Chapter 6

Fractional Impulsive Integrodifferential Equations of Sobolev Type with Nonlocal Conditions

6.1 Introduction

The Sobolev type semilinear integrodifferential equation serves as an abstract formulation of partial integrodifferential equations which arise in applications such as flow of fluid through fissured rocks [29], thermodynamics and shear in second order fluids and so on. Brill [37] and Showalter [111] investigated the existence problem for semilinear Sobolev type equations in Banach spaces. Balachandran et al. [22] established the existence of solutions for Sobolev type semilinear integrodifferential equation whereas Balachandran and Uchiyama [21] studied the existence of solution of nonlinear integrodifferential equation of Sobolev type in Banach spaces. The problem of existence of solutions of evolution equations with nonlocal condition was initiated by Byszewski [38] and subsequently studied by several authors for different kinds of problems [15, 23, 39] such as abstract fractional semilinear evolution equations, quasilinear delay integrodifferential equations and semilinear functional evolution equations. Motivated by this, in this chapter, we study the existence of solutions of abstract fractional integrodifferential equations of Sobolev type using fixed point technique and resolvent operators.
6.2 Fractional Impulsive Integrodifferential Equations of Sobolev Type

Fractional impulsive integrodifferential equations of Sobolev type form an unexplored topic so we made an attempt in this section to discuss the existence of solutions of such kinds of equations using fixed point techniques.

6.2.1 Preliminaries

Consider the following nonlinear fractional impulsive integrodifferential equation of Sobolev type of the form

\[\begin{align*}
C D^q (Bu(t)) + Au(t) &= f(t, u(t)) + \int_0^t k(t, s, u(s))ds, \quad t \in J, \quad t \neq t_k, \quad (6.2.1) \\
\Delta u \big|_{t=t_k} &= I_k(u(t^-_k)), \quad (6.2.2) \\
u(0) &= u_0, \quad (6.2.3)
\end{align*}\]

where \(0 < q < 1\), \(A\) and \(B\) are linear operators with domains contained in a Banach space \(X\) and ranges contained in a Banach space \(Y\) and the operators \(A : D(A) \subset X \rightarrow Y\) and \(B : D(B) \subset X \rightarrow Y\) satisfy the following hypotheses:

(H1) \(A\) and \(B\) are closed linear operators,

(H2) \(D(B) \subset D(A)\) and \(B\) is bijective,

(H3) \(B^{-1} : Y \rightarrow D(B)\) is compact,

(H4) \(B^{-1}A : X \rightarrow D(B)\) is continuous.

The nonlinear operators \(f : J \times X \rightarrow Y\) and \(k : \Omega \times X \rightarrow Y\) are given abstract functions, \(I_k : X \rightarrow Y\), \(k = 1, 2, \ldots, m\) and \(u_0 \in X\), \(0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a\),

\[\begin{align*}
\Delta u \big|_{t=t_k} &= u(t^+_k) - u(t^-_k); \\
u(t^+_k) &= \lim_{h \rightarrow 0^+} u(t_k + h) \text{ and} \\
u(t^-_k) &= \lim_{h \rightarrow 0^-} u(t_k + h),
\end{align*}\]

represent the right and left limits of \(u(t)\) at \(t = t_k\). Here \(\Omega = \{(t, s) : 0 \leq s \leq t \leq T\}\). It is easy to prove that the equations (6.2.1)-(6.2.3) are equivalent to the integral equation
By a local solution of the abstract Cauchy problem (6.2.1)-(6.2.3), we mean an abstract function $u$ such that the following conditions are satisfied:

(i) $u \in PC(J; X)$ and $u \in D(A)$ on $J'$;

(ii) $\frac{du}{dt}^q$ exists and continuous on $J'$, where $0 < q < 1$;

(iii) $u$ satisfies equation (6.2.1) on $J'$ and satisfies the conditions $\Delta u|_{t=t_k} = I_k(u(t^-_k))$, $u(0) = u_0 \in X$ or equivalently $u$ satisfies the integral equation (6.2.4).

We assume the following conditions to prove the existence of solution of the equations (6.2.1)-(6.2.3):

(H5) The functions $I_k : X \to Y$ are continuous and there exists a constant $L > 0$ such that

$$\|I_k(u) - I_k(v)\|_Y \leq L\|u - v\|_X, \quad \text{for each } u, v \in X \text{ and } k = 1, 2, \ldots, m.$$  

(H6) $f : J \times X \to Y$ is continuous and there exists a constant $L_1 > 0$ such that

$$\|f(t, u) - f(t, v)\|_Y \leq L_1\|u - v\|_X, \quad \text{for all } u, v \in X.$$  

(H7) $k : \Omega \times X \to Y$ is continuous and there exists a constant $L_2 > 0$ such that

$$\left\| \int_0^t [k(t, s, u) - k(t, s, v)] ds \right\|_Y \leq L_2\|u - v\|_X, \quad \text{for all } u, v \in X.$$  

Let $R = \|B^{-1}A\|$, $R^* = \|B^{-1}\|$, $N = \max_{t \in J} \|f(t, 0)\|$ and $N^* = \max_{t \in J} \left( \left\| \int_0^t k(t, s, 0) ds \right\| \right).$
6.2.2 Existence of Solutions

**Theorem 6.2.1.** If the hypotheses (H1)-(H7) are satisfied and if \( \gamma(m+1)(R+R^*(L_1+L_2))+mR^*L \leq \frac{1}{2} \), then the problem (6.2.1)-(6.2.3) has a unique solution continuous on \( J \).

