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DERIVATIONS ON PRIME AND SEMIPRIME NEAR-RINGS
Bell and Mason [11] studied derivations in near-rings and near fields. In this chapter we present some properties of prime and semiprime left near-rings, and also the properties of strong commutativity-preserving derivations in left near-rings.

In section 3.1, we extend some results concerning derivations of near-rings to the semiprime left near-ring $N$ and we prove that if $d$ acts as a homomorphism or an anti-homomorphism on a subset $A$ of $N$, then $d(A)={0}$. In section 3.2, we prove some results concerning strong commutativity-preserving derivation on $N$. We present a result that if $N$ has no zero divisors and admits a nonzero commuting strong commutativity-preserving derivation, then $N$ is a commutative ring with no idempotents except 0 or 1. Using this property we show that if $N$ admits a nonzero commuting derivation $d$ such that $[x,y] = [d(x), d(y)]$ for all $x,y$ in $U$ where $U$ is a right ideal of $N$, then $N$ is a commutative ring. In section 3.3, we consider a prime left near-ring with $U$ as a right ideal and prove that if a derivation $d$ on $N$ is such that $d([x,y]) = [x,y]$ or $d([x,y]) = -[x,y]$ or $d(\circ) = x \circ y$ or $d(\circ) = -(x \circ y)$ for all $x,y \in U$, then $N$ is a commutative ring. Also we give examples to show that the primeness hypothesis in the above result is necessary.
3.1. Derivations on Semiprime left near-rings


In this section we prove some results concerning derivations of semiprime left near-rings. We prove that if $d$ is a derivation of a semiprime near-ring $N$ which acts as a homomorphism or anti-homomorphism on a subset $A$, then $d(A) = \{0\}$.

We know that a left near-ring is a set $N$ with two operations $+$ and $\cdot$ such that $(N, +)$ is a group and $(N, \cdot)$ is a semigroup satisfying the left distributive law $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x,y,z \in N$. A near-ring $N$ is said to be prime if $x Ny = 0$ for $x,y \in N$ implies $x = 0$ or $y = 0$ and semiprime if $xNx = 0$ for all $x \in N$ implies $x = 0$. An additive mapping $d$ is said to be a homomorphism if $d(xy) = d(x) d(y)$ and anti-homomorphism if $d(xy) = d(y) d(x)$ for all $x,y \in N$.

Throughout this section, $N$ denotes a semiprime left near-ring with a derivation $d$ satisfying $d(xy) = xd(y) + d(x)y$ for all $x,y$ in $N$.

In order to prove the theorem, first we prove the following Lemmas.

Lemma 3.1.1: If $N$ is a left near-ring and $d$ is a derivation of $N$, then $(yd(x) + d(y)x)c = yd(x)c + d(y)xc$ for all $x,y,c \in N$. 

Proof: We have that
\[ d((yx)c) = yxd(c) + d(y)xc = yxd(c) + (yd(x) + d(y)x)c, \text{ for all } x,y,c \in N. \]

Also \( d(y(xc)) = yd(xc) + d(y)xc = y(xd(c) + d(x)c) + d(y)xc = yxd(c) + yd(x)c + d(y)xc \), for all \( x,y,c \in N. \)

Combining these two relations, we obtain
\[ (yd(x) + d(y)x)c = yd(x)c + d(y)xc, \text{ for all } x,y,c \in N. \]

\[ \Box \]

Lemma 3.1.2: Let \( N \) be a left near-ring, \( d \) a derivation of \( N \) and \( A \) a multiplicative subsemigroup of \( N \) which contains 0. If \( d \) acts as an anti-homomorphism on \( A \), then \( 0 \cdot a = 0 \) for all \( a \in A \).

Proof: Since \( a \cdot 0 = 0 \) for all \( a \in A \) and \( d \) acts as a anti-homomorphism on \( A \), we have \( 0 \cdot d(a) = 0 \) for all \( a \in A \).

Taking \( 0a \) instead of \( a \), we obtain \( 0 \cdot d(a) + 0 \cdot a = 0 \).

Hence \( 0 \cdot a = 0 \) for all \( a \in A \). \( \Box \)

Lemma 3.1.3: Let \( N \) be a left near-ring and \( A \) a multiplicative subsemigroup of \( N \). (i) If \( d \) acts as a homomorphism on \( A \), then
\[ d(y)xd(y) = yxd(y) = d(y)xy \text{ for all } x,y \in A, \]

(ii) If \( d \) acts as an anti-homomorphism on \( A \), then,
\[ d(y)xd(y) = d(y)yx = xy d(y) \text{ for all } x,y \in A. \]

Proof: (i) Let \( d \) acts as a homomorphism on \( A \).
Then \( d(xy) = xd(y) + d(x) \ y = d(x) \ d(y) \) for all \( x,y \in A \). \[3.1.3\]

Taking \( yx \) instead of \( x \) in the equation \(3.1.3\), we get
\[
yx \ d(y) + d(yx) \ y = d(yx) \ d(y) = d(y) \ d(xy).
\] \[3.1.4\]

By left distributive law, \( d(y) \ d(xy) = d(y) \ x \ d(y) + d(y) \ d(x) \ y = d(y) \ x \ d(y) + d(\ yx) \ y \) for all \( x,y \in A \).

