CHAPTER I
INTRODUCTION

Topology is an important branch of Mathematics. It has become a powerful instrument of mathematical research. The study of topological spaces, their continuous functions and general properties makes up one branch of topology known as ‘general topology’.

IDEAL TOPOLOGICAL SPACES

Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathasamy [155]. In 1990, Jankovic and Hamlett [64] once again initiated the applications of topological ideals in the generalization of most fundamental properties in general topology. The notion of $I_g$ - closed sets was introduced by Dontchev et al [41] in 1999. In 2007, Navaneethakrishnan and Joseph [99] further investigated and characterized $I_g$ - closed sets and $I_g$ - open sets by the use of the local function and obtained some of their properties. In 2010, Khan and Noiri [68] characterized $gL$ - closed sets in ideal topological spaces by the use of the semi local function and investigated some of their properties. In 2011, Khan and Hamza [69] introduced and investigated the notion of $I_{s\alpha}$ - closed sets in ideal topological spaces as generalization of $I_g$ - closed sets due to Dontchev et al [41].

GRILL TOPOLOGICAL SPACES

The idea of grills on a topological space was first introduced by Choquet [23] in 1947. In [137], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Hatir and Jafari [52] have defined new classes of sets in grill topological spaces. Quite recently, Ahmad Al-Omari and Noiri [6] introduced and investigated the notions of $\zeta\alpha$- open sets, $\zeta$ semi open sets and $\zeta\beta$ open sets in grill topological spaces.
SUPRA TOPOLOGICAL SPACES

In 1983, Mashhour et al [88] introduced supra topological spaces and studied $S$ - continuous maps and $S^*$ - Continuous maps. In 2008, Devi et al [32] introduced the concept of supra $\alpha$ -open sets, $S\alpha$ - continuous functions respectively. In 2010, Sayed et al [140] introduced and investigated several properties of supra $b$ - open sets and supra $b$ - continuity. In 2011, Ravi et al [132] introduced and investigated a new type of sets called supra $g$- closed and a new class of functions called supra $g$- continuous functions.

This thesis is an elaborated study of a new type of generalized closed sets in ideal topological spaces called $\omega I$ - closed sets, their respective continuous functions, closed functions, homeomorphism, compactness, connectedness, regular spaces and normal spaces. Further, the generalizations of closed sets and continuous functions via grills and supra topological spaces are studied.

In this chapter, the recent developments of topology contributed by various authors are mentioned and definitions cited by them are presented. Section 1 begins with the discussion of weak and strong forms of open sets and closed sets in general, ideal, grill and supra topological spaces. Section 2 deals with weak and strong forms of continuous functions, while section 3 is devoted to irresolute functions, closed and open functions. Some generalized homeomorphisms, some new type of compact spaces, connected spaces, regular spaces and normal spaces are discussed in section 4 while section 5 outlines the contribution of the author to various topological spaces. The last section describes the various notations used in the thesis. Throughout the thesis $(X, \tau), (Y, \sigma)$ and $(Z, \eta)$ denote topological spaces on which no separation axioms are discussed unless otherwise mentioned.

1.1 WEAK AND STRONG FORMS OF OPEN AND CLOSED SETS

After Levine’s work [75], where he introduced the notions of generalized closed sets (briefly $g$- closed), many mathematician have been published their articles using the notions related with generalization of closed sets [75,10,85,40,125,146,147,13,128,130] . Note that the complements of the various
types of closed (open) sets defined here are open (closed) sets of respective types.

We give the Definitions of some of them which are used in our present study.

**Definition 1.1.1** A subset $A$ of a topological space $(X, \tau)$ is said to be

(i) a semi-open set [74] if $A \subseteq \text{int}(cl(A))$ and a semi-closed set [24] if

\[ \text{int}(cl(A)) \subseteq A, \]

(ii) a pre-open set [89] if $A \subseteq \text{int}(cl(A))$ and a pre-closed set [89] if $\text{cl}(\text{int}(A)) \subseteq A.$

**Definition 1.1.2** A subset $A$ of a topological space $(X, \tau)$ is said to be

(i) a generalized closed (briefly $g$-closed) [75], if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau,$

(ii) a generalized semi-closed (briefly $gs$-closed) [11], if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau,$

(iii) a generalized pre-closed (briefly $gp$-closed) [81], if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau,$