**Proof.** Let \( Z = PC(J; X) \). Define the mapping \( \Phi_1 : Z \to Z \) by

\[
\Phi_1 u(t) = u_0 - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} B^{-1} Au(s) ds - \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} B^{-1} Au(s) ds \\
+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds + \sum_{0 < t_k < t} B^{-1} I_k(u(t_k^-))
\]

and we have to show that \( \Phi_1 \) has a fixed point. This fixed point is then a solution of the equations (6.2.1)-(6.2.3). Choose \( r \geq 2(\|u_0\| + \gamma(m+1)R^*(N+N^*)) \). Then we can show that \( \Phi_1 B_r \subset B_r \), where \( B_r := \{ u \in Z : \|u\| \leq r \} \). From the assumptions, we have

\[
\|\Phi_1 u(t)\| \leq \|u_0\| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}\|B^{-1} A\|\|u(s)\| ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1}\|B^{-1} A\|\|u(s)\| ds \\
+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}\|B^{-1}\|\left(\|f(s, u(s))\| + \|\int_{0}^{s} k(s, \tau, u(\tau)) d\tau\|\right) ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1}\|B^{-1}\|\left(\|f(s, u(s))\| + \|\int_{0}^{s} k(s, \tau, u(\tau)) d\tau\|\right) ds \\
+ \sum_{0 < t_k < t} \|B^{-1}\|\|I_k(u(t_k^-))\| \\
\leq \|u_0\| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}\|B^{-1} A\|\|u(s)\| ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1}\|B^{-1} A\|\|u(s)\| ds \\
+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}\|B^{-1}\|\left(\|f(s, u(s)) - f(s, 0)\| + \|f(s, 0)\|\right) ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1}\|B^{-1}\|\left(\|f(s, u(s)) - f(s, 0)\| + \|f(s, 0)\|\right) ds
\]
\[
\frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|B^{-1}\| \left( \left\| \int_0^s [k(s, \tau, u(\tau)) - k(s, \tau, 0)] d\tau \right\| + \left\| \int_0^s h(s, \tau, 0) d\tau \right\| \right) ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|B^{-1}\| \left( \left\| \int_0^s [k(s, \tau, u(\tau)) - k(s, \tau, 0)] d\tau \right\| + \left\| \int_0^s k(s, \tau, 0) d\tau \right\| \right) ds \\
+ \sum_{0 < t_k < t} \|B^{-1}\| \|I_k(u(t_k^-))\| \\
\leq \|u_0\| + \frac{T^q}{\Gamma(q + 1)} \left( (m + 1)r \left( R + R^*(L_1 + L_2) \right) + (m + 1)R^*(N + N^*) \right) + mR^*L \\
\leq \|u_0\| + r \left( \gamma(m + 1)(R + R^*(L_1 + L_2)) + mR^*L \right) + \gamma(m + 1)R^*(N + N^*) \\
\leq r.
\]

Thus \(\Phi_1\) maps \(B_r\) into itself. Now, for \(u, v \in Z\), we have

\[
\|\Phi_1 u(t) - \Phi_1 v(t)\| \leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|B^{-1}\| \|u(s) - v(s)\| ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|B^{-1}\| \|u(s) - v(s)\| ds \\
+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|B^{-1}\| \|f(s, u(s)) - f(s, v(s))\| ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|B^{-1}\| \|f(s, u(s)) - f(s, v(s))\| ds \\
+ \sum_{0 < t_k < t} \|B^{-1}\| \|I_k(u(t_k^-)) - I_k(v(t_k^-))\| \\
\leq \left( \frac{T^q}{\Gamma(q + 1)} (m + 1)(R + R^*(L_1 + L_2)) + mR^*L \right) \|u - v\| \\
\leq \left( \gamma(m + 1)(R + R^*(L_1 + L_2)) + mR^*L \right) \|u - v\|.
\]

Hence \(\Phi_1\) is a contraction mapping and therefore there exists a unique fixed point \(u \in B_r\) such that \(\Phi_1 u(t) = u(t)\). Any fixed point of \(\Phi_1\) is a solution of equations (6.2.1)-(6.2.3). \(\blacksquare\)

Now we discuss the existence of solution of the fractional impulsive Sobolev type equations (6.2.1)-(6.2.2) with the nonlocal condition of the form

\[ u(0) + g(u) = u_0, \quad (6.2.6) \]
where \( g : PC(J; X) \to X \) is a given function which satisfies the following condition:

(H8) \( g : PC(J; X) \to X \) is continuous and there exists a constant \( G > 0 \) such that

\[
\|g(u) - g(v)\| \leq G\|u - v\|_{PC}, \text{ for } u, v \in PC(J; X).
\]

**Theorem 6.2.2.** If the hypotheses (H1)-(H8) are satisfied and if \( \gamma(m + 1)(R + R^*(L_1 + L_2)) + mR^*L + G \leq \frac{1}{2} \), then the problem \([6.2.1]-[6.2.2]\) with the nonlocal condition \([6.2.6]\) have a unique solution continuous on \( J \).

**Proof.** We want to prove that the operator defined by \( \Phi_2 : Z \to Z \) by

\[
\Phi_2 u(t) = u_0 - g(u) - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1} B^{-1}Au(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} B^{-1}Au(s)ds
\]

\[
+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k-1}^{t_k} (t_k - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau))d\tau \right)ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau))d\tau \right)ds + \sum_{0 < t_k < t} B^{-1}I_k(u(t_k^-))
\]

has a fixed point. This fixed point is then a solution of the equations \([6.2.1]-[6.2.2]\) and \([6.2.6]\). Choose \( r \geq 2(\|u_0\| + \|g(0)\| + \gamma(m + 1)R^*(N + N^*)) \). Then we can easily show that \( \Phi_2 B_r \subset B_r \).

\[
\|\Phi_2 u(t) - \Phi_2 v(t)\| \leq \left( \gamma(m + 1)(R + R^*(L_1 + L_2)) + G + mR^*L \right)\|u - v\| \leq \frac{1}{2}.
\]

The result follows by the application of the contraction mapping principle. \( \blacksquare \)

Now we assume the following conditions instead of (H6) and apply Krasnoselskii fixed point theorem.

(H9) \( f : J \times X \to Y \) is continuous and there exists a continuous function \( \mu \in L^1(J) \) such that \( \|f(t, u)\| \leq \mu(t) \), for all \( (t, u) \in J \times X \).