Using this in the equation \(3.1.4\), we obtain
\[
d(y) \ x \ d(y) = y \ x \ d(y) \text{ for all } x,y \in A.
\]

Similarly, taking \( yx \) instead of \( y \) in the equation \(3.1.3\), we get
\[xd(yx) + d(x) \ yx = d(x) \ d(yx) = d(xy) \ d(x)\]. This implies \( xd(yx) + d(x) \ yx = (x \ d(y) + d(x) \ y) \ d(x) = xd(y) \ d(x) + d(x) \ y \ d(x) = xd(yx) + d(x) \ y \ d(x)\).

So \( d(x)yx = d(x) \ y \ d(x), \text{ for all } x,y \in A\).

By interchanging \( x \) and \( y \), we obtain
\[
d(y) \ xy = d(y) \ x \ d(y), \text{ for all } x, y \in A.
\]

(ii) Since \( d \) acts as an anti-homomorphism on \( A \), we have
\[d(xy) = xd(y) + d(x) \ y = d(y) \ d(x), \text{ for all } x, y \in A.\] \[3.1.5\]

Substituting \( xy \) for \( y \) in the equation \(3.1.5\), we obtain
\[xd(xy) + d(x)xy = d(xy)d(x) = (xd(y) + d(x)y)d(x) = xd(y)d(x) + d(x)y\ d(x) = x \ d(xy) + d(x) \ y \ d(x)\).

So \( d(x) \ xy = d(x) \ y \ d(x), \text{ for all } x,y \in A\).
Similarly, by taking \( xy \) instead of \( x \) in the equation 3.1.5, we obtain
\[
xyd(y) + d(xy)y = d(y)d(xy) = d(y)xd(y) + d(y)d(x) y = d(y) x d(y) + d(xy) y.
\]
Hence \( xyd(y) = d(y)xd(y) \), for all \( x, y \in A \).

**Theorem 3.1.1**: Let \( N \) be a semiprime left near-ring, and \( d \) a derivation on \( N \). Let \( A \) be a subset of \( N \) such that \( 0 \in A \) and \( NA \subseteq A \). If \( d \) acts as a homomorphism on \( A \) or as an anti-homomorphism on \( A \), then \( d(A) = \{0\} \).

**Proof**: Suppose that \( d \) acts as a homomorphism on \( A \).

By Lemma 3.1.3 (i), we have
\[
d(y) x d(y) = yx d(y), \text{ for all } x, y \in A.
\]

Left multiplying the equation 3.1.6 by \( d(z) \), where \( z \in A \), we obtain
\[
d(z) d(y) x d(y) = d(z) yx d(y).
\]

This implies that
\[
d(z) y x d(y) = d(z) y x d(y).
\]

That is\( (zd(y) + d(z)y) x d(y) = d(z) y x d(y) \).

Now by using the Lemma 3.1.1, we get
\[
zd(y)x d(y) + d(z) y xd(y) = d(z) yx d(y).
\]

So\( zd(y) x d(y) = 0 \), for all \( x, y, z \in A \).

By taking \( rx \) instead of \( x \), where \( r \in N \), we obtain
\[
zd(y)rx d(y) = 0, \text{ for all } x, y, z \in A, \quad r \in N.
\]

Hence \( xd(y) N xd(y) = \{0\} \), for all \( x, y \in A \).

By semiprimeness of \( N \), we get \( xd(y) = 0 \), for all \( x, y \in A \).
Now we substitute \( ry \) for \( y \) in the equation 3.1.7, where \( r \in N \). Then it leads to \( xd(ry) = 0 = xrd(y) + xd(r)y \), for all \( x, y \in A, r \in N \).  

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Right multiplying the equation 3.1.8 by \( d(z) \), where \( z \in A \), we get

\[
xr d(y) d(z) + xd(r)y d(z) = 0.
\]

By using the equation 3.1.7, it reduces to \( xrd(y)d(z) = 0 \) and \( xrd(yz) = 0 = xryd(z) + xrd(y)z \).

Again using the equation 3.1.7, we find that \( xrd(y)z = 0 \), for all \( x, y, z \in A, r \in N \).

Hence \( d(y)_zrd(y)_z = 0 \) and by semiprimeness, we get

\[
d(y)_z = 0 = d(y)_rz, \quad \text{for all } y, z \in A, r \in N.
\]

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Combining the equations 3.1.7 and 3.1.9, we get \( d(yz) = 0 \), for all \( y, z \in A \).

In particular, \( d(xrx) = 0 \), for all \( x \in A, r \in N \).

Since \( d \) acts as a homomorphism on \( A \), \( d(x) d(rx) = 0 \).

That is, \( d(x) rd(x) + d(x)d(r)x = 0 \), for all \( x \in A, r \in N \).

By using the equation 3.1.9, it reduces to \( d(x)_N d(x) = \{0\} \).

Hence by semiprimeness, we obtain \( d(x) = 0 \), for all \( x \in A \).

Now we assume that \( d \) acts as an anti-homomorphism on \( A \).

We note that \( 0 \cdot a = 0 \), for all \( a \in A \), by Lemma 3.1.2.