(iv) an $\alpha$-generalized closed (briefly $\alpha g$-closed) [86], if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau,$

(v) a generalized semi-pre-closed (briefly $gsp$-closed) [37], if $\text{spcl}(A) \subseteq U$

whenever $A \subseteq U$ and $U \in \tau,$

(vi) a $\delta$-generalized semi-closed (briefly $\delta g$-closed) [40], if $\text{cl}_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau,$

(vii) a regular generalized closed (briefly $rg$-closed) [125], if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau \in RO(X, \tau),$ 

(viii) an $\omega$-closed [147], if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in SO(X, \tau),$ 

(ix) a semi generalized closed set (briefly $sg$-closed) [14], if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in SO(X, \tau),$ 

(x) a generalized $\alpha$-closed (briefly $g \alpha$-closed) [85], if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \alpha O(X, \tau),$ 

(xi) a generalized pre regular closed(briefly $gpr$-closed) [54], if $\text{pcl}(A) \subseteq U$

whenever $A \subseteq U$ and $U \in RO(X, \tau),$ 

(xii) an $\alpha$-generalized semi closed (briefly $\alpha gs$-closed) [129], if $\alpha \text{cl}(A) \subseteq U$
whenever \( A \subseteq U \) and \( U \in \text{SO}(X, \tau) \).

**Definition 1.1.3** A space \((X, \tau)\) is called

(i) a \( T_{1/2} \) space \([75]\), if every \( g \)-closed subset of \( X \) is closed in \( X \),

(ii) \( T_\omega \) space \([147]\), if every \( \omega \)-closed set of \( X \) is closed in \( X \).

**Definition 1.1.4** \([142]\) The intersection of all semi-open subsets of a space \( X \) containing set \( A \) is known as semi kernel of \( A \) and is denoted by \( \text{sker}(A) \).

**Definition 1.1.5** \([72]\) An ideal \( I \) on a topological space \((X, \tau)\) is a collection of non-empty subsets of \( X \) which satisfies the following properties:

(i) \( A \in I \) and \( B \subseteq A \) implies \( B \in I \) and

(ii) \( A \in I \) and \( B \in I \) implies \( A \cup B \in I \).

An ideal topological space is a topological space \((X, \tau)\) with an ideal \( I \) on \( X \) and is denoted by \((X, \tau, I)\). Let \( Y \) be a non-empty subset of \( X \), then \((X, \tau_Y, I_Y)\) is an ideal topological subspace of an ideal topological space \((X, \tau, I)\). It is obvious that the simplest ideals are \( \{\phi\} \) and \( P(X) = \{A \mid A \subseteq X\} \).

**Definition 1.1.6** \([72]\) For a subset \( A \subseteq X \), \( A^*(I, \tau) = \{x \in X \mid U \cap A \notin I \text{ for every } U \in \tau(x) \} \) where \( \tau(x) = \{U \in \tau \mid x \in U\} \) is called the local function of \( A \) with respect to \( I \) and \( \tau \). \( X^* \) is often a proper subset of \( X \). The hypothesis \( X = X^* \) is equivalent to the hypothesis \( \tau \cap I = \phi \). For every ideal topological space \((X, \tau, I)\), there exists a topology \( \tau^*(I) \), finer than \( \tau \) generated by \( \beta(I, \tau) = \{U \setminus J : U \in \tau \text{ and } J \in I\} \), but in general \( \beta(I, \tau) \) is not always a topology. When there is no chance for confusion we simply write \( A^* \) instead \( A^*(I, \tau) \) and \( \tau^\ast \) for \( \tau^*(I) \). Additionally, \( cl^\ast(A) = A \cup A^*(I, \tau) \) defines a Kuratowski closure operator for \( \tau^\ast(I) \) (briefly \( \tau^\ast \)).

**Definition 1.1.7** A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to be \( * \)-dense in itself \([61]\) (resp. \( * \)-perfect \([61]\), \( \tau^* \)-closed \([64]\)) if \( A \subseteq A^* \) (resp. \( A = A^* \), \( A^* \subseteq A \)).

**Definition 1.1.8** A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to
(i) \(I\)- open [65] if \(A \subseteq \text{int}(A^*)\),
(ii) semi-\(I\)- open if [57] \(A \subseteq \text{cl}^*(\text{int}(A))\),
(iii) pre-\(I\)- open [39] if \(A \subseteq \text{int}(\text{cl}^* (A))\).