(H10) \( k : \Omega \times X \to Y \) is continuous and there exists a continuous function \( \mu^* \in L^1(J) \) such that \( \left\| \int_{0}^{t} k(t, s, u)ds \right\| \leq \mu^*(t) \), for all \( (t, s) \in \Omega, \ u \in X \).

**Theorem 6.2.3.** Assume that (H1)-(H5),(H7)-(H9) hold. If \( G + \gamma(m + 1)RR^* + mR^*L < 1 \), the fractional evolution equations \([6.2.1]-[6.2.2]\) with nonlocal condition \([6.2.6]\) has a solution on \( J \).
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**Proof.** Choose \( r \geq \frac{\|u_0\| + \|g(0)\| + \gamma R^*(m+1)(\mu_0 + \mu_1)}{1 - (G + R^*(\gamma(m+1)R + mL))} \) where \( \mu_0 = \sup_{t \in J} \mu(t) \), \( \mu_1 = \sup_{t \in J} \mu^*(t) \) and define the operators \( \mathcal{P} \) and \( \mathcal{Q} \) on \( B_r \) by

\[
\mathcal{P}u(t) = u_0 - g(u) - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1} B^{-1} Au(s) ds - \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} B^{-1} Au(s) ds
\]

\[
+ \sum_{0 < t_k < t} B^{-1} I_k(u(t_k^-)) \quad \text{and} \quad \mathcal{Q}u(t) = \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds.
\]

For any \( u, v \in B_r \), we have

\[
\|\mathcal{P}u(t) + \mathcal{Q}v(t)\|
\]

\[
\leq \|u_0\| + \|g(u) - g(0)\| + \|g(0)\| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1}\|B^{-1}A\|\|u(s)\| ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}\|B^{-1}A\|\|u(s)\| ds
\]

\[
+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1}\|B^{-1}\| \left( \|f(s, u(s))\| + \| \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \| \right) ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}\|B^{-1}\| \left( \|f(s, u(s))\| + \| \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \| \right) ds
\]

\[
+ \sum_{0 < t_k < t} \|B^{-1}\|\|I_k(u(t_k^-))\|
\]

\[
\leq \|u_0\| + \|g(0)\| + \gamma R^*(m+1)(\mu_0 + \mu_1) + r \left( G + R^*(\gamma(m+1)R + mL) \right)
\]

\[
\leq r.
\]

Hence we deduce that \( \|\mathcal{P}u + \mathcal{Q}v\| \leq r \).

Next, for any \( t \in J, u, v \in X \), we have

\[
\|\mathcal{P}u(t) - \mathcal{P}v(t)\| \leq \|g(u) - g(v)\| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1}\|B^{-1}A\|\|u(s) - v(s)\| ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}\|B^{-1}A\|\|u(s) - v(s)\| ds
\]
 Consequently \( \| \sum_{0<t_k<t} \| B^{-1} \| I_k(u(t^-_k)) - I_k(v(t^-_k)) \| \) 
\[ \leq G \| u - v \| + \gamma(m+1)R\star \| u - v \| + mR\star L \| u - v \| \]
\[ \leq \left( G + R\star (\gamma(m+1)R + mL) \right) \| u - v \|. \]

And since \( G + R\star (\gamma(m+1)R + mL) < 1 \), \( \mathcal{P} \) is a contraction mapping.

Now let us prove that \( \mathcal{Q} \) is continuous and compact. Let \( \{ u_n \} \) be a sequence in \( B_r \) such that \( u_n \to u \) in \( B_r \). Then
\[ f(s, u_n(s)) \to f(s, u(s)), k(s, \tau, u_n(\tau)) \to k(s, \tau, u(\tau)) \quad n \to \infty \]
because the function \( f \) is continuous on \( J \times X \) and \( k \) is continuous on \( \Omega \times X \). Now, for each \( t \in J \), we have
\[ \| \mathcal{Q} u_n(t) - \mathcal{Q} u(t) \| \leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_k}^{t} (t_k - s)^{q-1} \| f(s, u_n(s)) - f(s, u(s)) \| ds \]
\[ + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| f(s, u_n(s)) - f(s, u(s)) \| ds \]
\[ + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_k}^{t} (t_k - s)^{q-1} \| B^{-1} \| \left( \int_{0}^{s} k(s, \tau, u_n(\tau)) - k(s, \tau, u(\tau)) d\tau \right) ds \]
\[ \to 0 \quad \text{as} \quad n \to \infty. \]
Consequently \( \lim_{n \to \infty} \| \mathcal{Q} u_n(t) - \mathcal{Q} u(t) \| = 0. \) In other words, \( \mathcal{Q} \) is continuous.

Let us now note that \( \mathcal{Q} \) is uniformly bounded on \( B_r \). This follows from the inequality
\[ \| \mathcal{Q} u(t) \| \leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_k}^{t} (t_k - s)^{q-1} \| B^{-1} \| \left( \| f(s, u(s)) \| + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds \]
\[ + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| B^{-1} \| \left( \| f(s, u(s)) \| + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds \]
\[ \leq \gamma R\star (m+1)(\mu_0 + \mu_1). \]

We first prove that \( \{ \mathcal{Q} u(t) : u \in B_r \} \) is relatively compact in \( X \), for all \( t \in J \). Obviously \( \{ \mathcal{Q} u(0) : u \in B_r \} \) is compact. Fix \( t \in (0, T] \) and for each \( \varepsilon \in (0, t) \) and \( u \in B_r \), define the operator \( \mathcal{Q}^\varepsilon \) by
\[ \mathcal{Q}^\varepsilon u(t) = \frac{1}{\Gamma(q)} \sum_{0<t_k<t-\varepsilon} \int_{t_k}^{t-\varepsilon} (t_k - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds \]
\[ + \frac{1}{\Gamma(q)} \int_{t_k}^{t-\varepsilon} (t - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} k(s, \tau, u(\tau)) d\tau \right) ds. \]
relative compactness in $X$ with the nonlocal condition (6.2.6). and since it is compact at $t = 0$, we have its relative compactness in $X$ for all $t \in J$. Moreover, by using (H9) and (H10),

$$\|Q(t) - Q^r(t)\| \leq \frac{1}{\Gamma(q)} \sum_{t_{k-} < t < t_k} \int_{t_{k-}}^{t_k} (t_k - s)^{q-1} \|B^{-1}\| \left(\|f(s, u(s))\| + \int_0^s k(s, \tau, u(\tau)) d\tau\right) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_{k-}}^{t_1} (t_1 - s)^{q-1} \|B^{-1}\| \left(\|f(s, u(s))\| + \int_0^s k(s, \tau, u(\tau)) d\tau\right) ds$$

$$\leq \frac{R^*(m + 1)(\mu_0 + \mu_1)\epsilon^q}{\Gamma(q + 1)}.$$

From this, we deduce that $\{Q(t) : u \in B_r\}$ is relatively compact in $X$ for all $t \in (0, T]$ and since it is compact at $t = 0$, we have its relative compactness in $X$ for all $t \in J$.

