = 1, \ldots, q \right)$$

and

$$\beta_1, \ldots, \beta_s \left( \frac{\beta_j}{B_j} \neq 0, -1, -2, \ldots; j = 1, \ldots, s \right)$$

where $q \leq s + 1$ we define the Fox-Wright generalized hypergeometric function [10] (see also [25]),

$$q\psi_s[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z]$$

$$= q\psi_s[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}; z]$$

$$= \sum_{n=0}^{\infty} \left\{ \prod_{j=1}^{q} \Gamma(\alpha_j + A_j n) \right\} \left\{ \prod_{j=1}^{s} \Gamma(\beta_j + B_j n) \right\}^{-1} \frac{z^n}{n!}$$

(4.15)

$(A_j > 0 \ (j = 1, \ldots, q); \ B_j > 0 \ (j = 1, \ldots, s); 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{q} A_j \geq 0)$.

If $A_j = 1 \ (j = 1, \ldots, q)$ and $B_j = 1 \ (j = 1, \ldots, s)$, we have the relationship

$$w_q\psi_s[(\alpha_j, 1)_{1,q}; (\beta_j, 1)_{1,s}; z] = qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$$

(4.16)

where $qF_s$ is generalized hypergeometric function and

$$w = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_s)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_q)}, \ (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ z \in \mathcal{U})$$

Now let $q, s \in \mathbb{N}$ and suppose that $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ are also positive real numbers. Then, we define a function

$$q\phi_s[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}; z] = wz_q\psi_s[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}; z]$$

and consider a linear operator [23]
\[ L[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}] : \mathcal{A}_1 \to \mathcal{A}_1 \text{ defined by} \]

\[ L[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}]f(z) = q\phi_s[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}; z] * f(z). \quad (4.17) \]

For simplicity, we write

\[ L[\alpha]f(z) = L[(\alpha_1, A_1), \cdots, (\alpha_q, A_q); (\beta_1, B_1), \cdots, (\beta_s, B_s)]f(z) \quad (4.18) \]

We note that special cases of this operator were investigated by Dziok and Srivastava [9], by letting \( A_j = 1 \) (\( j = 1, \cdots, q \)) and \( B_j = 1 \) (\( j = 1, \cdots, s \)) in (4.17) and includes the Noor Integral operator [16].

Let us denote by \( LM(q, s; A, B; \mathcal{I}P, \xi) \) the class of functions \( f \) of the form

\[ f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \quad (a_1 > 0; a_n \geq 0; \ n \in \mathbb{N} \setminus \{1\}) \quad (4.19) \]

and satisfying the following

\[ \frac{1}{1 - \xi} \left( \frac{\mathcal{I}P L[\alpha]f(\mathcal{I}P)}{L[\alpha]f(\mathcal{I}P)} - \xi \right) < \frac{1 + A\mathcal{I}P}{1 + B\mathcal{I}P} \quad (-1 \leq B < A \leq 1) \text{ and } 0 \leq \xi < 1 \quad (4.20) \]

for all operator \( \mathcal{I}P \) such that \( \mathcal{I}P \neq 0 \) and \( \|\mathcal{I}P\| < 1, 0 \) being the null operator on \( \mathcal{I}H \), where \( \mathcal{I}P \) defined in Definition 0.1.23.

On the other hand, for real parameter \( \mathcal{E} \quad (0 < |\mathcal{E}| < 1) \), we define the following subclasses of the class \( LM(q, s; A, B; \mathcal{I}P, \xi) \)

\[ LM_{\mathcal{E}}(q, s; A, B; \mathcal{I}P, \xi) = \{ f : f \in LM(q, s; A, B; \mathcal{I}P, \xi) \text{ and } f(0) = f(\mathcal{E}) - \mathcal{E} = 0 \} \quad (4.21) \]

and

\[ LM_{\mathcal{E}}^*(q, s; A, B; \mathcal{I}P, \xi) = \{ f : f \in LM(q, s; A, B; \mathcal{I}P, \xi) \text{ and } f(0) = f'(\mathcal{E}) - 1 = 0 \} \quad (4.22) \]
In particular, for $q = s + 1$ and $\alpha_{s+1} = A_{s+1} = 1$, we have

$$LM(s; A, B; IP, \xi) = LM(s + 1, s; A, B; IP, \xi), \quad (4.23)$$

$$LM_{\epsilon}(s; A, B; IP, \xi) = LM_{\epsilon}(s + 1, s; A, B; IP, \xi) \quad (4.24)$$

$$LM_{\epsilon}^*(s; A, B; IP, \xi) = LM_{\epsilon}^*(s + 1, s; A, B; IP, \xi). \quad (4.25)$$

First we state a coefficient theorem.

**Theorem 4.2.10**: A function $f$ of the form (4.19) is in $LM(q, s; A, B; IP, \xi)$ if and only if

$$\sum_{n=2}^{\infty} [n(1-B) + (A-1) + \xi(B-A)]\sigma_n a_n < [(A-B) + \xi(B-A)]\sigma_1 a_1 \quad (4.26)$$

where

$$\sigma_n = \frac{\Gamma(\alpha_1 + A_1(n-1)) \cdots \Gamma(\alpha_q + A_q(n-1))}{\Gamma(\beta_1 + B_1(n-1)) \cdots \Gamma(\beta_s + B_s(n-1))(n-1)!}, \quad n \in \mathbb{N}. \quad (4.27)$$

**Proof**: Let $f \in LM(q, s; A, B; IP, \xi)$, that is we have that

$$\frac{1}{1-\xi} \left( \frac{f[\alpha_1]f(IP)}{L[\alpha_1]f(IP)} - \xi \right) = \frac{1 + AW(IP)}{1 + BW(IP)} \quad (-1 \leq B < A \leq 1 \text{ and } 0 \leq \xi < 1)$$

where $W(\emptyset) = \emptyset$ and $\|W(IP)\| < 1$ for all operators $IP \neq \emptyset$, thus we obtain

$$\left\| \frac{f[\alpha_1]f(IP)}{L[\alpha_1]f(IP)} - L[\alpha_1]f(IP) \right\| < 1. \quad (4.28)$$

By using (4.16), (4.17) and (4.18), we have

$$\left\| \sum_{n=2}^{\infty} (n-1)\sigma_n a_n IP^{n-1} \right\| < 1.$$

Setting $IP = r II$ ($0 < r < 1$), we obtain

$$\sum_{n=2}^{\infty} (n-1)\sigma_n a_n r^{n-1} < a_1 \frac{(A-B)(1-\xi)}{w} - \sum_{n=2}^{\infty} ((A-B)(1-\xi) - B(n-1))\sigma_n a_n r^{n-1}.$$
By letting $r \to 1^-$, we get the assertion (4.26).

Conversely, it suffices to show that

$$\|\mathcal{P}^\prime f_1 - L[\alpha_1]f(f_1)\| - \|(A-B)(1-\xi)L[\alpha_1]f(f_1) - B(\mathcal{P}^\prime f_1 - L[\alpha_1]f(f_1))\| < 0.$$ 

Choosing $\mathcal{P} = r\mathcal{I}$ $(0 < r < 1)$, we get the left hand side of the above normed inequality equal to

$$\| \sum_{n=2}^{\infty} (n-1)\sigma_n a_n \mathcal{P}^n \| - \left\| a_1 \frac{(A-B)(1-\xi)}{w} \mathcal{P} - \sum_{n=2}^{\infty} ((A-B)(1-\xi) - B(n-1))\sigma_n a_n \mathcal{P}^n \right\|

\leq \sum_{n=2}^{\infty} (n-1)\sigma_n a_n r^n - \left( a_1 \frac{(A-B)(1-\xi)}{w} r - \sum_{n=2}^{\infty} ((A-B)(1-\xi) - B(n-1))\sigma_n a_n r^n \right)

= \sum_{n=2}^{\infty} [n(1-B) + (A-1) + \xi(B-A)]\sigma_n a_n r^n - [(A-B) + \xi(B-A)]\sigma_1 a_1 r

< \sum_{n=2}^{\infty} [n(1-B) + (A-1) + \xi(B-A)]\sigma_n a_n - [(A-B) + \xi(B-A)]\sigma_1 a_1 \leq 0$$

then we have $f$ belongs to $LM(q, s; A, B; \mathcal{P}, \xi)$.

**Corollary 4.2.1**: A function $f$ of the form (4.19) is in $LM_{\mathcal{E}}(q, s; A, B; \mathcal{P}, \xi)$ if and only if

$$\sum_{n=2}^{\infty} [n(1-B) + (A-1) + \xi(B-A)]\sigma_n - [(A-B) + \xi(B-A)]\sigma_1 \mathcal{E}^{n-1} |a_n| \leq [(A-B) + \xi(B-A)]\sigma_1 \mathcal{E}^{n-1}$$

(4.29)

where $\sigma_n$ is defined by (4.27).