**Definition 1.1.9** A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be
(i) \(I_{g}\)- closed [41], if \(A^*(I, \tau) \subseteq U\) whenever \(A \subseteq U\) and \(U \in \tau\),
(ii) \(gI\)- closed [68], if \(A^*(I, \tau) \subseteq U\) whenever \(A \subseteq U\) and \(U \in \tau\),
(iii) \(I_{s}^* g\)- closed [69], if \(A^*(I, \tau) \subseteq U\) whenever \(A \subseteq U\) and \(U \in \text{SO}(X, \tau)\),

**Definition 1.1.10** [68] Let \((X, \tau, I)\) be an ideal space and \(A\) a subset of \(X\). Then,
\[
A_s(I, \tau) = \{x \in X / U \cap A \notin I \text{ for every } U \in \text{SO}(X, x)\}\]
where \(\text{SO}(X, x) = \{U \in \text{SO}(X) | x \in U\}\) is called the semi local function of \(A\) with respect to \(I\) and \(\tau\).

**Definition 1.1.11** [71] An ideal topological space \((X, \tau, I)\) is said to be \(T\)- dense if every subset of \(X\) is \(\ast\)- dense in itself.

**Definition 1.1.12** [58] If \((X, \tau, I)\) is an ideal topological space and \(A\) is a subset of \(X\), we denote by \(\tau|_A\) the relative topology on \(A\) and \(I_s = \{A \cap J : J \in I\}\) is obviously an ideal on \(A\).

**Definition 1.1.13** [51] An ideal topological space \((X, \tau, I)\) is said to be \(I\)- Alexandroff if any intersection of open set is \(\ast\)- open.

**Definition 1.1.14** [51] An ideal topological space \((X, \tau, I)\) is said to be \(I_{g}\)- Alexandroff if any intersection of open sets in \((X, \tau, I)\) is \(I_{g}\)- open.

**Definition 1.1.15** [51] Let \((X, \tau, I)\) be an ideal topological space. \((X, \tau, I)\) is said to be an \(F^*\)- space if every open subset of \((X, \tau, I)\) is \(\ast\)- closed.

**Definition 1.1.16** [51] A topological space \((X, \tau)\) is said to be an \(R^*\)- space if \(\text{cl}^*(\{x\}) \subset U\) for each \(x \in X\) and each open set \(U\) with \(x \in U\).

**Definition 1.1.17** [59] An ideal topological space \((X, \tau, I)\) is called semi-Hausdorff if for each two distinct points \(x \neq y\), there exist semi-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively such that \(U \cap V = \emptyset\). Then the points \(x\) and \(y\) are said to be semi-separated.
Definition 1.1.18 [59] An ideal topological space \((X, \tau, I)\) is called \(I\)-Hausdorff if for each two distinct points \(x \neq y\), there exist \(I\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively such that \(U \cap V = \emptyset\). Then the points \(x\) and \(y\) are said to be \(I\)-separated.

Definition 1.1.19 [59] An ideal topological space \((X, \tau, I)\) is called semi-\(I\)-complete if \(\tau^* = SIO(X, \tau)\), that is, a subset \(A\) of \(X\) is \(\tau^*\)-open if and only if it is semi-\(I\)-open.

Definition 1.1.20 [23] A non-null collection \(\zeta\) of subsets of a topological spaces \(X\) is said to be a grill on \(X\) if

(i) \(\emptyset \notin \zeta\),
(ii) \(A \in \zeta\) and \(A \subseteq B\) implies that \(B \in \zeta\),
(iii) \(A, B \subseteq X\) and \(A \cup B \in \zeta\) implies that \(A \in \zeta\) or \(B \in \zeta\).

Definition 1.1.21 [137] Let \((X, \tau, \zeta)\) be a topological space and \(\zeta\) be a grill on \(X\). A mapping \(\Phi: P(X) \to P(X)\) is defined as follows: \(\Phi(A) = \Phi_{\zeta}(X, \tau, \zeta) = \{x \in X \mid A \cap U \in \zeta\ \text{for all} \ U \in \tau(x) \ \text{for each} \ A \in P(X)\}\). The mapping \(\Phi\) is called the operator associated with the grill \(\zeta\) and the topology \(\tau\).