Now let us prove that $Qu, u \in B_r$, is equicontinuous. The functions $Q(t), u \in B_r$, are equicontinuous at $t = 0$. Let $u \in B_r, 0 < t_1 < t_2 \leq T$. We have

$$\|Q(t_2) - Q(t_1)\|$$

$$\leq \frac{1}{\Gamma(q)} \int_0^{t_1} (t_2 - s)^{q-1} \|B^{-1}\| \left(\|f(s, u(s))\| + \int_0^s k(s, \tau, u(\tau)) d\tau\right) ds$$

$$+ \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} \|B^{-1}\| \left(\|f(s, u(s))\| + \int_0^s k(s, \tau, u(\tau)) d\tau\right) ds$$

$$\leq \frac{1}{\Gamma(q)} \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1}) \|B^{-1}\| \left(\|f(s, u(s))\| + \int_0^s k(s, \tau, u(\tau)) d\tau\right) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|B^{-1}\| \left(\|f(s, u(s))\| + \int_0^s k(s, \tau, u(\tau)) d\tau\right) ds$$

$$\leq \frac{R^*(\mu_0 + \mu_1)}{\Gamma(q + 1)} |t_2 - t_1|^q.$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Thus we have proved that $Q(B_r)$ is relatively compact for $t \in J$. By Arzela-Ascoli’s theorem, $Q$ is compact.

Hence, by the Krasnoselskii theorem, there exists a solution of the problem (6.2.1)-(6.2.2) with the nonlocal condition (6.2.6). ■
6.2.3 Example

Consider the following nonlinear fractional impulsive integrodifferential equation of Sobolev type of the form

$$C^D_q u(t) + \frac{1}{30} u(t) = \frac{e^{-t}|u(t)|}{(29 + e^t)(1 + |u(t)|)} + \int_0^t e^{-\frac{1}{10}u(s)} ds, \ t \in J, \ t \neq \frac{1}{2}, \ (6.2.8)$$

$$\Delta u|_{t=\frac{1}{2}} = \frac{|u(\frac{1}{2}^-)|}{8 + |u(\frac{1}{2}^-)|}, \quad (6.2.9)$$

$$u(0) = u_0, \quad (6.2.10)$$

where $0 < q \leq 1$. Take $J := [0, 1]$. Set

$$B = I, \quad A = \frac{1}{30},$$

$$f(t, u) = \frac{e^{-t}|u(t)|}{(29 + e^t)(1 + |u(t)|)},$$

$$\int_0^t k(t, s, u(s)) ds = \int_0^t e^{-\frac{1}{10}u(s)} ds \ t \in J, \ u \in X.$$ 

Let $u, v \in X$ and $t \in J$. Then we have

$$\left\| \int_0^t k(t, s, u(s)) ds - \int_0^t k(t, s, v(s)) ds \right\| = \left| \int_0^t e^{-\frac{1}{10}u(s)} ds - \int_0^t e^{-\frac{1}{10}v(s)} ds \right| \leq \frac{1}{4}|u - v|,$$

$$\|f(t, u) - f(t, v)\| = \left| \frac{e^{-t}}{(29 + e^t)(1 + u)} - \frac{v}{(1 + v)} \right|$$

$$\leq \frac{e^{-t}}{(29 + e^t)(1 + u)(1 + v)} |u - v|$$

$$\leq \frac{e^{-t}}{(29 + e^t)} |u - v|$$

$$\leq \frac{1}{30} |u - v|$$

and

$$\|I_k(u) - I_k(v)\| = \left| \frac{u}{8 + u} - \frac{v}{8 + v} \right| = \frac{8|u - v|}{(8 + u)(8 + v)} \leq \frac{1}{8}|u - v|.$$ 

Hence the conditions (H1)-(H7) hold with $L = \frac{1}{8}$, $L_1 = \frac{1}{30}$ and $L_2 = \frac{1}{15}$. Choose $m = 1$ and we check that condition

$$\gamma(m + 1)(R + R^*(L_1 + L_2)) + mR^* L \leq \frac{1}{2}$$

is satisfied. Indeed,

$$\gamma(m + 1)(R + R^*(L_1 + L_2)) + mR^* L \leq \frac{1}{2} \iff \Gamma(q + 1) > \frac{8}{9} \quad (6.2.11)$$
which is satisfied for some $q \in (0, 1]$. Then, by Theorem 6.2.3, the problem has a unique solution on $[0,1]$ for the values of $q$ satisfying (6.2.11).

6.3 Fractional Integrodifferential Equations of Sobolev Type

Hernández et al. [59] discussed the recent developments in the theory of abstract fractional differential equations in which the resolvent operator play a key role in proving the existence results. Numerical experiments for fractional models on population dynamics are examined in [103] and some of the applications of nonlinear fractional differential equations with their approximations are found in [57]. In this section, we study the existence of solutions of fractional integrodifferential equations of Sobolev type using resolvent operators in Banach spaces.