**Proof**: Since $\frac{f(\mathcal{E})}{\mathcal{E}} = 1 = a_1 - \sum_{n=2}^{\infty} a_n \mathcal{E}^{n-1}$, the result follows upon substituting $a_1 = 1 + \sum_{n=2}^{\infty} a_n \mathcal{E}^{n-1}$ in Theorem 4.2.10. $\square$

168
Corollary 4.2.2: A function $f$ of the form (4.19) is in $LM^*_c(q, s; A, B; IP, \xi)$ if and only if
\[
\sum_{n=2}^{\infty} \left[ \frac{\sigma_n}{[(A-B) + \xi(B-A)]\sigma_1} - nE^{n-1} \right] a_n \leq 1, \tag{4.30}
\]
where $\sigma_n$ is defined by (4.27). These results are entirely new.

Theorem 4.2.11: Let a function $f$ of the form (4.19) belong to the class $LM(q, s; A, B; IP, \xi)$. If the sequence $\{ [(n(1-B) + (A-1) + \xi(B-A)]\phi_n \}_{n=2}^{\infty}$ is nondecreasing and positive, then
\[
a_1r - \frac{[(A-B) + \xi(B-A)]\sigma_1a_1}{[1-2B + A + \xi(B-A)]\sigma_2}r^2 \leq \|f(IP)\| \leq a_1r + \frac{[(A-B) + \xi(B-A)]\sigma_1a_1}{[1-2B + a + \xi(B-A)]\sigma_2}r^2. \tag{4.31}
\]
If the sequence $\left\{ \frac{[(n(1-B) + (A-1) + \xi(B-A)]\sigma_n}{n(1-B)+A-1+\xi(B-A)]\sigma_n} \right\}_{n=2}^{\infty}$ is nondecreasing and positive, then
\[
a_1 - \frac{2[(A-B) + \xi(B-A)]\sigma_1a_1}{[1-2B + A + \xi(B-A)]\sigma_2}r \leq \|f'(IP)\| \leq a_1 + \frac{2[(A-B) + \xi(B-A)]\sigma_1a_1}{[1-2B + A + \xi(B-A)]\sigma_2}r. \tag{4.32}
\]
where $\sigma_n$ is defined by (4.27). The result is sharp.

Proof: Since $\{ [(n(1-B) + (A-1) + \xi(B-A)]\phi_n \}_{n=2}^{\infty}$ is nondecreasing and positive, then
\[
\sum_{n=2}^{\infty} a_n \leq \frac{[(A-B) + \xi(B-A)]\sigma_1a_1}{[1-2B + A + \xi(B-A)]\sigma_2}. \tag{4.33}
\]
Also $\left\{ \frac{[(n(1-B)+A-1+\xi(B-A)]\sigma_n}{n(1-B)+A-1+\xi(B-A)]\sigma_n} \right\}_{n=2}^{\infty}$ is nondecreasing and positive, then
\[
\sum_{n=2}^{\infty} na_n \leq \frac{2[(A-B) + \xi(B-A)]\sigma_1a_1}{[1-2B + A + \xi(B-A)]\sigma_2}. \tag{4.34}
\]
By using (4.19), we can write
\[
\|f(IP)\| = \|a_1IP - \sum_{n=2}^{\infty} a_n IP^n\| \leq a_1r + \sum_{n=2}^{\infty} a_n r^n = a_1r + r^2 \sum_{n=2}^{\infty} a_n r^{n-2} \\
\leq a_1r + r^2 \sum_{n=2}^{\infty} a_n \leq a_1r + \frac{[(A-B) + \xi(B-A)]\sigma_1a_1}{[1-2B + A + \xi(B-A)]\sigma_2}r^2.
\]
\[ \| f(\mathcal{IP}) \| \geq a_1 r - \frac{[(A - B) + \xi(B - A)] \sigma_1 a_1}{[1 - 2B + A + \xi(B - A)] \sigma_2 r^2}. \]

Also, we have
\[
\| f'(\mathcal{IP}) \| = \| a_1 - \sum_{n=2}^{\infty} n a_n \mathcal{IP}^{n-1} \| \leq a_1 + \sum_{n=2}^{\infty} n a_n r^{n-1} \]
\[ = a_1 + r \sum_{n=2}^{\infty} n a_n r^{n-2} \leq a_1 + r \sum_{n=2}^{\infty} n a_n \]
\[ \leq a_1 + 2[(A - B) + \xi(B - A)] \sigma_1 a_1 r \]
\[ \frac{[1 - 2B + A + \xi(B - A)] \sigma_2 r}{[1 - 2B + A + \xi(B - A)] \sigma_2 r} \]

and
\[ \| f'(\mathcal{IP}) \| \geq a_1 - \frac{2[(A - B) + \xi(B - A)] \sigma_1 a_1}{[1 - 2B + A + \xi(B - A)] \sigma_2 r}. \]

For sharpness we can take the extremal function \( f_2 \) of the form
\[ f_2(z) = a_1 z - \frac{[(A - B) + \xi(B - A)] \sigma_1 a_1}{[1 - 2B + A + \xi(B - A)] \sigma_2} z^2. \]
\[ \square \quad (4.35) \]

**Corollary 4.2.3**: Let a function \( f \) of the form (4.19) belong to the class \( LM(s; A, B; \mathcal{IP}, \xi) \). If \( \beta_j \leq \alpha_j, B_j \leq A_j \) \((j = 1, \cdots, s)\), then the assertion (4.31) and (4.32) hold true.

**Proof**: If \( q = s \) and \( \beta_j \leq \alpha_j, B_j \leq A_j \) \((j = 1, \cdots, s)\), then
\[ \{ n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n \}_{n=2}^{\infty} \]
and
\[ \{ n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n \}_{n=2}^{\infty} \]
are nondecreasing. Thus, by Theorem 4.2.11, we have Corollary 4.2.3. This result is entirely new.  
\[ \square \]
**Theorem 4.2.12**: Let a function $f$ of the form (4.19) be in $LM_{E}(q, s; A, B; IP, \xi)$.

If the sequence

$$\{[n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n - [(A - B) + \xi(B - A)]\sigma_1 E^{n-1}\}_{n=2}^{\infty}$$

is nondecreasing and positive, then

$$\phi(r) \leq \| f(IP) \| \leq \frac{[1 - 2B + A + \xi(B - A)]\sigma_2 r + [(A - B) + \xi(B - A)]\sigma_1 r^2}{[1 - 2B + A + \xi(B - A)]\sigma_2 - [(A - B) + \xi(B - A)]\sigma_1 E}$$

where

$$\phi(r) = \begin{cases} r & r \leq E \\ \frac{[1 - 2B + A + \xi(B - A)]\sigma_2 r - [(A - B) + \xi(B - A)]\sigma_1 r^2}{[1 - 2B + A + \xi(B - A)]\sigma_2 - [(A - B) + \xi(B - A)]\sigma_1 E} & r > E \end{cases}$$

If the sequence

$$\{[n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n - [(A - B) + \xi(B - A)]\sigma_1 E^{n-1}\}_{n=2}^{\infty}$$

is nondecreasing and positive, then

$$a_1 = \frac{2[(A - B) + \xi(B - A)]\sigma_1 r}{(1 - 2B + A + \xi(B - A)]\sigma_2 - [(A - B) + \xi(B - A)]\sigma_1 E} \leq \| f'(IP) \|$$

$$\leq \frac{[1 - 2B + A + \xi(B - A)]\sigma_2 r + 2[(A - B) + \xi(B - A)]\sigma_1 r}{[1 - 2B + A + \xi(B - A)]\sigma_2 r - [(A - B) + \xi(B - A)]\sigma_1 r}$$

($\| IP \| = r$ $(0 < r < 1)$), and $\sigma_n$ is defined by (4.27).

The result is sharp.