According the Lemma 3.1.3(ii), we have
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\[ d(y)xd(y) = d(y)yx, \text{ for all } x, y \in A \] \hspace{1cm} 3.1.10

\[ \text{and } d(y)xd(y) = xyd(y), \text{ for all } x, y \in A. \] \hspace{1cm} 3.1.11

By replacing \( x \) by \( d(y)x \) in the equation 3.1.10, we get

\[ d(y)d(y)xd(y) = d(y)yd(y)x. \] This implies that \( d(y)^2xd(y) = d(y)yd(y)x. \)

so,

\[ yd(y)xd(y) + d(y)yxd(y) = d(y)yd(y)x. \] \hspace{1cm} 3.1.12

Substituting \( yx \) for \( x \) in the equation 3.1.10, we get

\[ d(y)yxd(y) = d(y)y^2x. \] \hspace{1cm} 3.1.13

Left multiplying the equation 3.1.10 by \( y \), we get

\[ yd(y)xd(y) = yd(y)yx. \] \hspace{1cm} 3.1.14

Replacing \( x \) by \( y \) in the equation 3.1.10, we obtain

\[ d(y) yd(y) = d(y)y^2. \]

Multiplying this right by \( x \), we get

\[ d(y)yd(y)x = d(y)y^2x, \text{ for all } x, y \in A. \] \hspace{1cm} 3.1.15

Using the equations 3.1.13, 3.1.14 and 3.1.15 in the equation 3.1.12, we find that

\[ yd(y)yx = 0, \text{ for all } x, y \in A. \]

Hence \( yd(y)yry = 0 \) and \( yd(y)yryd(y)y = 0 \), for all \( y \in A, r \in N \).

By semiprimeness, we get \( yd(y)y = 0 \), for all \( y \in A \).

Using this in the equation 3.1.14, we get

\[ yd(y)xd(y) = 0, \text{ for all } x, y \in A. \]
Applying this in the equation 3.1.11, we obtain

\[ yxyd(y) = 0, \text{ for all } x,y \in A. \]  \hspace{1cm} 3.1.16

Replacing \( x \) by \( d(y)x \) in the equation 3.1.16, we obtain

\[ yd(y)xyd(y) = 0 = yd(y)rxyd(y). \]

Hence \( xyd(y)rxyd(y) = 0, \text{ for all } x,y \in A, \ r \in N. \)

Since \( N \) is semiprime, we obtain \( xyd(y) = 0, \text{ for all } x,y \in A. \)  \hspace{1cm} 3.1.17

Using this equation in the equation 3.1.11, we obtain

\[ d(y)xd(y) = 0 = xd(y)rxd(y), \text{ for all } x,y \in A, \ r \in N. \]

Hence \( xd(y) = 0, \text{ for all } x,y \in A. \)  \hspace{1cm} 3.1.18

Therefore \( xd(ny)d(z)x = 0, \text{ for all } x,y,z \in A, \ n \in N. \)

That is, \( x(nd(y) + d(n)y) d(z)x = 0 = xnd(y)d(z)x + xd(n)yd(z)x. \)

Using the equation 3.1.18, this gives \( xnd(y)d(z)x = 0 \) and which implies that \( d(y)d(z)x = 0. \)

Hence \( d(y)d(z)x = 0, \text{ for all } x,y,z \in A. \)

Since \( d \) acts as an anti-homomorphism on \( A \), we have \( d(zy)x = 0, \text{ for all } x,y,z \in A, \) so that \( zd(y)x + d(z)yx = 0, \text{ for all } x,y,z \in A. \)

By using the equation 3.1.18, this reduces to \( d(z)yx = 0 = d(z)yrx. \)

Hence \( d(z)yrx) = 0, \text{ for all } x,y,z \in A, \ r \in N. \)

Replacing \( y \) by \( x \), we get \( d(z)xrd(z)x = 0, \text{ for all } x,z \in A. \)

Hence \( d(z)x = 0, \text{ for all } x,z \in A. \)
Recalling the equation 3.1.18, we now have $d(xy) = 0$, for all $x, y \in A$.

So, $d(rxy) = 0 = d(y)d(rx)$, for all $x, y \in A$, $r \in N$.

In particular, $d(x)d(rx) = 0$ and the remaining proof is same as in the first part of the Theorem. □

The consequences of the above theorem are given below.

**Corollary 3.1.1**: Let $N$ be a semiprime left near-ring, and $d$ a derivation on $N$. If $d$ acts as a homomorphism on $N$ or as an anti-homomorphism on $N$, then $d = 0$.

**Corollary 3.1.2**: Let $N$ be a left prime near-ring and $d$ a derivation on $N$. Let $A$ be a nonzero subset of $N$ such that $0 \in A$ and $NA \subseteq A$. If $d$ acts as a homomorphism on $A$ or as an anti-homomorphism on $A$, then $d = 0$.

**Proof**: By the theorem, we have $d(a) = 0$, for all $a \in A$.

Then $d(xa) = xd(a) + d(x)a = d(x)a = 0 = d(x)ya$, for all $a \in A$, $x, y \in N$.

So, $d(x)Na = \{0\}$.