6.3.1 Preliminaries

Consider the following nonlinear fractional integrodifferential equation of Sobolev type of the form

$$
\begin{align*}
\mathcal{C}D^q (Bu(t)) &= Au(t) + f(t), \quad t \in J, \\
u(0) &= u_0,
\end{align*}
$$

(6.3.1)

where $0 < q < 1$, $A$ is a closed linear unbounded operator in $X$, $B$ is a linear operator, $f \in C(J; X)$ and the operators are with domains contained in a Banach space $X$ and ranges contained in a Banach space $Y$, the operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypotheses:

(H11) $A$ and $B$ are closed linear operators,

(H12) $D(B) \subset D(A)$ and $B$ is bijective,

(H13) $B^{-1} : Y \to D(B)$ is continuous.

Equation (6.3.1) is equivalent to the following integral equation

$$
u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t B^{-1}Au(s) \frac{1}{(t-s)^{1-q}}ds + \frac{1}{\Gamma(q)} \int_0^t B^{-1}f(s) \frac{1}{(t-s)^{1-q}}ds, \quad t \in J.
$$

(6.3.2)
Above equation can also be written as the integral equation of the form
\[
    u(t) = h(t) + \frac{1}{\Gamma(q)} \int_0^t \frac{B^{-1}Au(s)}{(t-s)^{1-q}} ds, \quad t \geq 0,
\]
where \( h(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t \frac{B^{-1}f(s)}{(t-s)^{1-q}} ds \). Let us assume that the integral equation \( (6.3.3) \) has an associated resolvent operator \( S(t), t \geq 0, \) on \( X \).

Now we define the resolvent operator for the integral equation \( (6.3.3) \).

**Definition 6.3.1.** [101, Definition 1.1.3] A one parameter family of bounded linear operators \( \{S(t)\}_{t \geq 0} \) on \( X \) is called a resolvent operator for \( (6.3.3) \) if the following conditions hold:

(i) \( S(\cdot)x \in C([0, \infty); X) \) and \( S(0)x = x \) for all \( x \in X \),

(ii) \( S(t)D(B^{-1}A) \subset D(B^{-1}A) \) and \( B^{-1}AS(t)x = S(t)B^{-1}Ax \), for all \( x \in D(B^{-1}A) \) and every \( t \geq 0 \),

(iii) for every \( x \in D(B^{-1}A) \) and \( t \geq 0 \),
\[
    S(t)x = x + \frac{1}{\Gamma(q)} \int_0^t \frac{B^{-1}AS(s)x}{(t-s)^{1-q}} ds.
\]

For brevity take \( B^{-1}A = E \). Here we assume that the resolvent operator \( S(t), t \geq 0, \) is analytic \([101, \text{Chapter 2}]\) and there exists a function \( \varphi_E \) in \( L^1_{loc}([0, \infty); \mathbb{R}^+) \) such that
\[
    \|S'(t)x\| \leq \varphi_E(t)\|x\|_{X_E}, \quad \text{for all } t > 0.
\]

We have the following concept of solution using Definition 1.1.1 in [101].

**Definition 6.3.2.** A function \( u \in C(J; X) \) is called a mild solution of the integral equation \( (6.3.3) \) on \( J \) if \( \int_0^t (t-s)^{q-1}u(s)ds \in D(E) \), for all \( t \in J \), \( h(t) \in C(J; X) \) and
\[
    u(t) = \frac{E}{\Gamma(q)} \int_0^t \frac{u(s)}{(t-s)^{1-q}} ds + h(t), \quad \forall \ t \in J.
\]

The next result follows from Lemma [4.2.1] [Proposition I.1.2, Corollary II.2.6 and Proposition I.1.3] in [101], which plays a key role in the subsequent sections. The proof of the Lemma is straight forward from Lemma [4.2.1] by replacing the operator \( A \) by \( E \).

**Lemma 6.3.1.** Under the above conditions the following properties are valid:
(i) If $u$ is a mild solution of (6.3.3) on $J$, then the function $t \to \int_0^t S(t-s)h(s)ds$ is continuously differentiable on $J$, and
\[
u(t) = \frac{d}{dt} \int_0^t S(t-s)h(s)ds, \quad \forall \ t \in J.
\] (6.3.5)

(ii) If $h \in C^q(J; X)$ for some $q \in (0, 1)$, then the function defined by
\[
u(t) = S(t)(h(t) - h(0)) + \int_0^t S'(t-s)[h(s) - h(t)]ds + S(t)h(0), \quad t \in J,
\] (6.3.6)
is a mild solution of (6.3.3) on $J$.

(iii) If $h \in C(J; X_E)$, then the function $u : J \to X$ defined by
\[
u(t) = \int_0^t S'(t-s)h(s)ds + h(t), \quad t \in J,
\] (6.3.7)
is a mild solution of (6.3.3) on $J$.

### 6.3.2 Existence of Solutions

In this section, we study the existence of mild solutions for a class of abstract fractional integrodifferential equation of Sobolev type of the form
\[
^{C}D^q(Bu(t))) = Au(t) + f(t, u(t), \int_0^t k(t, s, u(s))ds), \quad t \in J,
\]
(6.3.8)
\[
u(0) + g(u) = u_0,
\] (6.3.9)
where $A, B$ are defined as in (6.2.1)-(6.2.3), $u_0 \in X$ and $f : J \times X^2 \to X, k : \Omega \times X \to X, g : C(J; X) \to X$ are continuous. For brevity, let us take $Ku(t) = \int_0^t k(t, s, u(s))ds$.

Now we introduce the concept of mild solution for the equations (6.3.8)-(6.3.9). These equations are equivalent to the following integral equation
\[
u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)}\int_0^t \frac{Eu(s) ds}{(t-s)^{1-q}} + \frac{1}{\Gamma(q)}\int_0^t B^{-1}f(s, u(s), Ku(s)) \frac{ds}{(t-s)^{1-q}}, \quad \forall \ t \in J.
\] (6.3.10)
Motivated by the Lemma 6.3.1 and the above representation (6.3.10), we introduce the concept of a mild solution.