**Proof**: Since $f \in LM_{E}(q, s; A, B; IP, \xi)$, then we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{[(A - B) + \xi(B - A)]\sigma_1}{[1 - 2B + A + \xi(B - A)]\sigma_2 - [(A - B) + \xi(B - A)]\sigma_1 E}$$
where the sequence (4.36) is positive and nondecreasing. Also, by (4.38), we have by using (4.30)
\[
\sum_{n=2}^{\infty} na_n \leq \frac{2[(A - B) + \xi(B - A)]\sigma_1}{[1 - 2B + A + \xi(B - A)]\sigma_2 - 2[(A - B) + \xi(B - A)]\sigma_1 E}.
\]

For \(IP = r\mathbb{I} (0 < r < 1)\), we have
\[
\|f(IP)\| = \|a_1 IP - \sum_{n=2}^{\infty} a_n IP^n\| \leq r \left(a_1 + \sum_{n=2}^{\infty} a_n r^{n-1}\right)
\]
\[
\leq r \left(1 + \sum_{n=2}^{\infty} a_n \mathcal{E}^{n-1} + \sum_{n=2}^{\infty} a_n r^{n-1}\right)
\]
\[
\leq r \left(1 + (\mathcal{E} + r) \sum_{n=2}^{\infty} a_n\right)
\]
\[
\leq \frac{[1 - 2B + A + \xi(B - A)]\sigma_2 r + [(A - B) + \xi(B - A)]\sigma_1 r^2}{[1 - 2B + A + \xi(B - A)]\sigma_2 - [(A - B) + \xi(B - A)]\sigma_1 E}
\]
and
\[
\|f(IP)\| = \left\|a_1 IP - \sum_{n=2}^{\infty} a_n IP^n\right\| \geq r \left(a_1 - \sum_{n=2}^{\infty} a_n r^{n-1}\right)
\]
\[
= r \left(1 + \sum_{n=2}^{\infty} a_n (\mathcal{E}^{n-1} - r^{n-1})\right),
\]
then \(\|f(IP)\| \geq r\) if \(r \leq \mathcal{E}\), while if \(r > \mathcal{E}\), we get the sequence \(\{\mathcal{E}^{n-1} - r^{n-1}\}_{n=2}^{\infty}\) is negative and decreasing, therefore we have
\[
\|f(IP)\| \geq r \left(1 + (\mathcal{E} - r) \sum_{n=2}^{\infty} a_n\right)
\]
\[
\geq \frac{[1 - 2B + A + \xi(B - A)]\sigma_2 r - [(A - B) + \xi(B - A)]\sigma_1 r^2}{[1 - 2B + A + \xi(B - A)]\sigma_2 - [(A - B) + \xi(B - A)]\sigma_1 E}.
\]
By the same way we can obtain the assertion (4.39).

For sharpness we can take the extremal function \(f_2(z)\) and \(f(z) = z\) where
\[
f_2(z) = \frac{[1 - 2B + A + \xi(B - A)]\sigma_2 z - [(A - B) + \xi(B - A)]\sigma_1 z^2}{[(1 - 2B + A + \xi(B - A)]\sigma_2 - [(A - B) + \xi(B - A)]\sigma_1 E}.
\]
\(\square\)
Remark 4.2.1: Let a function $f$ of the form (4.19) belong to the class $LM_{E}(s; A, B; IP, \xi)$. Then if $A_{1} \leq \alpha_{1}, \beta_{j} \leq \alpha_{j}$ ($j = 2, \cdots, s$), and $B_{j} = A_{j}$ ($j = 1, \cdots, s$), the assertion (4.37) holds true and if $\beta_{1} \leq \alpha_{1}$, the assertion (4.39) holds true.

Next we introduce the region of univalency in particular to that of star-likeness and convexity of $LM_{E}(q, s; A, B; IP, \xi)$.

Theorem 4.2.13: The radius of starlikeness for the class $LM_{E}(q, s; A, B; IP, \xi)$ is given by

$$r_{1} = \inf_{n \geq 2} \left[ \frac{(n(1-B) + (A-1) + \xi(B-A))\sigma_{n}}{n[(A-B) + \xi(B-A)]\sigma_{1}} \right]^{\frac{1}{2}}. \tag{4.40}$$

The result is sharp.

Proof: We shall show that

$$\left\| \frac{IP f'(IP)}{f(IP)} - 1 \right\| \leq 1, \quad (IP = r_{1}I \quad (0 < r_{1} < 1)).$$

It is easy to show that

$$\left\| \frac{IP f'(IP)}{f(IP)} - 1 \right\| < 1$$

if

$$\sum_{n=2}^{\infty} (n-1)a_{n}r_{1}^{n-1} \leq 1 + \sum_{n=2}^{\infty} a_{n}E^{n-1} - \sum_{n=2}^{\infty} a_{n}r_{1}^{n-1}$$

or

$$\sum_{n=2}^{\infty} na_{n}r_{1}^{n-1} - a_{n}E^{n-1} \leq 1. \tag{4.41}$$

By (4.29),

$$\sum_{n=2}^{\infty} a_{n} \left[ \frac{(n(1-B) + (A-1) + \xi(B-A))\sigma_{n}}{[(A-B) + \xi(B-A)]\sigma_{1}} - E^{n-1} \right] \leq 1.$$
Therefore, (4.41) will be true if
\[nr_1^{n-1} - \xi^{n-1} \leq \frac{n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n}{[(A - B) + \xi(B - A)]\sigma_1} - \xi^{n-1}\]
or
\[r_1 < \left[\frac{n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n}{n[(A - B) + \xi(B - A)]\sigma_1}\right]^{\frac{1}{n-1}}, \quad n \geq 2.
\]

The result is sharp for the function
\[f(z) = \frac{n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n z - [(A - B) + \xi(B - A)]\sigma_1 z^n}{n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n - [(A - B) + \xi(B - A)]\sigma_1 \xi^{n-1}}.
\]
(4.42) \hfill \Box

Here we also have to mention that the radius of convexity for the class
\(LM_\xi(q,s;A,B;\mathcal{P},\xi)\) is given by
\[r_2 = \inf_{n \geq 2} \left[\frac{n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n}{[(A - B) + \xi(B - A)]\sigma_1 n^2}\right]^{\frac{1}{n-1}}.
\]
(4.43)

The result is sharp for the function given by (4.42).

The proof of this theorem is similar to that of Theorem 4.2.13 and hence omitted.

Here we introduce an integral operator due to Bernardi [6]
\[L_c[f] = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) dt.
\]
(4.44)

where \(c\) is real number such that \(c > -1\).

**Theorem 4.2.14** : If \(f \in LM(q,s;A,B;\mathcal{P},\xi)\), then so \(L_c[f]\).

**Proof** : We can write
\[L_c[f] = a_1z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0,\]
where \( b_n = \frac{c+1}{c+n} a_n \). Therefore

\[
\sum_{n=2}^{\infty} [n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n b_n
\]

\[
= \sum_{n=2}^{\infty} [n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n \frac{c+1}{c+n} a_n
\]

\[
\leq [n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n a_n \leq [(A - B) + \xi(B - A)] \sigma_1 a_1.
\]

By assumption \( f(z) \in \text{LM}(q, s; A, B; \mathcal{I}, \xi) \). Thus \( L_c[f] \in \text{LM}(q, s; A, B; \mathcal{I}, \xi) \).

\( \square \)

**Theorem 4.2.15** : Let \( f_1(z) = z \) and

\[
f_n(z) = \frac{[n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n z - [(A - B) + \xi(B - A)] \sigma_1 z^n}{[n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n - [(A - B) + \xi(B - A)] \sigma_1 \mathcal{E}^{n-1}}.
\]

Then \( f(z) \in \text{LM}_c(q, s; A, B; \mathcal{I}, \xi) \) if and only if it can be expressed in the form

\[ f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \text{ where } \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1. \]

**Proof** : Let \( f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \) where \( \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1. \) Therefore

\[
f(z) = \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \frac{[n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n z}{[n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n - [(A - B) + \xi(B - A)] \sigma_1 \mathcal{E}^{n-1}}
\]

\[
- \sum_{n=2}^{\infty} \lambda_n \frac{[(A - B) + \xi(B - A)] \sigma_1 z^n}{[(A - B) + \xi(B - A)] \sigma_n - [(A - B) + \xi(B - A)] \sigma_1 \mathcal{E}^{n-1}}.
\]

Now, by (4.29) we have

\[
\sum_{n=2}^{\infty} \lambda_n \frac{[(A - B) + \xi(B - A)] \sigma_1}{[n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n - [(A - B) + \xi(B - A)] \sigma_1 \mathcal{E}^{n-1}} \times
\]

\[
\left[ \frac{[n(1 - B) + (A - 1) + \xi(B - A)] \sigma_n}{[(A - B) + \xi(B - A)] \sigma_1} - \mathcal{E}^{n-1} \right] = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 < 1.
\]

This shows that \( f(z) \in \text{LM}_c(q, s; A, B; \mathcal{I}, \xi) \).