By primeness of $N$, we get $d(x) = 0$ or $a = 0$, for all $a \in A$, $x \in N$.

Since $a$ is nonzero, we have $d(x) = 0$, for all $x \in N$. □
3.2 Strong Commutativity – Preserving Derivations on Near-Rings

Bell and Mason [12] studied strong commutativity – preserving derivations in near-rings. In this section, we generalize the results of [12] by assuming that a left near-ring \( N \) has a nonzero right ideal \( U \). We prove that if \( N \) is a left near-ring with \( U \) as a right ideal which contains no zero devisors of \( N \) and \( N \) admits a nonzero derivation \( d \) which is commuting on \( U \) and \( [x,y] = [d(x), d(y)] \), for all \( x, y \in U \), then \( N \) is commutative.

We know that a strong commutativity-preserving derivation (scp-derivation) is defined as \( [x,y] = [d(x),d(y)] \), for all \( x, y \in N \). \( N \) is said to be zero-symmetric if \( 0 \cdot x = 0 \), for all \( x \in N \). A mapping \( d : N \rightarrow N \) is said to be commuting on \( N \) if \( [d(x), x] = 0 \), for all \( x \in N \). An element \( c \in N \) for which \( d(c) = 0 \) is called a constant and \( N \) is called prime if \( aNb = \{0\} \) implies that \( a = 0 \) or \( b = 0 \).

Throughout this section \( N \) denotes a zero-symmetric left near-ring with a derivation \( d \) satisfying \( d(xy) = xd(y) + d(x)y \), for all \( x, y \in N \) and \( Z \) denotes the center of \( N \).

To prove the main theorem we require the following Lemmas.

**Lemma 3.2.1:** If \( d \) is a scp-derivation on \( N \), then constants are in \( Z \). If \( N \) also has 1, then \( (N,+ ) \) is abelian.
Proof: For $c$ constant, we have $[c,y] = [d(c),d(y)] = [0,d(y)] = 0$, for all $y \in N$.

In particular, if $N$ has 1, then $1 + 1 \in Z$.

Hence $[1 + 1, x + y] = 0$, for all $x, y \in N$, from which we have that $(N, +)$ is abelian. □

Lemma 3.2.2: Let $d$ be be a derivation on $N$, and suppose $u \in N$ is not a left zero divisor. If $[u,d(u)] = 0$, then $(x,u) = x + u - x - u$ is a constant for every $x \in N$.

Proof: From $u(u + x) = u^2 + ux$, we obtain

$$ud(u + x) + d(u)(u + x) = ud(u) + d(u)u + ud(x) + d(u)x.$$  
This implies that $ud(x) + d(u)u = d(u)u + ud(x)$.

Since $d(u)u = ud(u)$, we obtain $u(d(x) + d(u) - d(x) - d(u)) = 0 = ud((x,u))$.

Thus $d((x,u)) = 0$. □

Theorem 3.2.1: If $N$ has no zero divisors and admits a nonzero commuting scp-derivation, then $N$ is a commutative ring with no idempotents except 0 or 1.

Proof: For all $x, y \in N$, we have $[x,y] = [d(x),d(y)]$.

We replace $y$ by $xy$ in the above equation. Then we get $[x,xy] = [d(x),d(xy)]$.

This implies that $x[x,y] = [d(x),xd(y) + d(x)y]$ for all $x,y \in N$.

So,

$$x[x,y] = d(x)xd(y) + d(x)2y - d(x)y d(x) - xd(y) d(x).$$
Since $d$ is commuting and hence $(N,+)$ is abelian. Therefore we have
\[ x[x,y] = x[d(x),d(y)] + d(x) [d(x),y] = x[x,y] + d(x) [d(x),y]. \]
Hence $d(x) [d(x),y] = 0$, for all $x, y \in N$; and since $N$ has no zero divisors, we get $[d(x),y] = 0$, for all $x, y \in N$.

In particular, $[d(x), d(y)] = 0$; and therefore $[x,y] = 0$, for all $x, y \in N$.

Thus $N$ is a commutative ring.

We know that if $N$ admits a commuting scp-derivation, then all idempotents $e$ are central.

Therefore, if $e^2 = e \neq 0$, then $e$ is central.

Since $e(ex - x) = 0$ for all $x \in N$, $e$ is a left identity element.

Since $e \in Z$, it follows that $e = 1$. \[\Box\]

Now we also have the following.

**Corollary 3.2.1**: A near-field with a scp-derivation is a field.

**Corollary 3.2.2**: A near-domain admitting a nonzero scp-derivation is a commutative ring (and hence an ordinary integral domain). \[\Box\]

**Corollary 3.2.3**: If $N$ has no nonzero nilpotent elements and admits a commuting scp-derivation, then $N$ is a commutative ring.

**Proof**: By Lemma 4 of [11], there exists a family of completely prime ideals $\{P_\alpha / \alpha \in \Lambda\}$ such that $N$ is a subdirect product of the near-rings
\(N/ P_\alpha\), and such that for each \(\alpha \in \Lambda\), the definition \(\tilde{d}_\alpha (x + P_\alpha) = d(x) + P_\alpha\) yields a derivation \(\tilde{d}_\alpha\) on \(N/ P_\alpha\). Let \(\bar{N}\) denote a typical \(N/ P_\alpha\); and \(\bar{N}\) has no zero divisors of zero.