**Definition 6.3.3.** A function $u \in C(J; X)$ is said to be a mild solution of (6.3.8)-(6.3.9) on $J$ if \[\int_0^t u(s)(t-s)^{q-1}ds \in D(E), \text{ for all } t \in J, \text{ and satisfies the integral equation } (6.3.10).\]
Suppose that there exists a resolvent operator $S(t)$, $t \geq 0$, which is differentiable and the functions $f, g$ and $k$ are continuous in $X_E$. Then

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t B^{-1}f(s, u(s), Ku(s)) \, ds$$

$$+ \int_0^t S'(t-s) \left( u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^s B^{-1}f(\tau, u(\tau), Ku(\tau)) \, d\tau \right) \, ds.$$ 

Assume the following conditions:

(H14) The function $f : J \times X^2 \to X_E$ is completely continuous; there exists a constant $L_1 > 0$ such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_1(\|x_1 - x_2\| + \|y_1 - y_2\|), \quad \forall \, (t, x_i, y_i) \in J \times X^2, \; i = 1, 2.$$ 

(H15) The function $k : \Omega \times X \to X_E$ is continuous and there exists a constant $L_2 > 0$ such that

$$\left\| \int_0^t [k(t, s, x_1) - k(t, s, x_2)] \, ds \right\| \leq L_2\|x_1 - x_2\|, \quad \forall \, (t, s, x_i) \in \Omega \times X, \; i = 1, 2.$$ 

(H16) There exists a constant $G > 0$, for the function $g : C(J; X) \to X_E$, such that

$$\|g(x_1) - g(x_2)\| \leq G\|x_1 - x_2\|, \quad \forall \, x_i \in X, \; i = 1, 2.$$ 

(H17) $2\left(1 + \|\varphi_E\|_{L^1}\right)(\gamma R L_1(1 + L_2) + G) < 1.$

Let $N = \max_{t \in J} f(t, 0, 0)$, $N^* = \max_{t \in J} \left[ \int_0^t k(t, s, 0) \, ds \right]$ and $R = \|B^{-1}\|$.

**Theorem 6.3.1.** Assume that $u_0 \in D(E)$ and $f, g, k$ satisfies the assumptions (H11)-(H17). Then there exists a mild solution of (6.3.8)-(6.3.9) on $J$.

**Proof.** First we transform the existence of solutions of (6.3.8)-(6.3.9) into a fixed point problem. Let $Z = C(J; X)$. By considering Lemma 6.3.1(iii), we introduce the map $\Psi : Z \to Z$ by

$$\Psi u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t B^{-1}f(s, u(s), Ku(s)) \, ds$$

$$+ \int_0^t S'(t-s) \left( u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^s B^{-1}f(\tau, u(\tau), Ku(\tau)) \, d\tau \right) \, ds.$$
Now we decompose $\Psi$ as $\Psi_1 + \Psi_2$ on $B_r(0; Z)$, where

$$
\Psi_1 u(t) = u_0 - g(u) + \int_0^t S'(t-s)(u_0 - g(u))\,ds,
\Psi_2 u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{B^{-1} f(s, u(s), K u(s))}{(t-s)^{1-q}}\,ds + \int_0^t S'(t-s) \frac{1}{\Gamma(q)} \int_0^s \frac{B^{-1} f(\tau, u(\tau), K u(\tau))}{(s-\tau)^{1-q}}\,d\tau\,ds.
$$

Obviously $k(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, u(s), K u(s))}{(t-s)^{1-q}}\,ds \in C(J; X_{E'})$ and let $B_r(0; Z) = \{z \in Z : \|z\| \leq r\}$. Choose $r \geq 2(1 + \|\varphi_E\|_{L^1})(\|u_0\| + \|g(0)\| + \gamma R(L_1N^* + N)).$

For any $u, v \in Z$, we have

$$
\|\Psi_1 u(t) + \Psi_2 v(t)\| \\
\leq \|u_0\| + \|g(u) - g(0)\| + \|g(0)\| + \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^t \frac{\|f(s, v(s), K v(s)) - f(s, 0, 0)\|}{(t-s)^{1-q}}\,ds \\
+ \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^t \frac{\|f(s, 0, 0)\|}{(t-s)^{1-q}}\,ds + \int_0^t S'(t-s) \left(\|u_0\| + \|g(u) - g(0)\| + \|g(0)\|\right) \\
+ \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^s \frac{\|f(\tau, v(\tau), K v(\tau)) - f(\tau, 0, 0)\|}{(s-\tau)^{1-q}}\,d\tau + \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^s \frac{\|f(\tau, 0, 0)\|}{(s-\tau)^{1-q}}\,d\tau\right)\,ds \\
\leq \|u_0\| + Gr + \|g(0)\| + \frac{RNT^q}{\Gamma(q+1)} + \frac{RL_1 T^q}{\Gamma(q+1)} \left(\|v(s)\| + \|\int_0^t k(t, s, v(s))\,ds\right) \\
+ \int_0^t \|S'(t-s)\| \left(\|u_0\| + Gr + \|g(0)\| + \frac{RL_1 T^q}{\Gamma(q+1)} \left(\|v(\tau)\| + \|\int_0^s k(s, \tau, v(\tau))\,d\tau\right)\right) \\
+ \frac{RNT^q}{\Gamma(q+1)} \,ds \\
\leq \|u_0\| + Gr + \|g(0)\| + \gamma RN + \gamma RL_1 \left(\|v(s)\| + \|\int_0^t k(t, s, v(s)) - k(t, s, 0)\,ds\right) \\
+ \|\int_0^t k(t, s, 0)\,ds\| + \int_0^t \|S'(t-s)\| \left(\|u_0\| + Gr + \|g(0)\| + \gamma RN \\
+ \gamma RL_1 \left(\|v(\tau)\| + \|\int_0^s [k(s, \tau, v(\tau)) - k(s, \tau, 0)]\,d\tau\right) + \|\int_0^s k(s, \tau, 0)\,d\tau\right)\right)\,ds \\
\leq \|u_0\| + Gr + \|g(0)\| + \gamma R \left(N + L_1 (r + L_2 r + N^*)\right) + \|\varphi_E\|_{L^1} \left(\|u_0\| + Gr \\
+ \|g(0)\| + \gamma R \left(N + L_1 (r + L_2 r + N^*)\right)\right) \\
\leq \left(1 + \|\varphi_E\|_{L^1}\right) \left(\|u_0\| + Gr + \|g(0)\| + \gamma R \left(L_1 r (1 + L_2) + L_1 N^* + N\right)\right) \\
\leq r.
$$

Thus $\Psi$ maps $B_r(0; Z)$ into itself and so $\Psi_1 u + \Psi_2 v \in B_r$. 