175
Conversely, let \( f(z) \in LM_E(q, s; A, B; P, \xi) \), then
\[
a_n \leq \frac{[(A - B) + \xi(B - A)]\sigma_1}{n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n - [(A - B) + \xi(B - A)]\sigma_1 E^{n-1}, \quad n \geq 2.
\]
Putting
\[
\lambda_n = \frac{[n(1 - B) + (A - 1) + \xi(B - A)]\sigma_n - [(A - B) + \xi(B - A)]\sigma_1 E^{n-1}}{[(A - B) + \xi(B - A)]\sigma_1}, \quad n \geq 2
\]
and \( \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n \), we obtain
\[
f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z). \quad \Box
\]

**SECTION 2**

4.3 Some Applications of Fractional Calculus to Certain Subclasses of Univalent and Uniformly Multivalent Holomorphic Functions

Let \( T(n) \) denote the class of functions of the form
\[
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0, n \in \mathbb{N}) \tag{4.45}
\]
which are holomorphic and univalent in the open disk
\[
U = \{ z : z \in \mathbb{C}, |z| < 1 \}.
\]

In the beginning of this section we introduce the class \( \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \) consisting of all functions in the form \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \) \( (a_k \geq 0, k \geq 2) \) satisfying the condition
\[
(1 - \lambda)J_{\eta, z}^{\mu, \beta, \eta} f(z) + \lambda(J_{\eta, z}^{\mu+1, \beta+1, \eta+1} f(z)) \prec \frac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}, \quad z \in U \tag{4.46}
\]
\(-1 \leq A < B \leq 1, 0 < B \leq 1, 0 \leq \alpha < 1, \lambda \geq 0, 0 \leq \beta < 1, 0 \leq \mu < 1, \eta > \max\{\mu, \beta\} - 2\), with \(J_{0,z}^{\mu,\beta,\eta} f(z)\) is the Saigo type fractional calculus operator defined in Definition 0.1.21.

Or equivalently,

\[
\frac{(1 - \lambda)J_{0,z}^{\mu,\beta,\eta} f(z) + \lambda(J_{0,z}^{\mu+1,\beta+1,\eta+1} f(z)) - 1}{B[(1 - \lambda)J_{0,z}^{\mu,\beta,\eta} f(z) + \lambda(J_{0,z}^{\mu+1,\beta+1,\eta+1} f(z)) - [B + (A - B)(1 - \alpha)]]} \leq 1.
\]

(4.47)

If \(\mu = \beta\), we have \(D_z^\mu f(z) = J_{0,z}^{\mu,\mu,\eta} f(z)\) \((0 \leq \mu < 1)\) and let

\[
J_{0,z}^{\mu,\beta,\eta} f(z) = \frac{\Gamma(\beta - \mu + \eta + 2)}{\Gamma(\eta + 2)\Gamma(\beta + 1)} z J_{0,z}^{\mu,\beta,\eta} (z^{\beta - 1}) f(z).
\]

By taking \(\mu = \beta = \nu, \eta = 1\), we have

\[
J_{0,z}^{\mu,\mu} f(z) = \frac{\Gamma(3)}{\Gamma(3)\Gamma(\nu + 1)} z J_{0,z}^{\mu,1} (z^{\nu - 1}) f(z).
\]

(4.48)

So we have

\[
D_2^v f(z) = \frac{z}{\Gamma(v + 1)} D_2^v (z^{v - 1} f(z)) = z - \sum_{k=2}^{\infty} a_k B_k(\lambda) z^k,
\]

where

\[
B_k(\lambda) = \frac{\Gamma(v + k)}{\Gamma(v + 1)\Gamma(k)}, \quad (\text{see e.g. [4]}).
\]

The relation (4.49) is Ruscheweyh derivative of order \(v\). Therefore, we get a special case of class \(M_\eta(A, B, \alpha, \lambda, v)\) which was studied by R. Aghalary and S. R. Kulkarni [1].

Next we derive the coefficient inequality for the class \(M_\eta(A, B, \alpha, \lambda, \mu, \beta)\).

**Theorem 4.3.1**: Let \(f \in T(n)\). Then a necessary and sufficient condition for
\( f \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \) is

\[
\sum_{k=n+1}^{\infty} \frac{(1 + B)(1 - \lambda + \lambda(k - \beta))}{\phi_k(\mu, \beta, \eta)} a_k \leq (B - A)(1 - \alpha) + (1 + B) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)
\]

(4.50)

where, \(-1 \leq A < B \leq 1, 0 < B \leq 1, 0 \leq \alpha < 1, 1 - \lambda \beta \geq \phi_1(\mu, \beta, \eta), \lambda \geq 0, 0 \leq \beta < 1, 0 \leq \mu < 1, \eta > \max\{\mu, \beta\} - 2, \) and

\[
\phi_k(\mu, \beta, \eta) = \frac{\Gamma(1 + k - \beta) \Gamma(1 + k + \eta - \mu)}{\Gamma(1 + k) \Gamma(1 + k + \eta - \beta)}, \quad k \in \mathbb{N}.
\]

(4.51)

**Proof**: Let \( f \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). Then in view of the fact \( \text{Re}(z) \leq |z| \) and by (0.14), we have

\[
\text{Re} \left( \frac{\sum_{k=n+1}^{\infty} \frac{1 - \lambda + \lambda(k - \beta)}{\phi_k(\mu, \beta, \eta)} a_k z^{k - \beta - 1} - \frac{1}{\phi_1(\mu, \beta, \eta)} z^{-\beta} + \frac{\lambda \beta}{\phi_1(\mu, \beta, \eta)} z^{-\beta} + 1}{(B - A)(1 - \alpha) - B \sum_{k=n+1}^{\infty} \frac{1 - \lambda + \lambda(k - \beta)}{\phi_k(\mu, \beta, \eta)} a_k z^{k - \beta - 1} + \frac{z^{-\beta}}{\phi_1(\mu, \beta, \eta)} B - \frac{\lambda \beta}{\phi_1(\mu, \beta, \eta)} B - B} \right) \leq 1.
\]

Choose the values of \( z \) on the real axis so that upon clearing the denominator in the last expression and letting \( z \to 1^- \) through real values, we have

\[
\sum_{k=n+1}^{\infty} \frac{1 - \lambda + \lambda(k - \beta)}{\phi_k(\mu, \beta, \eta)} a_k \leq (B - A)(1 - \alpha) - B \sum_{k=n+1}^{\infty} \frac{1 - \lambda + \lambda(k - \beta)}{\phi_k(\mu, \beta, \eta)} a_k + (1 + B) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right).
\]

Therefore, we have (4.50).

Conversely, if (4.50) holds true, then we obtain

\[
\left| \frac{\mu, \beta, \eta}{z} f(z) + \lambda z \left( \frac{\mu, \beta, \eta}{z} f(z) \right)' - 1 \right| B \frac{\left[ \frac{\mu, \beta, \eta}{z} f(z) + \lambda z \left( \frac{\mu, \beta, \eta}{z} f(z) \right)' \right]' - [B + (A - B)(1 - \alpha)]}{178}
\]
\[
\begin{align*}
&\leq \left[ \sum_{k=n+1}^{\infty} \frac{1 - \lambda + \lambda(k - \beta)}{\phi_k(\mu, \beta, \eta)} a_k z^{k-\beta-1} - \frac{1}{\phi_1(\mu, \beta, \eta)} z^{-\beta} + \frac{\lambda\beta}{\phi_1(\mu, \beta, \eta)} z^{-\beta} + 1 \right] / \left[ (B - A)(1 - \alpha) - B \sum_{k=n+1}^{\infty} \frac{1 - \lambda + \lambda(k - \beta)}{\phi_k(\mu, \beta, \eta)} a_k z^{k-\beta-1} + \frac{z^{-\beta}}{\phi_1(\mu, \beta, \eta)} B - \frac{\lambda\beta}{\phi_1(\mu, \beta, \eta)} z^{-\beta} B - B \right] < 1.
\end{align*}
\]

The function \( f(z) \) given by
\[
f(z) = z - \left[ (B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda\beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \right] \phi_k(\mu, \beta, \eta) z^k, \quad (k = 2, 3, \ldots)
\]
(4.52)
is an extremal function for assertion of the Theorem 4.3.1.

\[\square\]

**Deduction 4.3.1**: Let \( f(z) \in T(n) \) be in the class \( \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). Then
\[
a_k \leq \frac{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda\beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \right] \phi_k(\mu, \beta, \eta)}{(1 + B)(1 - \lambda + \lambda(k - \beta))} \quad (k = 2, 3, \ldots).
\]
(4.53)

**Theorem 4.3.2**: Let \( f_i(z) = z - \sum_{k=n+1}^{\infty} a_{k,i} z^k \ (a_{k,i} \geq 0, i = 1, \ldots, m) \), 
\( n, m \in \mathbb{N} \) be in the class \( \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). Then the function
\[
h(z) = \sum_{i=1}^{m} c_i f_i, \quad \left( \sum_{i=1}^{m} c_i = 1, c_i \geq 0, i = 1, \ldots, m \right)
\]
(4.54)
is in the class \( \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \).