If \(\tilde{d}_\alpha\) is nonzero, then \(\bar{N}\) is a commutative ring by Theorem 3.2.1.

If \(\tilde{d}_\alpha\) is trivial, then from the definition of scp-derivation we have that \(\bar{N}\) is commutative, hence distributive.

But then \((\bar{N}, +)\) is abelian, so that
\[
\tilde{x}^2 + \tilde{x}\tilde{y} - \tilde{x}^2 - \tilde{x}\tilde{y} = 0, \text{ for all } \tilde{x}, \tilde{y} \in \bar{N}; \text{ and cancelling } \tilde{x} \text{ we obtain that } (\bar{N}, +) \text{ is abelian.} \qed
\]

**Theorem 3.2.2:** Let \(U\) be a nonzero ideal of \(N\) which contains no zero divisors of \(N\). If \(N\) admits a nonzero derivation \(d\) such that \([x, d(x)] = 0\), for all \(x \in U\) and \([x, y] = [d(x), d(y)]\), for all \(x, y \in U\), then \(N\) is a commutative ring.

**Proof:** By Lemma 3.2.2, we have the additive group commutator
\[(x, a) = x + a - x - a \text{ is constant for all } a \in U \text{ and } x \in N.\]

Since \(U\) is an ideal, we have \((x, a)y = (xy, ay)\) is also constant for arbitrary \(y \in N\).

Hence \((x, a) d(N) = \{0\}\).

Since \(U\) has no zero divisors and \((x, a) \in U\), we obtain \((x, a) = 0; \text{ and therefore } (U, +) \text{ is abelian.}\)
Now for any arbitrary \( a \in U/\{0\} \) and \( x,y \in N \), we have \((ax,ay) = a(x,y) = 0\); and hence \((N,+)\) is abelian.

Now by the proof of theorem 3.2.1, we have

\[
d(x) [d(x),y] = 0, \text{ for all } x,y \in U.
\]

Since \([d(x),y] \in U\), we conclude that \([d(x),y] = 0\) or \(d(x) = 0\).

Thus \([d(x),y] = 0\), for all \( x,y \in U \).

In particular, for all \( x,y \in U \) we have \([d(x), yd(y)] = 0 = y[d(x), d(y)]\).

Therefore, \( 0 = [d(x), d(y)] = [x,y], \) for all \( x,y \in U \).

Using this, if \( a \in A/\{0\} \) and \( x,y \in N \), then we have

\[
axay - ayax = 0 = a^2(xy - yx) = a^2[x,y]; \quad \text{so } [x,y] = 0.
\]

Therefore, \( N \) is a commutative ring. \( \square \)

We need the following results to prove the theorem 3.2.3.

**Lemma 3.2.4:** If \( N \) has 1 and admits an SCP-derivation, then

\((zx+z)y = zxy + zy \) for all \( x,y,z \in N \).

**Proof:** Since \( d(1) = 0 \), we have \([x+1,y] = [d(x+1), d(y)] = [d(x), d(y)] = [x,y] \). This implies that \((x+1)y = xy + y \), for all \( x,y \in N \).

By left multiplying with \( z \), we get the result. \( \square \)

**Theorem 3.2.3:** Let \( N \) be a near-ring such that \( aN = N \) for all \( a \in N/\{0\} \).

If \( N \) admits an SCP- derivation, then \( N \) is a division ring.
Proof: We easily prove that $N$ has no zero divisors. If $y \in N \setminus \{0\}$, then there exists $e \in N$ such that $ye = y$, $ye^2 = ye$ and $y(e^2-e) = 0$.

Thus, $e$ is a nonzero idempotent, which must be a left identity.

Since $d$ is an SCP-derivation, we have

$$ed(e) + d(e)e = d(e); \text{ hence } d(e) + d(e)e = d(e) \text{ and } d(e)e = 0.$$

Thus $d(e)N = d(e)eN = \{0\}$, so $d(e) = 0$.

Therefore $e \in Z$ by the Lemma 3.2.1, hence $N$ has 1.

Now, by the Lemmas 3.2.1 and 3.2.4 it follows that $N$ is a ring which must be a division ring.  

We need the following Theorem to prove the Theorem 3.2.5.

Theorem 3.2.4: Let $R$ be a prime ring and $U$ a nonzero right ideal of $R$. If $R$ admits a derivation $d$ such that $[x,y] = [d(x), d(y)]$ for all $x,y \in U$, then $R$ is commutative.

Proof: Suppose we assume that $d$ is non zero; otherwise $U$ is commutative and so is $R$.

For all $x,y \in U$, we have $[x,xy] = [d(x), d(xy)]$.