Definition 1.1.22 Let \((X, \tau)\) be a topological space and \(\zeta\) be any grill on \(X\). Then a subset \(A\) of \(X\) is called

(i) \(\Phi\)-open [60] if \(A \subseteq \text{int}(\Phi(A))\),
(ii) \(\zeta\)-semi-open [6] if \(A \subseteq \psi(\text{int}(A))\),
(iii) \(\zeta\)-pre open if [60] \(A \subseteq \text{int}(\psi(A))\).

Definition 1.1.23 [33] Let \((X, \tau)\) be a topological space and \(\zeta\) be any grill on \(X\). Then a subset \(A\) of \(X\) is called \(\zeta\)-\(g\)-closed if \(\Phi(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

Definition 1.1.24 [139] Let \(X\) be a space and \((\emptyset \neq)A \subseteq X\). Then \([A] = \{B \subseteq X : A \cap B \neq \emptyset\}\) is a grill on \(X\) called principal grill generated by \(A\).
Definition 1.1.25 [88,140] A subfamily $\mu$ of $X$ is said to be a supra topology on $X$ if

(i) $X, \emptyset \in \mu$, 

(ii) if $A_i \in \mu$ for all $i \in I$ then $\bigcup A_i \in \mu$.

The pair $(X, \mu)$ is called supra topological space. The elements of $\mu$ are called supra open sets in $(X, \mu)$ and complement of a supra open set is called supra closed set.

Definition 1.1.26 [140]

(i) The supra closure of a set $A$ is denoted by $cl^\mu(A)$ and is defined as $cl^\mu(A) = \bigcap \{B : B$ is a supra closed and $A \subseteq B\}$.

(ii) The supra interior of a set $A$ is denoted by $int^\mu(A)$, and defined as $int^\mu(A) = \bigcup \{B : B$ is a supra open set and $A \supseteq B\}$.

Definition 1.1.27 [88] Let $(X, \tau)$ be a topological space and $\mu$ be a supra topology on $X$. We call $\mu$ a supra topology associated with $\tau$ if $\tau \subseteq \mu$.

Definition 1.1.28 Let $(X, \mu)$ be a supra topological space. A subset $A$ of $X$ is called

(i) supra semi open set [140], if $A \subseteq cl^\mu(int^\mu(A))$.

(ii) supra $b$ - open set [140], if $A \subseteq cl^\mu(int^\mu(A)) \cup int^\mu(cl^\mu(A))$.

(iii) supra $\alpha$ -open set [32], if $A \subseteq int^\mu(cl^\mu(int^\mu(A)))$.

The complement of above mentioned open sets are called their respective closed sets.

Definition 1.1.29 Let $(X, \mu)$ be a supra topological space. A set $A$ of $X$ is called

(i) supra generalized closed set (simply $g^\mu$ - closed) [132] if $cl^\mu(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is supra open.

The complement of supra generalized closed set is supra generalized open set.

(ii) supra semi – generalized closed set (simply $sg^\mu$ - closed [66] if $scl^\mu(A) \subseteq U$ and $U$ is supra semi open.

The complement of supra semi – generalized closed set is supra semi generalized open set.
(iii) supra generalized – semi closed set (simply $gs^\mu$ - closed)[66] if $scl^\mu(A) \subseteq U$

whenever $A \subseteq U$ and $U$ is supra open

The complement of supra generalized – semi closed set is supra generalized semi – open set.

**Definition 1.1.30** [132] Let $A$ and $B$ be subsets of $X$. Then the set $A$ and $B$ are said to be supra separated if $cl^\mu(A) \cap B = A \cap cl^\mu(B) = \emptyset$.

### 1.2 STRONG AND WEAK FORMS OF CONTINUOUS FUNCTIONS

One of the important and basic topics in theory of classical point set topology and several branches of mathematics, which have been researched by many authors, is continuity of functions. Strong and weak forms of continuous functions have been introduced and studied by several topologists [73, 12, 13, 126, 90, 1]. This concept has been extended to the setting of $I$-continuity of functions. Jankovic and Hamlett [65] introduced the notion of $I$-open sets in ideal topological spaces. Dontchev [39] introduced the notion of pre-$I$-open sets and obtained a decomposition of continuity. The notion of semi-$I$-open sets to obtain decomposition of continuity was introduced by Hatir and Noiri [58]. Recently, by using $Is^*g$ - closed sets, Khan and Noiri [71] introduced $Is^*g$ - continuous functions, strongly $Is^*g$ - continuous functions and weakly $Is^*g$ - continuous functions. Here we give definitions of strong and weak forms of continuous functions which are used in our study.