**Proof**: From Definition of \( h(z) \), we have
\[
h(z) = z - \sum_{k=n+1}^{\infty} \left[ \sum_{i=1}^{m} c_i a_{k,i} \right] z^k,
\]
and from Theorem 4.3.1
\[
\sum_{k=n+1}^{\infty} \frac{(B + 1)(1 - \lambda + \lambda(k - \beta))}{\phi_k(\mu, \beta, \eta)} \left[ \sum_{i=1}^{m} c_i a_{k,i} \right]
\]
179
\[ \leq [(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \sum_{i=1}^{m} c_i \]

\[ = (B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right), \text{ since } \sum_{i=1}^{m} c_i = 1. \]

This proves that \( h(z) \in M_{\eta}(A, B, \alpha, \lambda, \mu, \beta). \) \( \square \)

We note that \( M_{\eta}(A, B, \alpha, \lambda, \mu, \beta) \) is a convex set.

**Theorem 4.3.3**: Let \( f(z) \in M_{\eta}(A, B, \alpha, \lambda, \mu, \beta). \) Then for \( |z| = r < 1 \)

\[
\begin{align*}
& r - \frac{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \phi_{n+1}(\mu, \beta, \eta)]r^{n+1}}{(B + 1)(1 - \lambda + \lambda((n + 1) - \beta))} \leq |f(z)| \\
& \leq r + \frac{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \phi_{n+1}(\mu, \beta, \eta)]r^{n+1}}{(B + 1)(1 - \lambda + \lambda((n + 1) - \beta))} 
\end{align*}
\]

(4.55)

Furthermore

\[
\begin{align*}
r^{1-\beta} - A_n r^{n+1-\beta} & \leq \left| J_{0,z}^\mu f(z) \right| \leq r^{1-\beta} + A_n r^{n+1-\beta}, 
\end{align*}
\]

(4.56)

where

\[
A_n = \frac{(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)}{(B + 1)(1 - \lambda + \lambda((n + 1) - \beta))} 
\]

(4.57)

\((-1 \leq A < B \leq 1, 0 < B \leq 1, 0 \leq \alpha < 1, 1 - \lambda \beta \geq \phi_1(\mu, \beta, \eta), \lambda \geq 0, 0 \leq \beta \leq \mu < 1 \text{ and } \eta > \mu - 2).\)

**Proof**: Let \( f(z) \in M_{\eta}(A, B, \alpha, \lambda, \mu, \beta), (z \in U). \) Then by Theorem 4.3.1, we obtain

\[
\begin{align*}
& \frac{(B + 1)(1 + \lambda(n - \beta))}{\phi_{n+1}(\mu, \beta, \eta)} \sum_{k=n+1}^{\infty} a_k \leq \frac{(B + 1)(1 - \lambda + \lambda(k + \beta))}{\phi_{n+1}(\mu, \beta, \eta)} \sum_{k=n+1}^{\infty} a_k \\
& \leq (B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \\
\end{align*}
\]

which immediately yields

\[
\sum_{k=n+1}^{\infty} a_k \leq \frac{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)] \phi_{n+1}(\mu, \beta, \eta)}{(B + 1)(1 + \lambda(n - \beta))}.
\]

180
Now

\[ |f(z)| \geq |z| - \sum_{k=n+1}^{\infty} a_k |z|^k \geq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \]
\[ \geq |z| - \frac{(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)}{(B + 1)(1 + \lambda(n - \beta))} |z|^{n+1}. \]

On the other hand, we note that

\[ |f(z)| \leq |z| + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \]
\[ \leq |z| + \frac{(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)}{(B + 1)(1 + \lambda(n - \beta))} |z|^{n+1}. \]

Next,

\[ |J_{\mu, \beta, \eta} f(z)| \geq \frac{1}{\phi_1(\mu, \beta, \eta)} |z|^{1 - \beta} - |z|^{n+1 - \beta} \sum_{k=n+1}^{\infty} \frac{1}{\phi_k(\mu, \beta, \eta)} a_k \]
\[ \geq \frac{1}{\phi_1(\mu, \beta, \eta)} |z|^{1 - \beta} - \frac{(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)}{(B + 1)(1 + \lambda(n - \beta))} |z|^{n+1 - \beta} \]

and

\[ |J_{0, \mu, \beta, \eta} f(z)| \leq \frac{1}{\phi_1(\mu, \beta, \eta)} |z|^{1 - \beta} + \frac{(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)}{(B + 1)(1 + \lambda(n - \beta))} |z|^{n+1 - \beta} \]

where \( \phi_k(\mu, \beta, \eta) \) is given by (4.51).

\[ \square \]

**Corollary 4.3.2**: Under the hypothesis of Theorem 4.3.3, \( f(z) \) is included in a disc with center at the origin and radius \( \rho \) given by

\[ \rho = 1 + \frac{[\mu, \beta, \eta]}{(B + 1)(1 + \lambda(n - \beta))}. \]

We claim that this result is entirely new and not found in the literature.
Theorem 4.3.4: Let \( \psi_n(z) = z \)

\[
\psi_k(z) = z - \frac{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)] \phi_k(\mu, \beta, \eta)}{(1 + B)(1 - \lambda + \lambda(k - \beta))} z^k, \quad k \geq n+1.
\]

Then \( f(z) \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{k=n}^{\infty} c_k \psi_k(z),
\]

where

\[
c_k \geq 0 \quad \text{and} \quad \sum_{k=n}^{\infty} c_k = 1. \quad (4.58)
\]

Proof: Let \( f(z) \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). Then by (4.53) and taking

\[
c_k = \frac{(1 + B)(1 - \lambda + \lambda(k - \beta))}{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)] \phi_k(\mu, \beta, \eta)}, \quad (k \geq n+1)
\]

we have \( c_k \geq 0 \) and if we put \( c_n = 1 - \sum_{k=n+1}^{\infty} c_k \) we obtain

\[
f(z) = z - \sum_{k=n+1}^{\infty} c_k z^k
\]

\[
= z - \sum_{k=n+1}^{\infty} \frac{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)] \phi_k(\mu, \beta, \eta)}{(1 + B)(1 - \lambda + \lambda(k - \beta))} c_k z^k
\]

\[
= z - \sum_{k=n+1}^{\infty} c_k(z - \psi_k(z))
\]

\[
= (1 - \sum_{k=n+1}^{\infty} c_k)z + \sum_{k=n+1}^{\infty} c_k \psi_k(z) = \sum_{k=n}^{\infty} c_k \psi_k(z).
\]

Conversely, let

\[
f(z) = \sum_{k=n}^{\infty} c_k \psi_k(z)
\]

\[
= z - \sum_{k=n+1}^{\infty} \frac{[(B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right)] \phi_k(\mu, \beta, \eta)}{(1 + B)(1 - \lambda + \lambda(k - \beta))} c_k z^k.
\]
Then we have (in view of (4.58))
\[
\sum_{k=n+1}^{\infty} \frac{(1+B)(1-\lambda+\lambda(k-\beta))}{\phi_k(\mu, \beta, \eta)} \left( \frac{[\phi_1(\mu, \beta, \eta)-1]}{(1+B)(1-\lambda+\lambda(k-\beta))} \right) c_k \\
\leq (B-A)(1-\alpha) + (B+1) \left( \frac{1-\lambda\beta}{\phi_1(\mu, \beta, \eta)} - 1 \right).
\]

It follows therefore from Theorem 4.3.1 that \( f(z) \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). □

We do mention that the extreme points of the class \( \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \) are the functions \( \psi_n \) and \( \psi_k, (k \geq n+1, n \in \mathbb{N}) \) given by Theorem 4.3.4.

In the next discussion, we concentrate upon getting the radii of close-to-convexity, starlikeness and convexity.

**Theorem 4.3.5**: Let \( f(z) \in T(n) \) and let \( f(z) \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). Then \( f(z) \) is close-to-convex of order \( \delta (0 \leq \delta < 1) \) in
\[
|z| < r = \inf_k \left[ \frac{(1+B)(1-\lambda+\lambda(k-\beta))(1-\delta)}{\phi_k(\mu, \beta, \eta)k \left[ (B-A)(1-\alpha) + (B+1) \left( \frac{1-\lambda\beta}{\phi_1(\mu, \beta, \eta)} \right) \right]} \right]^{1/\delta},
\]

(4.59)

where \(-1 \leq A < B \leq 1, 0 \leq B < 1, 0 \leq \alpha < 1, 1-\lambda\beta \geq \phi_1(\mu, \beta, \eta), \lambda \geq 0, 0 \leq \beta \leq \mu < 1, \eta > \mu - 2, k \geq n+1, n \in \mathbb{N} \) and \( z \in \mathcal{U} \).