This implies that $x[x,y] = [d(x), xd(y) + d(x)y]$

$$= [d(x), xd(y)] + [d(x), d(x)y]$$

$$= x[d(x), d(y)] + [d(x), x]d(y) + d(x)[d(x), y]$$

$$= x[x,y] + [d(x), x]d(y) + d(x)[d(x), y].$$
Hence \[ (d(x), x) d(y) + d(y) [d(x), y] = 0 \text{ for all } x, y \in U. \] 3.1.19

By replacing \( y \) be \( yr \), we obtain
\[ (d(x), x) (yd(r) + d(y)r) + d(x) (y[d(x), r] + [d(x), y]r) = 0. \]

On comparing with 3.1.19, it yields
\[ (d(x), x) yd(r) + d(x)y[d(x), r] = 0 \text{ for all } x, y \in U \text{ and } r \in \mathbb{R}. \] 3.1.20

By taking \( r = d(x) \), we obtain
\[ (d(x), x) yd^2(x) = 0. \]

So \[ (d(x), x) Ud^2(x) = \{0\} = (d(x), x) Ur d^2(x) \text{ for all } x \in U. \]

Hence for each \( x \in U \), either \( d^2(x) = 0 \) or \( (d(x), x) U = \{0\} \).

Suppose that \( d^2(x) = 0 \). Then for each \( y \in U \), \([x, yd(y)] = [d(x), d(yd(x))] = [d(x), d(y)d(x)]\) and it follows that \( y[x, d(x)] = 0. \)

Therefore \( U[x, d(x)] = \{0\} \), hence \( [x, d(x)] = 0. \)

On the other hand, if \( (d(x), x) U = \{0\} \), then from the equation 3.1.20, we obtain
\[ d(x)U [d(x), r] = \{0\} = d(x) Ur [d(x), r]. \]

Hence either \( d(x) \in \mathbb{Z} \), in which case \( [x, d(x)] = 0 \), or \( d(x) U = \{0\} \).

Let us assume that there exists \( y \in U \) such that \( d(y) \in \mathbb{Z}\{0\}. \)

Then for each \( x \in U \) for which \( d(x) U = \{0\} \), the equation 3.1.19 yields
\[ (d(x), x) d(y) = 0. \]

Since \( d(y) \) is not a zero divisor, we get \( (d(x), x) = 0. \)

Hence in this case \( (d(x), x) = 0 \) for all \( x \in U \), and \( R \) is commutative by Theorem 4 of [10].
Now, we will prove the case where for each \( x \in U \), either \( d^2(x) = 0 \) or \( d(x)U = \{0\} \). The sets of elements of \( U \) for which these two conditions hold are additive subgroups of \( U \) whose union is \( U \); consequently, we must have either \( d^2(U) = \{0\} \) or \( d(U)U = \{0\} \).

If the first of these holds, the computation above shows that \([x, d(x)] = 0\), for all \( x \in U \), so that commutativity of \( R \) again follows from Theorem 4 of [10].

If \( d(U)U = \{0\} \), then from the condition \([x, yz] = [d(x), d(yz)]\), we obtain \( yd(x) d(z) = 0 \) for all \( x, y, z \in U \).

Therefore \( U[d(x), d(z)] = \{0\} = U[x, z] \), for all \( x, z \in U \), and so we conclude that \( U \) is commutative.

Hence \( R \) is commutative. \( \square \)

Now we prove the following Theorem.

**Theorem 3.2.5:** Let \( N \) be a prime near-ring and \( U \) a nonzero right ideal of \( N \) which is distributively generated near-ring with identity. If \( N \) admits a derivation \( d \) such that \([x, y] = [d(x), d(y)]\) for all \( x, y \in U \), then \( N \) is a commutative ring.

**Proof:** Let \( e \) be the identity element of \( U \).

Since \( ex = x \) for all \( x \in U \), we have \( ed(x) + d(e)x = d(x) \); hence \( ed(e)U = \{0\} \) and \( ed(e) = 0 \).

Thus for each \( x \in U \), we have \( xd(e) = xed(e) = 0 \),
so that \( Ud(e) = \{0\} \) and \( d(e) = 0 \).

But \( d(e+e) = 0 \). So by the Lemma 3.2.1, we get that both \( e \) and \( e+e \) commute with elements of \( U \) and hence, \( (U,+) \) is abelian.

Thus for all \( a \in U \) and all \( x, y \in N \), we have \( a(x+y-x-y) = 0 = a(x,y) \), hence \( (N,+) \) is abelian.

Now since \( N \) is distributively generated near-ring with identity and \( (U,+) \) is abelian, we have that \( U \) is distributive.

Let \( x, y \in N \) and \( a, b \in U \). Then \( (ax+ay)b = axb + ayb \).

That is, \( a(x+y)b = axb + ayb \),
\[
a((x+y)b - (xb+yb)) = 0.
\]

Hence \( (x+y)b - (xb+yb) = 0 \),

i.e., elements of \( U \) are distributive in \( N \).

Replacing \( b \) by \( bz \) for \( z \in N \), we obtain
\[
(x+y)bz = xzb + yzb, \text{ for all } x, y, z \in N.
\]

Now by interchanging \( b \) and \( z \) in the above equation, we get \( (x+y)zb = xzb + yzb \) and which implies that \( (x+y)z - (xz+yz)b = 0 \).

Hence \( ((x+y)z - (xz+yz))U = \{0\} \).

This implies that \( (x+y)z - (xz+yz) = 0 \).

Therefore, \( N \) is distributive.

Now we proved that \( N \) is a ring, and hence commutative by the Theorem 3.2.4.