**Definition 1.2.1** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be function and $f$ is said to be

(i) semi-continuous [74] if $f^{-1}(V)$ is semi-closed in $X$ for every closed set $V$ of $Y$.

(ii) $g$ - continuous [12] if $f^{-1}(V)$ is $g$ - closed in $X$ for every closed set $V$ of $Y$.

(iii) $sg$ - continuous [145] if $f^{-1}(V)$ is $sg$ - closed in $X$ for every closed set $V$ of $Y$.

(iv) $rg$ - continuous [125] if $f^{-1}(V)$ is $rg$ - closed in $X$ for every closed set $V$ of $Y$.

**Definition 1.2.2** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $f$ is said to be

(i) strongly continuous [76] if $f^{-1}(V)$ is both open and closed in $X$ for each subset $V$ of $Y$,
(ii) weakly continuous [73] if for each open set \( V \) of \( Y \) containing \( f(x) \), there exists an open set \( U \) containing \( x \) such that \( f(U) \subseteq cl(V) \).

**Definition 1.2.3** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function and \( f \) is said to be

(i) strongly \( g \)-continuous [146] if \( f^{-1}(V) \) is both open and closed in \( X \) for each subset \( V \) of \( Y \),

(ii) perfectly \( g \)-continuous [146] if for each open set \( V \) of \( Y \) containing \( f(x) \), there exists an open set \( U \) containing \( x \) such that \( f(U) \subseteq cl(V) \).

**Definition 1.2.4** Let \((X, \tau, I)\) be an ideal topological space. A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is called

(i) semi-\( I \)-continuous [58] if the inverse image of each open set in \( Y \) is semi-\( I \)-open in \((X, \tau, I)\).

(ii) pre-\( I \)-continuous [39] if the inverse image of each open set in \( Y \) is pre-\( I \)-open in \((X, \tau, I)\).

(iii) \( I_{s*g} \)-continuous [71] if for every open set \( V \) of \((Y, \sigma)\), \( f^{-1}(V) \) is \( I_{s*g} \)-open in \((X, \tau, I)\).

**Definition 1.2.5** Let \((X, \tau, \zeta)\) be a grill topological space. A function \( f : (X, \tau, \zeta) \rightarrow (Y, \sigma) \) is called

(i) \( \zeta \) semi continuous [52] if the inverse image of each open set in \( Y \) is \( \zeta \) semi open in \((X, \tau, \zeta)\).

(ii) \( \zeta \) pre continuous [52] if the inverse image of each open set in \( Y \) is \( \zeta \) pre open in \((X, \tau, \zeta)\).

**Definition 1.2.6** Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \( \mu \) be an associated supra topology with \( \tau \). A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called

(i) supra continuous [32] if the inverse image of each open set in \( Y \) is supra open in \( X \).

(ii) supra \( \alpha \)-continuous [32] if \( f^{-1}(V) \) is supra \( \alpha \)-open in \( X \) for every open set \( V \) of \( Y \).

(iii) supra semi-continuous [140] if \( f^{-1}(V) \) is supra semi-open in \( X \) for every open
set \( V \) of \( Y \).

(iv) supra \( b \)-continuous [140] if the inverse image of each open set in \( Y \) is supra \( b \)-open in \( X \).

(v) supra \( g \)-continuous [132] if the inverse image of each open set in \( Y \) is supra \( g \)-open in \( X \).

### 1.3 Irresolute Functions, Closed Functions and Open Functions

Crossely and Hildebrand [25] introduced and investigated irresolute functions which are stronger than semi-continuous functions but are independent of continuous functions. Since then various strong and weak forms of irresolute functions have been introduced and studied by many researchers ([18, 40, 77, 101, 127]). Yuksel et al [157] introduced the notion of \( \alpha - I \)-irresolute functions and \( \alpha - pre - I \)-irresolute functions. Recently, Acikgoz et al [4] introduced and investigated the notions of \( \alpha - I \)-preirresolute functions and \( \beta - I \)-preirresolute functions. Here we recall the following Definitions.