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From the assumptions (H16), we see that, for any \( u \in Z \),
\[
\left\| \int_0^t S'(t-s)(u_0+g(u))ds \right\| \leq \|\varphi_E\|_{L^1}(\|u_0\| + Gr + \|g(0)\|),
\]
which implies that the function \( s \to S'(t-s)(u_0+g(u)) \) is integrable on \( J \), for all \( t \in J \) and \( \Psi_1 u \in Z \). Moreover, for \( u, v \in Z \) and \( t \in J \), we get
\[
\left\| \Psi_1 u(t) - \Psi_1 v(t) \right\| \leq \|g(u) - g(v)\| + \int_0^t \|S'(t-s)\| (\|g(u) - g(v)\|) ds
\leq G\|u - v\| + \|\varphi_E\|_{L^1} G\|u - v\|
\leq G(1 + \|\varphi_E\|_{L^1})\|u - v\|.
\]

By (H17), \( \Psi_1 \) is a contraction on \( B_r(0; Z) \).

Now we show that the operator \( \Psi_2 \) is completely continuous. Note that the function
\[
\begin{align*}
s \to \int_0^t S'(t-s) \int_0^s \frac{B^{-1}\|f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-q}} d\tau ds
\end{align*}
\]
is integrable on the assumptions of (H14) and (H15) as shown above. First we show that \( \Psi_2 \) is uniformly bounded. Now, for \( t \in J \),
\[
\left\| \Psi_2 u(t) \right\| \leq \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^t \frac{\|f(s, u(s), Ku(s))\|}{(t-s)^{1-q}} ds
+ \int_0^t \|S'(t-s)\| \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^s \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-q}} d\tau ds
\leq (1 + \|\varphi_E\|_{L^1}) \gamma R(L_1 r(1 + L_2) + L_1 N^* + N).
\]
This shows that \( \Psi_2 \) is uniformly bounded.

Let \( \{u_n\} \) be a sequence in \( B_r(0; Z) \) such that \( u_n \to u \) in \( B_r(0; Z) \). Since the functions \( f \) and \( k \) are continuous,
\[
f(s, u_n(s), Ku_n(s)) \to f(s, u(s), Ku(s)), \text{ as } n \to \infty.
\]
Now, for each \( t \in J \), we have
\[
\left\| \Psi_2 u_n(t) - \Psi_2 u(t) \right\|
\leq \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^t \frac{\|f(s, u_n(s), Ku_n(s)) - f(s, u(s), Ku(s))\|}{(t-s)^{1-q}} ds
+ \int_0^t \|S'(t-s)\| \frac{\|B^{-1}\|}{\Gamma(q)} \int_0^s \frac{\|f(\tau, u_n(\tau), Ku_n(\tau)) - f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-q}} d\tau ds
\to 0 \text{ as } n \to \infty.
\]
From the above, it is clear that \( \Psi_2 \) is continuous.
We need to prove that the set \( \{ \Psi_2(u) : u \in B_r(0; Z) \} \) is relatively compact in \( X \) for all \( t \in J \). Obviously \( \{ \Psi_2(u) : u \in B_r(0; Z) \} \) is compact. Fix \( t \in (0, T] \) and \( u \in B_r(0; Z) \) and define the operator \( \Psi_2^\varepsilon \) by

\[
\Psi_2^\varepsilon u(t) = \frac{1}{\Gamma(q)} \int_0^{t-\varepsilon} S(t-s) \frac{B^{-1}f(s, u(s), Ku(s))}{(t-s)^{1-q}} ds + \int_0^t S'(t-s) \frac{B^{-1}f(t, u(t), Ku(t))}{(s-t)^{1-q}} d\tau ds.
\]

Since, by (H14), \( f \) is completely continuous, the set \( X_\varepsilon = \{ \Psi_2^\varepsilon u(t) : u \in B_r(0; Z) \} \) is precompact in \( X \), for every \( \varepsilon > 0 \), \( 0 < \varepsilon < t \). Moreover, for every \( u \in B_r(0; Z) \), we have

\[
\|\Psi_2 u(t) - \Psi_2^\varepsilon u(t)\| \leq \frac{1}{\Gamma(q)} \int_{t-\varepsilon}^t \frac{\|B^{-1}\|}{(t-s)^{1-q}} \|f(s, u(s), Ku(s))\| ds + \int_{t-\varepsilon}^t \frac{\|B^{-1}\|}{(s-t)^{1-q}} \|f(t, u(t), Ku(t))\| d\tau ds.
\]

This shows that precompact sets \( X_\varepsilon \) are arbitrarily close to the set \( \{ \Psi_2 u(t) : u \in B_r(0; Z) \} \). Hence the set \( \{ \Psi_2 u(t) : u \in B_r(0; Z) \} \) is precompact in \( X \).