The result is sharp, the extremal function \( f(z) \) is given by (4.52).

**Proof**: It is enough to show that \( |f'(z) - 1| \leq 1 - \delta \) for \( |z| < r \) where \( r \) is given by (4.59)
\[
|f'(z) - 1| < 1 - \delta \quad \text{if} \quad \sum_{k=n+1}^{\infty} \frac{k a_k}{1-\delta} |z|^{k-1} \leq 1.
\]

By Theorem 4.3.1, the last expression holds true if
\[
\left( \frac{k}{1-\delta} \right) |z|^{k-1} \leq \frac{(1+B)(1-\lambda+\lambda(k-\beta))}{(B-A)(1-\alpha) + (B+1) \left( \frac{1-\lambda\beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \phi_k(\mu, \beta, \eta)},
\]

183
then

\[ |z| \leq \left[ \frac{(1 + B)(1 - \lambda + \lambda(k - \beta))(1 - \delta)}{k \left[ (B - A)(1 - \alpha) + (1 + B) \left(\frac{1 - \lambda \beta}{\phi_k(\mu, \beta, \eta)} - 1\right)\right] \phi_k(\mu, \beta, \eta)} \right]^{\frac{1}{k - 1}}. \]

**Theorem 4.3.6**: Let \( f(z) \in \mathcal{T}(n) \) and \( f(z) \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). Then \( f(z) \) is starlike of order \( \delta (0 \leq \delta < 1) \) in \( |z| < r_1 \), where

\[
\text{r}_1 = \inf_k \left[ \frac{(1 - \delta)(1 + B)(1 - \lambda + \lambda(k - \beta))}{\phi_k(\mu, \beta, \eta) \left[ (B - A)(1 - \alpha) + (B + 1) \left(\frac{1 - \lambda \beta}{\phi_k(\mu, \beta, \eta)} - 1\right)\right] (k - \delta)} \right]^{\frac{1}{k - 1}}. \]  

(4.60)

The result is sharp.

**Proof**: We want to show that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta.
\]

The rest of the details involved are obvious and may be omitted. \(\square\)

By Theorem 0.2.3 [Alexander’s theorem], we have the following result.

**Corollary 4.3.3**: Let \( f(z) \in \mathcal{T}(n) \) and \( f(z) \in \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \). Then \( f(z) \) is convex of order \( \delta (0 \leq \delta < 1) \) in \( |z| < r_2 \), where

\[
\text{r}_2 = \inf_k \left[ \frac{(1 - \delta)(1 + B)(1 - \lambda + \lambda(k - \beta))}{\phi_k(\mu, \beta, \eta) \left[ (B - A)(1 - \alpha) + (B + 1) \left(\frac{1 - \lambda \beta}{\phi_k(\mu, \beta, \eta)} - 1\right)\right] k(k - \delta)} \right]^{\frac{1}{k - 1}}. \]  

(4.61)

This result is entirely new.

**Theorem 4.3.7**: Let \( f(z) \in \mathcal{T}(n) \) be in the class \( \mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta) \) and let \( \sigma > 0 \). Then the function \( I^\sigma f(z) \) (Jung-Kim-Srivastava operator) defined by

\[
I^\sigma f(z) = \frac{2^\sigma}{z \Gamma(\sigma)} \int_0^z (\log \frac{z}{t})^{\sigma - 1} f(t)dt = z - \sum_{k=n+1}^{\infty} \left( \frac{2}{k + 1} \right)^\sigma a_k z^k \]  

(4.62)
belongs to the class $\mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta)$.

**Proof**: Note that

$$I^\sigma f(z) = z - \sum_{k=n+1}^\infty \left(\frac{2}{k+1}\right)^\sigma a_k z^k.$$ 

Therefore if $I^\sigma f(z) = z - \sum_{k=n+1}^\infty b_k z^k$, then $b_k = \left(\frac{2}{k+1}\right)^\sigma a_k$ and so

$$\sum_{k=n+1}^\infty \frac{(1+B)(1-\lambda+\lambda(k-\beta))}{\phi_k(\mu, \beta, \eta)} b_k$$

$$= \sum_{k=n+1}^\infty \frac{(1+B)(1-\lambda+\lambda(k-\beta))}{\phi_k(\mu, \beta, \eta)} \left(\frac{2}{k+1}\right)^\sigma a_k$$

$$\leq \sum_{k=n+1}^\infty \frac{(1+B)(1-\lambda+\lambda(k-\beta))}{\phi_k(\mu, \beta, \eta)} a_k$$

$$\leq (B-A)(1-\alpha) + (B+1) \left(\frac{1-\lambda\beta}{\phi_1(\mu, \beta, \eta)} - 1\right).$$

Then by Theorem 4.3.1 we get the required result. □

**Theorem 4.3.8**: Let $f(z) \in T(n)$ be in the class $\mathcal{M}_\eta(A, B, \alpha, \lambda, \mu, \beta)$. Let $\sigma > 0$. Then the function $I^\sigma f(z)$ given by (4.62) is starlike in $|z| < R$ where

$$R = \inf_k \left[ \frac{(1+B)(1-\lambda+\lambda(k-\beta))(k+1)^\sigma}{k \left[(B-A)(1-\alpha) + (B+1) \left(\frac{1-\lambda\beta}{\phi_1(\mu, \beta, \eta)} - 1\right)\right] \phi_k(\mu, \beta, \eta)(2)^\sigma} \right]^\frac{1}{\sigma},$$

$$n \in \mathbb{N}.$$ (4.63)

**Proof**: By Definition of $I^\sigma f(z)$, we have

$$I^\sigma f(z) = z - \sum_{k=n+1}^\infty \left(\frac{2}{k+1}\right)^\sigma a_k z^k$$

and it is starlike if

$$\sum_{k=n+1}^\infty k \left(\frac{2}{k+1}\right)^\sigma a_k |z|^{k-1} \leq 1.$$
The last expression is obtained by the same technique of Theorem 4.3.6 and it is true if
\[
k \left( \frac{2}{k+1} \right)^{\sigma} |z|^{k-1} \leq \frac{(1 + B)(1 - \lambda + \lambda(k - \beta))}{\phi_k(\mu, \beta, \eta) \left[ (B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \right]}
\]
that is, if
\[
|z| \leq \left[ \frac{(1 + B)(1 - \lambda + \lambda(k - \beta))(k + 1)^{\sigma}}{k\phi_k(\mu, \beta, \eta) \left[ (B - A)(1 - \alpha) + (B + 1) \left( \frac{1 - \lambda \beta}{\phi_1(\mu, \beta, \eta)} - 1 \right) \right] 2^{\sigma}} \right]^{1/\sigma} .
\]
This completes the proof of theorem. \qed

Denote by \( A^*_p \) the subclass of functions \( f(z) \) of the form
\[
f(z) = z^p + \sum_{k=p+1}^{2p-1} q_{k-p+1} z^{k-p+1} - 2F_1(a, b; c; z), \quad |z| < 1 \quad (4.64)
\]
where \( q_{k-p+1} = \frac{(a, k-p+1)(b, k-p+1)}{(c, k-p+1)(k-p+1)!} \). Therefore \( f(z) \) can be expressed in the form
\[
f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad |z| < 1, \quad (4.65)
\]
where
\[
a_k = \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(k+1)}, \quad k \geq p + 1. \quad (4.66)
\]
We denote by \( n-UCV^p_\delta(\lambda, \beta) \) the class of uniformly convex multivalent functions in \( U \) and \( n-ST^p_\delta(\lambda, \beta) \) the class of \( n \)-starlike multivalent functions in \( U \) which are defined in Definition 0.1.4 and Definition 0.1.5 respectively.