**Definition 1.3.1** Let \( f : (X, \tau) \to (Y, \sigma) \) be a function and \( f \) is said to be

(i) irresolute [25] if \( f^{-1}(V) \) is semi-open in \( X \) for every semi-open set \( V \) of \( Y \).

(ii) \( \alpha g \)-irresolute [29] if \( f^{-1}(V) \) is \( \alpha g \)-closed in \( X \) for every \( \alpha g \)-closed set \( V \) of \( Y \).

**Definition 1.3.2** A function \( f : (X, \tau, I) \to (Y, \sigma, J) \) is called

(i) semi-\( I \)-irresolute [58], if \( f^{-1}(V) \) is semi-\( I \)-open in \( (X, \tau, I) \) for every semi-\( I \)-open set \( V \) of \( (Y, \sigma, J) \).

(ii) \( \alpha - I \)-Irresolute [157], if \( f^{-1}(V) \) is \( \alpha - I \)-open in \( (X, \tau, I) \) for every \( \alpha \)-open set \( V \) of \( (Y, \sigma, J) \).

(iii) pre-\( I \)-irresolute [4], if \( f^{-1}(V) \) is pre-\( I \)-open in \( (X, \tau, I) \) for every pre open set \( V \) of \( (Y, \sigma, J) \).

**Definition 1.3.3** A function \( f : (X, \tau, \zeta_1) \to (Y, \sigma, \zeta_2) \) is called

(i) grill irresolute [6], if \( f^{-1}(V) \) is \( \zeta \)-semi open in \( (X, \tau, \zeta) \) for every \( \zeta \)-semi open

**Definition 1.3.4** [79] A space $(X, \tau)$ is called Semi-Hausdorff if for each two distinct points $x \neq y$, there exist semi-open sets $U$ and $V$ containing $x$ and $y$ respectively such that $U \cap V = \emptyset$.

**Definition 1.3.5** An ideal topological space $(X, \tau, I)$ is called

(i) $I$-Hausdorff [36] if for each two distinct points $x \neq y$, there exist $I$-open sets $U$ and $V$ containing $x$ and $y$ respectively such that $U \cap V = \emptyset$.

(ii) semi-$I$-Hausdorff [59] if for each two distinct points $x \neq y$, there exist semi-$I$-open sets $U$ and $V$ containing $x$ and $y$ respectively such that $U \cap V = \emptyset$.

Generalized closed mapping were introduced and studied by Malghan [82], Noiri [110], Biswas [16], Mashhour [90], Sundaram [146], Crossley and Hilderbrand [24], have defined and studied semi closed maps and semi-open maps, pre-open maps and weakly pre-open maps, $\alpha$-open and $\alpha$-closed maps, generalized open maps, pre-semi-open maps respectively. Here we recall some of the definitions which are used in our study.

**Definition 1.3.6** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $f$ is said to be

(i) semi-closed [107] if $f(V)$ is semi-closed in $Y$ for every closed set $V$ of $X$, 
(ii) pre-closed [50] if $f(V)$ is pre-closed in $Y$ for every closed set $V$ of $X$, 
(iii) $g$-closed [82] if $f(V)$ is $g$-closed in $Y$ for every closed set $V$ of $X$, 
(iv) $\alpha$-open [90] if $f(V)$ is $\alpha$-open in $Y$ for every open set $V$ of $X$, 
(v) $\beta$-open [1] if $f(V)$ is $\beta$-open in $Y$ for every open set $V$ of $X$.

**Definition 1.3.7** Let $(X, \tau, I)$ be an ideal topological space. A function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is said to be
(i) semi-$I$-open [58] if $f(V)$ is semi-$I$-open in $Y$ for every open set $V$ of $X$,
(ii) pre-$I$-open [92] if $f(V)$ is pre-$I$-open in $Y$ for every open set $V$ of $X$,
(iii) $\alpha$-$I$-open [92] if $f(V)$ is $\alpha$-$I$-open in $Y$ for every open set $V$ of $X$.

**Definition 1.3.8** [51] A function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is said to be *-closed if $f(A)$ is *-closed in $(Y, \sigma, J)$ for every *-closed subset $A$ of $(X, \tau, I)$.