Thus the proof is completed. \( \square \)
3.3. Derivations on Prime Near-Rings with Right Ideals

Daif and Bell [22] established that a prime ring $R$ must be commutative if it admits a derivation $d$ such that either $d([x,y]) = [x,y]$ for all $x,y$ in $U$ or $d([x,y]) = -[x,y]$ for all $x,y$ in $U$, where $U$ is a nonzero ideal of $R$. Boua and Oukhtite [15] proved that a prime near-ring with a nonzero derivation $d$ satisfying certain differential identities must be commutative.

In this section, we extend the above results to prime left near-rings with a right ideal $U$. We prove that a prime left near-ring $N$ is commutative if it admits a derivation $d$ such that $d([x,y]) = [x,y]$ or $d([x,y]) = -[x,y]$ or $d(xoy) = (xoy)$ or $d(xoy) = -(xoy)$ for all $x,y$ in $U$, where $U$ is a right ideal of $N$. $N$ is called a zero symmetric left near-ring if $0.x = 0$ for all $x \in N$. $N$ is said to be prime if $xNy = 0$ for all $x,y \in N$ implies $x = 0$ or $y = 0$. For all $x,y \in N$ the symbol $[x,y]$ denote the commutator $xy-yx$ and the symbol $xoy$ denote the anticommutator $xy+yx$. The symbol $Z$ denotes the center of $N$.

Throughout this section $N$ denotes a zero symmetric left near-ring with a derivation $d$ satisfying $d(xy) = xd(y) + d(x)y$ for $x,y \in U$ where $U$ is a nonzero right ideal of $N$.

**Theorem 3.3.1:** Let $N$ be a prime left near-ring, $U$ a right ideal of $N$ and if $N$ admits a nonzero derivation $d$ such that $d([x,y]) = [x,y]$ for all $x,y \in U$, then $N$ is commutative.
**Proof:** We have \( d([x,y]) = [x,y] \) for all \( x,y \in U \).

Replacing \( y \) by \( xy \), we get \( d([x,xy]) = [x, xy] \).

That is, \( d(x[x,y]) = x[x,y] \), for all \( x,y \in U \).

But \( d(x[x,y]) = xd([x,y]) + d(x) [x,y] \). According to the equation 3.3.1, we obtain \( d(x[x,y]) = x[x,y] + d(x) [x,y] \). It follows from the two expressions that \( d(x) [x,y] = 0 \), for all \( x,y \in N \).

Replacing \( y \) by \( yz \), we obtain

\[
d(x)y [x,z] + d(x) [x,y]z = 0 \text{ for all } x,y,z \in U.
\]

Using the equation 3.3.2, we obtain that \( d(x) y [x,z] = 0 \), for all \( x,y,z \in U \).

Replacing \( y \) by \( yr \), for \( r \in N \), we obtain

\[
d(x)yr [x,z] = 0 \text{ for all } x,y,z \in U, r \in N.
\]

By interchanging \( y \) and \( r \) in the above equation we get

\[
d(x)ry [x,z] = 0.
\]

So \( d(x)N [x,z] = \{0\} \) for all \( x,z \in U \).

Since \( N \) is prime, the equation 3.3.3 reduces to

\[
d(x) = 0 \text{ or } [x,z] = 0 \text{ for all } x,z \in U.
\]

From the equation 3.3.4 it follows that for each fixed \( x \in U \), we have \( d(x) = 0 \) or \( x \in Z \), that is non constants are central.

If \( d(x) = d(y) = 0 \), for all \( x,y \in U \), then

\[
d([x,y]) = d(xy-yx) = xd(y) + d(x)y - yd(x) - d(y)x = xy - yx.
\]

Hence \([x,y] = 0 \), for all \( x,y \in U \). Therefore \( x \) is in the center of \( U \).
Hence \((U,\cdot)\) is commutative.

If \(a \in U/\{0\}\), and \(u,v \in N\), then \(auv - avu = 0; a^2(\mu v - \mu v) = 0\).

Hence \(U(\mu v - \mu v) = 0\).

So \([u,v] = 0\), for all \(u,v \in N\).

Hence \((N,\cdot)\) is commutative.

Since \(N\) is distributive, we have \((x+y)(z+z) = x(z+z) + y(z+z) = xz + xz + yz + yz = z(x+x+y+y)\), for all \(x,y \in N\).

On the other hand,

\((x+y)(z+z) = (x+y)z + (x+y)z = xz + yz + xz + yz = z(x+y+x+y)\).

So, \(z(x+x+y+y-z-y-x) = 0\),

Hence \(N(x+x+y+y-z-y-x) = 0\).

Since \(N\) is prime, we get \(x+x+y+y = x+y+x+y\) and therefore, \(x+y = y+x\), for all \(x,y \in N\).

Hence \((N,+)\) is abelian. So \(N\) is a commutative ring. \(\square\)

Now we give an example to show that the primeness hypothesis in Theorem 3.3.1 is necessary even in the case of arbitrary rings.

**Example 3.3.1:** Let \(R\) be a commutative ring which is not a zero ring and consider \(N = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \bigg/ x,y \in R \right\} \).
If we define \( N \rightarrow N \) by 
\[
\begin{pmatrix}
0 & 0 \\
x & y
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 \\
x & 0
\end{pmatrix},
\]
then clearly \( d \) is a nonzero derivation of \( N \).

