Chapter 2
Certain Differential Inequalities
Implying Univalence

2.1 Introduction

For $f \in A$, we define the differential operator $I(\alpha; f)$ as

$$I(\alpha; f)(z) = (1 - \alpha)f'(z) + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right),$$

where $\alpha$ is some real number. Let $\mathcal{H}_\alpha(\beta)$ denote the class of normalized functions $f$, analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$, which satisfy the condition

$$\Re[I(\alpha; f)(z)] > \beta, \ z \in E,$$

where $\alpha$ and $\beta$ are pre-assigned real numbers.

The class $\mathcal{H}_\alpha(0)$ was first studied by Al-Amiri and Reade [3], in 1975. They established that for $\alpha \leq 0$, each function $f$ in $\mathcal{H}_\alpha(0)$ satisfies $\Re(f'(z)) > 0$ in $E$ and so is close-to-convex and hence univalent in $E$ (Noshiro [70], Warchawski [138]). The question of univalence for $\alpha > 0$ (except for $\alpha = 1$ when, obviously, $f$ is convex)

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Note: The contents of this chapter have appeared in Singh, S., Gupta, S. and Singh, S. ([110], [111], [114], [115]).
remained unanswered. Ahuja and Silverman [2] noticed that the convex function
\( f(z) = z/(1-z) \) is not in \( \mathcal{H}_\alpha(0) \) for any real \( \alpha \neq 1 \). In fact, for \( z = e^{i\theta} \neq 1 \),
\[
\Re \left[ I(\alpha; f)(z) \right] = \Re \left[ \frac{1-\alpha}{(1-z)^2} + \frac{1+z}{1-z} \right] = -\frac{(1-\alpha)\cos \theta}{2(1-\cos \theta)},
\]
which is negative for \( \theta = \theta_0 = \pi/3 \) when \( \alpha < 1 \), and for \( \theta = \theta_0 = 2\pi/3 \) when \( \alpha > 1 \).
Thus \( \mathcal{H}_1(0) \not\subset \mathcal{H}_\alpha(0), \alpha \neq 1 \) and even for convex functions \( f \), \( \Re(f'(z)) \) need not be positive in \( \mathbb{E} \).

Recently, this problem was pursued by Singh, Singh and Gupta [126] and they established that for \( 0 < \alpha < 1 \), the class \( \mathcal{H}_\alpha(\alpha) \) consists of univalent functions. They also proved that the functions \( f \) in \( \mathcal{H}_\alpha(1/2) \) satisfy \( \Re(f'(z)) > 1/2 \) for all \( z \) in \( \mathbb{E} \) and for all \( \alpha \geq 0 \).

The present chapter is divided in six sections. Section 2.2 contains some preliminary results which we shall require to prove our main results. In Section 2.3, we prove that if \( f \in \mathcal{H}_\alpha(\beta) \), then \( \Re(f'(z)) > 0 \) in \( \mathbb{E} \) for all real numbers \( \alpha \) and \( \beta \) satisfying \( \alpha \leq \beta < 1 \). Further, it is shown that our result generalizes the result of Singh, Singh and Gupta [126] and improves the result of Al-Amiri and Reade [3]. We also claim that our result is the best possible one in the sense that \( \beta \) cannot be replaced by any real number less than \( \alpha \).

In Section 2.4, it is shown that if we restrict the range of \( \alpha \) to \((0,2]\), then the region of variability of the operator \( I(\alpha; f) \) can be extended to get the same conclusion i.e. \( \Re f'(z) > 0 \) in \( \mathbb{E} \). In fact, we prove that for \( 0 < \alpha \leq 2 \) if \( f \in \mathcal{A} \) with \( f'(z) \neq 0 \) in \( \mathbb{E} \), satisfies the condition
\[
I(\alpha; f)(z) \prec F_1(z),
\]
then \( f \) is univalent in \( \mathbb{E} \), where \( F_1 \) is a conformal mapping of the unit disk \( \mathbb{E} \) with \( F_1(0) = 1 \) and
\[
F_1(\mathbb{E}) = \mathbb{C} \setminus \left\{ w \in \mathbb{C} : \Re(w) = \alpha, \ |\Im(w)| \geq \sqrt{\alpha(2-\alpha)} \right\}.
\]
The problem of starlikeness for members of the class $H_\alpha(\beta)$ still remains open for all real $\alpha$. In Section 2.5, functions $f$ satisfying the differential inequality

$$|I(\alpha; f)(z) - 1| < \mu, \ z \in \mathbb{E},$$

are studied and univalence, starlikeness and strongly starlikeness of such functions is obtained for $0 < \alpha < 2$ and for certain positive real number $\mu$.

We close this chapter by obtaining the largest radius $r_{\alpha, \beta} < 1$ such that $\Re(f'(z)) > 0$, $z \in \mathbb{E}$ implies that $f \in H_\alpha(\beta)$ in $|z| = r < r_{\alpha, \beta}$ for certain real numbers $\alpha$ and $\beta$.

### 2.2 Preliminaries

To prove our results, we shall need the following lemmas.

**Lemma 2.2.1.** Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ and let $\phi: \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2$, $v = v_1 + iv_2$ ($u_1, u_2, v_1, v_2$ are real), let $\phi$ satisfy the following conditions:

(i) $\phi(u, v)$ is continuous in $\mathbb{D},$

(ii) $(1, 0) \in \mathbb{D}$ and $\Re(\phi(1, 0)) > 0$ and

(iii) $\Re(\phi(iu_2, v_1)) \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1 + u_2^2)/2$.