**Definition 1.3.9** [62] Let $(X, \tau, \zeta')$ be a grill topological space. Let $f : (X, \tau, \zeta') \rightarrow (Y, \sigma, \lambda)$ be a function and $f$ is said to be $\zeta'\gamma$-open if for each $U \in \tau$, $f(U)$ is $\zeta'\gamma$-open in $Y$.

**Definition 1.3.10** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\mu$ be an associated supra topology with $\tau$. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) supra open (resp. supra closed) [132] if the image of each open (resp. closed) set in $X$ is supra open (resp. supra closed) in $Y$.
(ii) supra $g$-open (resp. supra $g$-closed) [132] if the image of each open (resp. closed) set in $X$ is supra $g$-open in $Y$.
(iii) supra $b$-open (resp. supra $b$-closed) [140] if the image of each open (resp. closed) set in $X$ is supra $b$-open (resp. supra $b$-closed) in $Y$.

### 1.4 GENERALIZED HOMEOMORPHISM

The notion of homeomorphism plays a very important role in topology. Many researchers have generalized the notions of homeomorphism in topological spaces. Biswas [16], Crossely and Hildebrand [25] have introduced and studied semi-Homeomorphism which are strictly weaker than homeomorphism in topological spaces. Maki et al [84] have introduced and investigated $g$-homeomorphism and $gc$-homeomorphism in topological spaces. Sheik John [142] introduced and studied $\omega$-homeomorphisms and $\omega^*$-homeomorphisms in topological spaces. Recently several topologists have introduced and studied several types of generalized homeomorphisms in topological spaces ([10, 28, 54, 101, 130, 142]).
**Definition 1.4.1** A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) a generalized homeomorphism (g-homeomorphism) [84] if $f$ is both $g$-continuous and $g$-open.

(ii) a gc-homeomorphism [84] if both $f$ and $f^{-1}$ are gc-irresolute functions.

(iii) a generalized semi-homeomorphism (sg-homeomorphism) [28] if $f$ is both $sg$-continuous and $sg$-open.

(iv) a semi-generalized homeomorphism (gs-homeomorphism) [28] if $f$ is both $gs$-continuous and $gs$-open.

Using semi-open covers Di Maio and Noiri [35] have introduced and studied a new class of compact spaces called $s$-closed spaces. The concept of compactness modulo an ideal was defined by Newcomb [105] and has also been studied by Rancin [131], this concept has been further investigated by Hamlett and Jankovic [55]. Newcomb [105] also defined the concept of countable compactness modulo an ideal. In 1991, Hamlett et al [56] have further studied the latter concept under the term countably $I$-compact. In 2001, Nasef [8] used $\gamma$-open sets to define some classes of compactness in terms of ideals.

**Definition 1.4.2** A collection $\{A_{\alpha} : \alpha \in \Delta \}$ of $g$-open (resp. $\omega$-open [142]) sets in a topological space $(X, \tau)$ is called $g$-open (resp. $\omega$-open) cover of a subset $B$ if $B \subseteq \bigcup_{\alpha \in \Delta} A_{\alpha}$.

**Definition 1.4.3** A topological space $(X, \tau)$ is called GO-compact [12] ($\omega$-compact [142]) if every $g$-open (resp. $\omega$-open) cover of $(X, \tau)$ has a finite subcover.

**Definition 1.4.4** [56,105] A subset $A$ of an ideal topological space $(X, \tau, I)$ is called compact modulo an ideal or $I$-compact if for every $I$-open cover $\{W_{\alpha} : \alpha \in \Delta \}$ of $(X, \tau, I)$ there exists a finite subset $\Delta_0$ of $\Delta$ such that $(A - \bigcup \{W_{\alpha} : \alpha \in \Delta_0\}) \in I$. The space $(X, \tau, I)$ is $I$-compact if $X$ is $I$-compact as a subset.

**Definition 1.4.5** [56] A space $(X, \tau, I)$ is called $I$-compact if $X$ is $I$-compact as a subset.
Definition 1.4.6 [8] A space $(X, \tau, I)$ is called $\gamma$-compact modulo an ideal or $\gamma I$-compact if for every cover $\{W_\alpha : \alpha \in \Delta\}$ by $\gamma$-open sets of $(X, \tau, I)$, there exists a finite subset $\Delta_\gamma$ of $\Delta$ such that $(X - \cup \{W_\alpha : \alpha \in \Delta_\gamma\}) \in I$.