On the other hand, if \( a = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} \) where \( r \neq 0 \), then \( aNa = \{0\} \) which proves that \( N \) is not prime.

Moreover, \( d \) satisfies the condition
\[
d([A,B]) = [A,B] \text{ for all } A,B \in U,
\]
but \( N \) is a noncommutative ring.

**Theorem 3.3.2:** Let \( N \) be a prime left near-ring, and \( U \) is a right ideal of \( N \) and if \( N \) admits a nonzero derivation \( d \) such that \( d([x,y]) = -[x,y] \), for all \( x,y \in U \), then \( N \) is a commutative ring.

**Proof:** Replacing \( y \) by \( xy \) in the defining equation
\[
d([x,y]) = -[x,y] = -xy + yx, \text{ for all } x,y \in U,
\]
we have 
\[
d([x,xy]) = -x^2y + xyx = x(-xy + yx).
\]
On the other hand,
\[
d([x,xy]) = d(x[x,y]) = xd([x,y]) + d(x) [x,y]
\]
\[
= x(-xy + yx) + d(x) [x,y].
\]
It follows from the two expressions that
\[
d(x) [x,y] = 0, \text{ for all } x,y \in U.
\]
The rest of the proof follows from the Theorem 3.3.1. \( \square \)
Now we prove the theorems by replacing the product \([x,y]\) by \(xoy\).

**Theorem 3.3.3:** Let \(N\) be a prime left near ring and \(U\) a nonzero right ideal of \(N\). If \(N\) admits a nonzero derivation \(d\) such that \(d(xoy) = xoy\) for all \(x,y \in U\), then \(N\) is a commutative ring.

**Proof:** By the hypothesis, we have

\[
d(xoy) = xy + yx, \quad \text{for all } x,y \in U. \tag{3.3.6}
\]

Replacing \(y\) by \(xy\) in 3.3.6, we get

\[
d(xo(xy)) = x^2 y + xyx \text{ for all } x,y \in U. \tag{3.3.7}
\]

So, \(d(x(xoy)) = x^2 y + yx \) which gives

\[
x d(xoy) + d(x) (xoy) = x^2 y + yx = x(xoy).
\]

Hence, \(d(x) (xoy) = 0\) for all \(x,y \in U\).

i.e., \(d(x) (xy+yx) = 0\).

\[
\Rightarrow \quad d(x) xy = -d(x)yx \text{ for all } x,y \in U. \tag{3.3.8}
\]

Substituting \(yz\) by \(y\), we obtain that \(d(x)xyz = -d(x)yzx\).

This implies, \((-d(x)yx)z = d(x)yz(-x),\)

\[
d(x)yz(-x) = d(x)yz(-x) \text{ for all } x,y,z \in U. \tag{3.3.9}
\]

Taking \(-x\) instead of \(x\) in the equation 3.3.9 gives

\[
d(-x)y (x)z = d(-x)yz (x) \text{ for all } x,y,z \in U.
\]

So that \(d(-x)y (xz-zx) = 0\).

Therefore \(d(-x) N [x,z] = 0\) for all \(x,z \in U\). \tag{3.3.10}
By primeness, we obtain either \( d(-x) = 0 \) or \( x \in \mathbb{Z} \).

Accordingly, \( d(x) = 0 \) or \([x,z] = 0\) for all \( x, z \in U \).

The rest of the proof follows by the Theorem 3.3.1. \( \square \)

The primeness hypothesis in the Theorem 3.3.3 is necessary even in the case of arbitrary rings. We illustrate this by the following example.

**Example 3.3.2:** Let \( S \) be any ring.

Let us consider the ring \( N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & o \end{pmatrix} | x, y, z \in S \right\} \).

We define a mapping \( d : N \to N \) such that

\[
d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & o \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

If we get \( a = \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) with \( s \neq 0 \), then

\( aNa = 0 \) proving that \( N \) is not prime.

Moreover, it can be easily seen that \( d \) is a nonzero derivation such that

\( d(AoB) = AoB \) for all \( A, B \in U \),

but \( N \) is a noncommutative ring.
**Theorem 3.3.4:** Let $N$ be prime left near-ring and $U$ is a nonzero right ideal of $N$. If $N$ admits a nonzero derivation $d$ such that $d(xoy) = - (xoy)$ for all $x,y \in U$, then $N$ is a commutative ring.

**Proof:** Replacing $y$ by $xy$ in $d(xoy) = - (xoy)$, for all $x,y \in U$, we obtain $d(x(xoy)) = -x(xoy)$, for all $x,y \in U$.

Since $d(x(xoy)) = d(x)(xoy) + xd(xoy) = d(x)(xoy) + x( - (xoy)) = d(x)(xoy) - x(xoy)$, then the equation 3.3.11 reduces to

$$d(x)(xoy) - x(xoy) = -x(xoy), \text{ for all } x,y \in U.$$ 

Hence $d(x)(xoy) = 0$, for all $x,y \in U$.

The rest of the proof as in the proof of Theorem 3.3.3. □