The class \( n-ST^p_\delta(\lambda, \beta) \) is a naturally way emerged as the class of functions with the property that \( g \in n-UCV^p_\delta(\lambda, \beta) \) if and only if \( zg' \in n-ST^p_\delta(\lambda, \beta) \). This class is introduced by Kanas and Wiśniowska [13].
**Definition 4.3.1**: For \( f \in A_p^n \), the fractional derivative of \( f \) of order \( \delta \) is defined by

\[
D_{z}^{\delta}f(z) = \frac{1}{\Gamma(2 - \delta)} z^{p - \delta} - \sum_{k=2}^{\infty} \frac{\Gamma(k + p)}{\Gamma(k + p - \delta)} a_k z^{k - \delta}, \quad 0 \leq \delta < 1. \tag{4.67}
\]

By making use of (4.67), Srivastava and Owa [23] introduced the operator

\[
L_{z}^{\delta}f(z) = \Gamma(2 - \delta)z^{\delta}D_{z}^{\delta}f(z), \quad 0 \leq \delta < 1
\]
and for \( \delta = 0 \) we have \( L_{z}^{0}f(z) = f(z) \). Now let \( f(z) \) be in the form (4.65), then

\[
L_{z}^{\delta}f(z) = \Gamma(2 - \delta)z^{\delta}D_{z}^{\delta}f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(k + p)}{\Gamma(k + p - \delta)} a_k z^k \tag{4.68}
\]

\[
= z^p - \sum_{k=p+1}^{\infty} \zeta^p(k, \delta) a_k z^k \tag{4.69}
\]
where

\[
\zeta^p(k, \delta) = \frac{\Gamma(2 - \delta)\Gamma(k + p)}{\Gamma(k + p - \delta)}.
\]

**Definition 4.3.2**: A function \( f(z) \in A_p^n \) is said to be in the class \( n - ST_{\delta}^p(\lambda, \beta) \) if satisfies the inequality

\[
Re \left\{ \frac{z(L_{z}^{\delta}f(z))'}{L_{z}^{\delta}f(z)} \right\} \geq n \left| \frac{z(L_{z}^{\delta}f(z))'}{L_{z}^{\delta}f(z)} + 1 - p\lambda \right| + \beta, \tag{4.70}
\]
where \( 0 \leq \beta < p, 0 \leq \lambda < \frac{1}{p}, n \geq 0, z \in U, 0 \leq \delta < 1 \) and \( p \in \mathbb{N} \).

**Theorem 4.3.9**: The function \( f(z) \) defined by (4.65) is in the class \( n - ST_{\delta}^p(\lambda, \beta) \) if and only if

\[
\sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)] a_k \leq p - \beta. \tag{4.71}
\]
\textbf{Proof}: Let }f \in n - ST^P_{\delta}(\lambda, \beta). \text{ Then by Theorem 0.2.1, we have }

\[ \text{Re} \left\{ \frac{z(L^k f(z))'}{L^k f(z)} (1 + ne^{i\theta}) (npe^{i\theta} + \beta) \right\} \geq 0, \]

where \(-\pi \leq \theta \leq \pi\). By using (4.69), we have

\[ \text{Re} \left\{ \frac{(pz^p - \sum_{k=p+1}^{\infty} k\zeta^p(k, \delta)a_k z^k)(1 + ne^{i\theta})}{\lambda p z^p - \sum_{k=p+1}^{\infty} \lambda k\zeta^p(k, \delta)a_k z^k + (1 - p\lambda)z^p - \sum_{k=p+1}^{\infty} (1 - p\lambda)\zeta^p(k, \delta)a_k z^k} - (npe^{i\theta} + \beta) \right\} \geq 0. \]

The last inequality holds for all \(z \in U\). Choosing values of \(z\) on the real axis, and letting \(z \to 1^-\) through real values, then we have

\[ \text{Re} \left\{ \frac{p(1 + ne^{i\theta}) - \sum_{k=p+1}^{\infty} k\zeta^p(k, \delta)(1 + ne^{i\theta})a_k - (npe^{i\theta} + \beta)(1 - \sum_{k=p+1}^{\infty} (\lambda k + 1 - p\lambda)\zeta^p(k, \delta)a_k)}{1 - \sum_{k=p+1}^{\infty} (\lambda k + 1 - p\lambda)\zeta^p(k, \delta)a_k} \right\} \geq 0. \]

By mean value theorem we obtain

\[ \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]a_k \leq p - \beta. \]

Conversely, suppose (4.71) holds true, then by using the Theorem 0.2.2 it is enough to show that

\[ A = \left| \frac{z(L^k f(z))'}{L^k f(z)} + (1 - p\lambda) \right| - \left( p + n \right) \left| \frac{z(L^k f(z))'}{L^k f(z)} + (1 - p\lambda) \right| - \left( p + n \right) \left| \frac{z(L^k f(z))'}{L^k f(z)} + (1 - p\lambda) \right| - \beta \right) \right| = B \]

For letting \(e^{i\theta} = \frac{M}{|M|}\), where

\[ M = \lambda \frac{z(L^k f(z))'}{L^k f(z)} + 1 - p\lambda, \]

188
we may write

\[
B = \frac{1}{|M|} \left| \frac{z(L_δ f(z))'}{L_δ^2 f(z)} - (p - \beta) \left( \frac{z(L_δ f(z))'}{L_δ^2 f(z)} + 1 - p\lambda \right) - ne^{i\theta} \right|
\]

\[
= \frac{1}{|M|} \left| \frac{pz^p - \sum_{k=p+1}^{\infty} k\zeta^p(k, \delta)a_k z^k}{z^p - \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)a_k z^k} - (p - \beta) \left( \frac{z^p - \sum_{k=p+1}^{\infty} (\lambda k + 1 - p\lambda)\zeta^p(k, \delta)a_k z^k}{z^p - \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)a_k z^k} \right) \right|
\]

\[
> \frac{1}{|M|} \left[ \frac{2p - \beta - \sum_{k=p+1}^{\infty} [k(1 + n) + (\lambda k + 1 - p\lambda)(p - \beta - np)]\zeta^p(k, \delta)a_k}{1 - \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)a_k} \right]
\]

\[
A = \frac{1}{|M|} \left| \frac{z(L_δ f(z))'}{L_δ^2 f(z)} - (p + \beta) \left( \frac{z(L_δ f(z))'}{L_δ^2 f(z)} + 1 - p\lambda \right) - ne^{i\theta} \right|
\]

\[
= \frac{1}{|M|} \left| \frac{z(L_δ f(z))'}{L_δ^2 f(z)} - p \left( \frac{z(L_δ f(z))'}{L_δ^2 f(z)} + 1 - p\lambda \right) \right|
\]

\[
< \frac{1}{|M|} \left[ \frac{\beta + \sum_{k=p+1}^{\infty} [k(1 + n) - (\lambda k + 1 - p\lambda)(p + \beta + np)]\zeta^p(k, \delta)a_k}{1 - \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)a_k} \right]
\]

It is easy to verify that $B - A > 0$, if (4.71) holds, therefore the proof is complete.

\[\square\]

**Deduction 4.3.4**: If $f(z) \in n - ST_δ^p(\lambda, \beta)$, then

\[a_k \leq \frac{(p - \beta)}{\zeta^p(k, \delta)[k(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}.
\]

**Corollary 4.3.5**: If $f(z) \in 0 - ST_0^1(\alpha, \beta)$ if and only if

\[
\sum_{k=2}^{\infty} (k - 1)(1 - \lambda\beta)a_k < 1 - \beta \quad (0 \leq \beta < 1, 0 \leq \lambda < 1)
\]
this is a class studied by Altintas and Owa [3].

**Corollary 4.3.6**: \( f(z) \in 0 - ST_0^1(0, \beta) \) if and only if

\[
\sum_{k=2}^{\infty} (k-1)a_k \leq 1 - \beta \quad (0 \leq \beta < 1)
\]

this is a class studied by Silverman [22].

In the next Theorem we obtain distortion theorem for the class \( n - ST_{\delta}^p(\lambda, \beta) \).