Definition 1.4.7 [34] A space is called Lindelof if every open cover contains a countable subcover.

Definition 1.4.8 A topological space $(X, \tau)$ is called GO-connected [12] ($\omega$-connected [142]) if $(X, \tau)$ cannot be written as a disjoint union of non-empty $g$-open (resp. $\omega$-open) sets.

Munshi [97] introduced $g$-normal and $g$-regular spaces using $g$-closed sets in topological spaces. Sheik john [142] introduced and investigated $\omega$-regular and $\omega$-normal spaces in topological spaces. Navaneethakrishnan et al [100] introduced and investigated $I_g$-regular and $I_g$-normal spaces in ideal topological spaces.

Definition 1.4.9 A topological space $(X, \tau)$ is said to be Semi-normal [46] (resp. $S^*$-normal [67], $g$-normal [97] and $\omega$-normal [142]) if for each pair of disjoint semi-closed (resp. semi-closed, $g$-closed and $\omega$-closed) sets $A$ and $B$, there exists disjoint semi-open (resp. open and open) sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Definition 1.4.10 [100] An ideal space $(X, \tau, I)$ is said to be an $I_g$-normal if for every pair of disjoint closed sets $A$ and $B$, there exist $I_g$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Definition 1.4.11 A topological space $(X, \tau)$ is said to be s-regular [78] (resp. $g$-regular [97] and $\omega$-regular [142]) if for each closed (resp. $g$-closed and $\omega$-closed) set $F$ of $(X, \tau)$ and each point $x \in F^c$, there exist disjoint semi-open (resp. open and open) sets $U$ and $V$ such that $F \subseteq U$ and $x \in V$.

Definition 1.4.12 [100] An ideal space $(X, \tau, I)$ is said to be $I_g$-regular if for each pair consisting of a point $x$ and a closed set $B$ not containing $x$, there exist disjoint $I_g$-open sets $U$ and $V$ such that $x \in U$ and $B \subseteq V$. 

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1.5 CONTRIBUTION OF THE AUTHOR

In the light of the above work, the author has obtained interesting generalizations of closed sets and continuous maps in various topological spaces on the following topics:

(i) $\omega l$ - closed sets in ideal topological spaces,

(ii) $\omega l$ - continuous functions in ideal topological spaces,

(iii) $\omega l$ - homeomorphism in ideal topological spaces,

(iv) $\zeta \omega$ - closed sets in grill topological spaces,

(v) $\zeta \omega$ - continuous functions in grill topological spaces,

(vi) $\omega^\mu$ - closed sets and $\omega^\mu$ - continuous functions in supra topological spaces.

The rest of the thesis is the detailed study of the above topics.
1.6 NOTATIONS

\[ P(X) \] - the power set of \( X \)

\[ \Lambda \] - the index set

\[ (X - A) \text{ or } A^c \] - the complement of \( A \)

\[ \text{int}(A) \] - interior of \( A \)

\[ \text{int}^*(A) \] - interior of \( A \) with respect to ideal

\[ \text{int}^\zeta(A) \] - interior of \( A \) with respect to grill

\[ \text{int}^\mu(A) \] - supra interior of \( A \)

\[ \text{cl}(A) \] - closure of \( A \)

\[ \text{cl}^*(A) \] - closure of \( A \) with respect to ideal

\[ \text{cl}^\zeta(A) \] - closure of \( A \) with respect to grill

\[ \text{cl}^\mu(A) \] - supra closure of \( A \)

\[ SO(X, \tau) \] - the set of all semi open subsets of \((X, \tau)\)

\[ SIO(X, \tau) \] - the set of all semi-I-open subsets of \((X, \tau, I)\)

\[ \zeta SO(X, \tau) \] - the set of all \( \zeta \) semi open subsets of \((X, \tau, \zeta)\)

\[ SO(X, \mu) \] - the set of all supra semi open subsets of \((X, \mu)\)

\[ I_f \] - the ideal of finite subsets of \( X \)

\[ I_c \] - the ideal of countable subsets of \( X \)

\[ I_n \] - the ideal of nowhere dense subsets of \( X \)

\[ I_{cd} \] - the ideal of closed discrete subsets of \( X \)