Next let us prove that \( \Psi_2(B_r(0; Z)) \) is equicontinuous. Note that the functions \( \Psi_2 u, u \in B_r(0; Z) \), are equicontinuous at \( t = 0 \). For \( t < t+h \leq T \), \( h > 0 \), we have

\[
\|\Psi_2 u(t+h) - \Psi_2 u(t)\| \leq \frac{1}{\Gamma(q)} \int_0^{t+h} \frac{\|B^{-1}\|}{(t+h-s)^{1-q}} \|f(s, u(s), Ku(s))\| ds + \int_0^{t+h} \frac{\|B^{-1}\|}{(t-h-s)^{1-q}} \|f(s, u(s), Ku(s))\| ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_0^{t-h} S'(t+h-s) \int_0^{s} \frac{B^{-1}f(t, u(t), Ku(t))}{(t+h-\tau)^{1-q}} d\tau ds - \int_0^{t} S'(t-s) \int_0^{s} \frac{f(t, u(t), Ku(t))}{(t-\tau)^{1-q}} d\tau ds \leq \frac{1}{\Gamma(q)} \int_0^{t+h} \frac{\|B^{-1}\|}{(t+h-s)^{1-q}} \|f(s, u(s), Ku(s))\| ds + \int_0^{t-h} \frac{\|B^{-1}\|}{(t-h-s)^{1-q}} \|f(s, u(s), Ku(s))\| ds
\]

\[
+ \int_0^{h} S'(t+h-s) \int_0^{s} \frac{B^{-1}f(t, u(t), Ku(t))}{(t+h-\tau)^{1-q}} d\tau ds - \int_0^{t} S'(t-s) \int_0^{s} \frac{f(t, u(t), Ku(t))}{(t-\tau)^{1-q}} d\tau ds
\]

which tends to zero as \( h \to 0 \); since, by (H14), \( f \) is completely continuous, the set \( \{\Psi_2 u : u \in B_r(0; Z)\} \) is equicontinuous. Thus we have proved that \( \Psi_2(B_r(0; Z)) \) is relatively compact for \( t \in J \). By Arzela-Ascoli’s theorem, \( \Psi_2 \) is compact. Hence, by the Krasnoselskii fixed point theorem [112], there exists a fixed point \( u \in Z \) such that \( \Psi u = u \) which is a mild solution to the problem (6.3.8) with the nonlocal condition (6.3.9).
6.3.3 Application

Consider the following partial integrodifferential equation with fractional temporal derivative of the form

\[ \frac{\partial^q}{\partial t^q} \left( (u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x)) = \frac{\partial^2}{\partial x^2} u(t, x) + \int_0^t a_1(t-s) e^{-u(s,x)} ds + a_2(t) \sin(u(t, x)), \right. \]

\[ \left. t > 0, \quad (6.3.11) \right. \]

\[ u(t, 0) = u(t, \pi) = 0, \quad (t, x) \in J \times [0, \pi], \quad (6.3.12) \]

\[ u(0, x) + \sum_{i=1}^{n} \int_0^{t_i} b_i(\tau) u(\tau, x) d\tau = z(x), \quad (6.3.13) \]

where \( q \in (0, 1) \), \( z \in L^2[0, \pi] \) and \( a_i, b_i \in L^2(J) \). Take \( X = Y = L^2[0, \pi] \) and define the operators \( A : D(A) \subset X \to Y \) and \( B : D(B) \subset X \to Y \) by

\[ Aw = w'' \quad \text{and} \quad Bw = w - w'', \]

where each of the domains \( D(A) \) and \( D(B) \) is given by

\[ \{w \in X : w, w' \text{are absolutely continuous, } w'' \in X, \ w(0) = w(\pi) = 0\}. \]

Then \( A \) and \( B \) can be written respectively as \[79\]

\[ Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A), \]

\[ Bw = \sum_{n=1}^{\infty} (1 + n^2)(w, w_n) w_n, \quad w \in D(B), \]

where \( w_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \ n = 1, 2, \cdots \), is the orthogonal set of vectors \( A \). Furthermore, for \( w \in X \), we have

\[ B^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1 + n^2}(w, w_n) w_n, \]

\[ Ew = B^{-1}Aw = \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2}(w, w_n) w_n. \]

Now, from \[101\], we know that the integral equation

\[ u(t) = f(t) + \frac{1}{\Gamma(q)} \int_0^t \frac{E(u(s))}{(t-s)^{1-q}} ds, \quad s \geq 0, \]
has an associated analytic resolvent operator \( S(t), t \geq 0 \), on \( X \) given by

\[
S(t) = \begin{cases} 
\frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{q-1} (\lambda^q - E)^{-1} d\lambda, & t > 0, \\
I & t = 0,
\end{cases}
\]  

(6.3.14)

where \( \Gamma_{r,\theta} \) denotes a contour consisting of the rays \( \{re^{i\theta} : r \geq 0\} \) and \( \{re^{-i\theta} : r \geq 0\} \), for some \( \theta \in (\pi, \frac{\pi}{2}) \) and \( E = B^{-1}A \). It is easy to see that \( S(t) \) is differentiable [Proposition 2.15 in [11], Theorem 2.2 in [101]] and there exists a constant \( M > 0 \) such that

\[
\|S'(t)x\| \leq M \|x\|, \quad x \in D(E), \quad t > 0.
\]

To represent the differential equations (6.3.11)-(6.3.13) in the abstract form (6.3.8)-(6.3.9), we introduce the functions \( f : J \times X^2 \to Y \), \( g : Z \to X \) and \( k : \Omega \times X \to Y \) defined by

\[
f(t, w, Kw)(x) = w(x) + a_2(t) \sin w(x) + Kw(x),
\]

\[
Kw(x) = k(t, s, w(x)) = a_1(t - s)e^{-w(x)} \quad \text{and}
\]

\[
g(w(x)) = \sum_{i=1}^{n} \int_{0}^{t_i} b_i(\tau)w(\tau, x)d\tau.
\]

Note that \( \|g(u(x)) - g(v(x))\| \leq \sum_{i=1}^{n} t_i \|b_i\||u - v| \). Here \( \|\varphi E\|_{L^1} = M \), \( L_1 = (1 + \sup_{t \in J} \|a_2(t)\| + L_2) \), \( L_2 = \sup_{t \in J} \|a_1(t)\|, G = \sum_{i=1}^{n} t_i \|b_i\| \) and choose \( t_i \) such that \( r \geq 2(1 + M)(\gamma R(L_1N^* + N)) \) and \( 2(1 + M)(\gamma RL_1(1 + L_2) + G) < 1 \). Thus the conditions (H11)-(H17) of Theorem 6.3.1 are satisfied. Hence there is a function \( u \in C(J, L^2[0, \pi]) \) which is a mild solution of (6.3.11)-(6.3.13) on \( J \).

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