**Theorem 4.3.10**: Let \( f(z) \in n - ST_{\delta}^p(\lambda, \beta) \), then for \(|z| \leq r < 1\), we have

\[
\begin{align*}
pr^p - r^{p+1} & \leq \frac{(p - \beta)}{\zeta^p(p+1, \delta)[k(1+n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]} \\
& \leq r^p + r^{p+1} \frac{(p - \beta)}{[(p + 1)(1+n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}
\end{align*}
\]

and

\[
pr^{p-1} - (p + 1)r^p \leq \frac{(p - \beta)}{[(p + 1)(1+n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}
\]

\[
\leq |(L_2^p f(z))'| \leq pr^{p-1} - (p+1)r^p \frac{(p - \beta)}{[(p + 1)(1+n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}.
\]

**Proof**: In view of (4.71), we have

\[
\zeta^p(p+1, \delta)[(p + 1)(1+n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)] a_{p+1}
\]

\[
\leq \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1+n - \lambda(np + \beta)) - (1 - p\lambda)(np + b)] a_k
\]

\[
\leq p - \beta,
\]

then

\[
\sum_{k=p+1}^{\infty} a_k \leq \frac{p - \beta}{\zeta^p(p+1, \delta)[(p + 1)(1+n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}
\]
Hence
\[ |L^\delta_z f(z)| \leq |z|^p + |z|^{p+1} \zeta^p(p + 1, \delta) \sum_{k=p+1}^{\infty} a_k \]
\[ \leq r^p + r^{p+1} \zeta^p(p + 1, \delta) \sum_{k=p+1}^{\infty} a_k \]
\[ \leq r^p + r^{p+1} \frac{p - \beta}{[(p + 1)(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]} \]
and
\[ |L^\delta_z f(z)| \geq r^p - r^{p+1} \frac{p - \beta}{[(p + 1)(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}. \]

Also, we have
\[ |(L^\delta_z f(z))'| \leq pr^{p-1} + (p + 1)r^p \zeta^p(p + 1, \delta) \sum_{k=p+1}^{\infty} a_k \]
\[ \leq pr^{p-1} + r^p \frac{p - \beta}{[(p + 1)(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]} \]
and
\[ |L^\delta_z f(z)'| \geq pr^p - (p + 1)r^p \frac{p - \beta}{[(p + 1)(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}. \]

**Corollary 4.3.7**: Let the function \( f(z) \) given by (4.65) be in the class \( n - ST^p_\delta(\lambda, \beta) \). Then
\[ |D^\delta_z f(z)| \geq \frac{r^{p-\delta}}{\Gamma(2 - \delta)} - \frac{r^{p-\delta+1}}{\Gamma(2 - \delta) [(p + 1)(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]} \]
\[ |D^\delta_z f(z)| \leq \frac{r^{p-\delta}}{\Gamma(2 - \delta)} + \frac{r^{p-\delta+1}}{\Gamma(2 - \delta) [(p + 1)(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]}. \]

**Corollary 4.3.8**: Let the function \( f(z) \) given by (4.65) be in the class \( n - ST^p_\delta(\lambda, \beta) \). Then
\[ | \int_0^z f(t)dt | \geq \frac{r^{p+1}}{p + 1} \frac{r^{p+2}}{p + 2} \left( \frac{p - \beta}{[(p + 1)(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta)]} \right) \]
\[ \left\| \int_0^z f(t) dt \right\| \leq \frac{r^{p+1}}{p+1} + \frac{r^{p+2}}{p+2} \left( \frac{p-\beta}{[p+1](1+n-\lambda(np+\beta)) - (1-p\lambda)(np+\beta)} \right). \]

All the above results are entirely new.

**Remark 4.3.1**: Under the hypothesis of Corollary 4.3.7, \( D_2^\delta f(z) \) is included in a disc with its center at the origin and radius \( r \) given by

\[ r = \frac{1}{\Gamma(2-\delta)} \left( 1 + \frac{p-\beta}{[p+1](1+n-\lambda(np+\beta)) - (1-p\lambda)(np+\beta)} \right). \]

**Theorem 4.3.11**: Let the functions \( f_i(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,i} z^k, \quad (a_{k,i} \geq 0, \quad k \in \mathbb{N}, k \geq p+1, \quad i = 1, \cdots, \ell) \) be in the class \( n - ST_\delta^p(\lambda, \beta) \). Then the function

\[ g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k, \quad (b_k \geq 0, k \in \mathbb{N}, k \geq p+1) \]

also in the class \( n - ST_\delta^p(\lambda, \beta) \) where \( b_n = \frac{1}{\ell} \sum_{i=1}^{\ell} a_{k,i} \).

**Proof**: Since \( f_i(z) \in n - ST_\delta^p(\lambda, \beta) \), then by Theorem 4.3.9, we have

\[ \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1+n-\lambda(np+\beta)) - (1-p\lambda)(np+\beta)] a_k \leq p - \beta. \]

Now consider \( g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k \), so that

\[ \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1+n-\lambda(np+\beta)) - (1-p\lambda)(np+\beta)] a_k \left[ \frac{1}{\ell} \sum_{i=1}^{\ell} a_{k,i} \right] \]

\[ = \frac{1}{\ell} \sum_{i=1}^{\ell} \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1+n-\lambda(np+\beta)) - (1-p\lambda)(np+\beta)] a_{k,i} \]

\[ \leq \frac{1}{\ell} \sum_{i=1}^{\ell} (p - \beta) \leq p - \beta. \]

This completes the proof of theorem. \( \square \)
Theorem 4.3.12: Let \( f_i(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,i} z^k \), \( a_{k,i} \geq 0, k \in \mathbb{N} \), 
\( k \geq p + 1, i = 1, \ldots, \ell \) be in the class \( n_i - ST_\delta^p(\lambda_i, \beta_i) \). Then the function 
\( g(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,i} z^k \) is in the class \( n - ST_\delta^p(\lambda, \beta) \), where 
\[ \beta = \min_{1 \leq i \leq \ell} \{ \beta_i \}, n = \min_{1 \leq i \leq \ell} \{ n_i \} \text{ and } \lambda_i = \min_{1 \leq i \leq \ell} \{ \lambda_i \}. \]

Proof: By assumption \( f_i(z) \in n_i - ST_\delta^p(\lambda_i, \beta_i) \) for every \( i = 1, \ldots, \ell \), so by Theorem 4.3.9, we have 
\[
\sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1 + n_i - \lambda_i np + \beta_i)) - (1 - p\lambda_i)(n_ip - \beta_i)] \left[ \frac{1}{\ell} \sum_{i=1}^{\ell} a_{k,i} \right] \\
= \frac{1}{\ell} \sum_{i=1}^{\ell} \sum_{k=p+1}^{\infty} \zeta^p(k, \delta)[k(1 + n_i - \lambda_i np + \beta_i)) - (1 - p\lambda_i)(n_ip - \beta_i)] \\
\leq \frac{1}{\ell} \sum_{i=1}^{\ell} (p - \beta_i) \leq (p - \beta),
\]
then \( g(z) \in n - ST_\delta^p(\lambda, \beta) \). □

Theorem 4.3.13: Let \( f \in n - ST_\delta^p(\lambda, \beta) \). Then \( f \) is closed under convex combination.

Proof: Let \( f \) and \( g \) be in the class \( n - ST_\delta^p(\lambda, \beta) \),
\[
f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, g(z) = z^p - \sum_{k=p+1}^{\infty} a'_k z^k,
\]
consider the function \( h(z) = \lambda f(z) + (1 - \lambda)g(z) \), \( 0 \leq \lambda \leq 1 \), then 
\[
h(z) = z^p - \sum_{k=p+1}^{\infty} [\lambda a_k + (1 - \lambda) a'_k] z^k.
\]
By Theorem 4.3.9, we have 
\[
\sum_{k=p+1}^{\infty} \zeta^p(k, \delta) \left[ k(1 + n - \lambda(np + \beta)) - (1 - p\lambda)(np + \beta) \right] \frac{1}{p - \beta} [\lambda a_k + (1 - \lambda) a'_k]
\]
\[
\begin{align*}
&= \lambda \sum_{k=p+1}^{\infty} \zeta^p(k, \delta) \frac{k(1+n - \lambda np + \beta)}{p - \beta} a_k + \\
&(1 - \lambda) \sum_{k=p+1}^{\infty} \zeta^p(k, \delta) \frac{k(1+n - \lambda np + \beta) - (1 - p \lambda)(np + \beta)}{p - \beta} a'_k \leq 1,
\end{align*}
\]

then \( h(z) \in n - ST^p_\delta(\lambda, \beta) \). \qed

**Theorem 4.3.14**: Let \( f \in n - ST^p_\delta(\lambda, \beta) \). Then

\[
L^\delta zf(z) = \exp \left( \int_0^z \frac{n - E(t) \beta}{n - E(t)t} dt \right), \quad |E(z)| < 1, z \in \mathcal{U}.
\]

**Proof**: For \( n = 0 \) obvious.

Let \( n \neq 0 \), for \( f \in n - ST^p_\delta(\lambda, \beta) \) and \( w = \frac{z(L^\delta zf(z)')'}{L^\delta zf(z)} \) we have

\[
\text{Re } w > n|w-1| + \beta.
\]

Thus

\[
\left| \frac{w-1}{w-\beta} \right| < \frac{1}{n},
\]

or

\[
\frac{w-1}{w-\beta} = \frac{E(z)}{n}, \quad \text{where } |E(z)| < 1, z \in \mathcal{U}.
\]

This yields

\[
\frac{z(L^\delta zf(z)')'}{L^\delta zf(z)} - 1 = \frac{E(z)}{n}
\]

or equivalently

\[
\frac{(L^\delta zf(z)')'}{L^\delta zf(z)} = \frac{n - \beta E(z)}{z(n - E(z))}.
\]

Then we obtain the desired result by integration. This completes the proof of theorem. \qed
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