Let $p(z) = 1 + p_1z + p_2z^2 + \cdots$ be regular in the open unit disk $\mathbb{E}$, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If

$$\Re[\phi(p(z), zp'(z))] > 0, \ z \in \mathbb{E},$$

then $\Re(p(z)) > 0$ in $\mathbb{E}$.

**Lemma 2.2.2.** Let $q$ be univalent in $\mathbb{E}$ and let $\theta$ and $\phi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either
(i) $h$ is convex, or
(ii) $Q_1$ is starlike.

In addition, assume that
(iii) $\Re \left( \frac{zh'(z)}{Q_1(z)} \right) > 0$ for all $z$ in $\mathbb{E}$.

If $p$ is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] < \theta[q(z)] + zq'(z)\phi[q(z)], \ z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and $q$ is the best dominant.

Lemma 2.2.3. Suppose $w$ is a nonconstant analytic function in $\mathbb{E}$ with $w(0) = 0$.
If $|w(z)|$ attains its maximum value at a point $z_0 \in \mathbb{E}$ on the circle $|z| = r < 1$, then
$z_0w'(z_0) = mw(z_0)$, where $m \geq 1$, is some real number.

Lemma 2.2.1 is due to Miller [49], Lemma 2.2.2 is due to Miller and Mocanu [57, p.132] and Lemma 2.2.3 is due to Jack [34].

2.3 Univalence of Functions in $\mathcal{H}_{\alpha}(\beta)$

Theorem 2.3.1. Let $\alpha$ and $\beta$ be real numbers such that $\alpha \leq \beta < 1$. If $f \in \mathcal{A}$ satisfies

$$\Re \left[ (1 - \alpha) f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > \beta, \ z \in \mathbb{E}, \quad (2.1)$$

then $\Re(f'(z)) > 0$ in $\mathbb{E}$. So, $f$ is close-to-convex and hence univalent in $\mathbb{E}$. The result is sharp in the sense that the constant $\beta$ on the right hand side of (2.1) cannot be replaced by a real number smaller than $\alpha$.

Proof. Let $f'(z) = p(z)$ where $p$, $p(0) = 1$, is analytic in $\mathbb{E}$. Then,

$$\Re \left[ (1 - \alpha) f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] = (1 - \alpha)p(z) + \alpha \left( 1 + \frac{zpf'(z)}{p(z)} \right).$$
Thus, condition (2.1) is equivalent to
\[
\Re \left[ \frac{1 - \alpha}{1 - \beta} p(z) + \frac{\alpha}{1 - \beta} \frac{z p'(z)}{p(z)} + \frac{\alpha - \beta}{1 - \beta} \right] > 0, \quad z \in \mathbb{E}.
\] (2.2)

If \( \mathbb{D} = (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \), define \( \Phi(u, v) : \mathbb{D} \to \mathbb{C} \) as under:
\[
\Phi(u, v) = \frac{1 - \alpha}{1 - \beta} u + \frac{\alpha}{1 - \beta} v + \frac{\alpha - \beta}{1 - \beta}.
\]

Then \( \Phi(u, v) \) is continuous in \( \mathbb{D}, \quad (1, 0) \in \mathbb{D} \) and \( \Re(\Phi(1, 0)) = 1 > 0 \). Further, in view of (2.2), we get \( \Re[\Phi(p(z), zp'(z))] > 0, \quad z \in \mathbb{E} \). Let \( u = u_1 + iu_2, v = v_1 + iv_2 \) where \( u_1, u_2, v_1 \) and \( v_2 \) are all real. Then, for \((iu_2, v_1) \in \mathbb{D}\), with \( v_1 \leq -\frac{1 + u_2^2}{2} \), we have
\[
\Re[\Phi(iu_2, v_1)] = \Re \left[ \frac{1 - \alpha}{1 - \beta} u_2 i + \frac{\alpha}{1 - \beta} \frac{v_1}{u_2 i} + \frac{\alpha - \beta}{1 - \beta} \right]
= \frac{\alpha - \beta}{1 - \beta}
\leq 0.
\]

In view of Lemma 2.2.1, proof now follows.

To show that the constant \( \beta \) on the right hand side of (2.1) cannot be replaced by a real number smaller than \( \alpha \), we consider the function \( f_0(z) = ze^z \) which belongs to the class \( \mathcal{A} \). Using Mathematica 7.0, we have plotted, in Figure 2.3.1, the image of the unit disk under the function \( (1 - \alpha)f_0'(z) + \alpha \left( 1 + \frac{zf''_0(z)}{f'_0(z)} \right) \) taking \( \alpha = -1 \). From this figure, we observe that minimum real part of \( (1 - \alpha)f_0'(z) + \alpha \left( 1 + \frac{zf''_0(z)}{f'_0(z)} \right) \) is smaller than -1 (the chosen value of \( \alpha \)). In Figure 2.3.2, we have plotted the image of unit disk under the function \( f_0'(z) \). It is obvious that \( \Re f_0'(z) \neq 0 \) for all \( z \) in \( \mathbb{E} \). For example, the point \( z = -\frac{1}{2} + i\frac{\pi}{4} \) is an interior point of \( \mathbb{E} \), but at this point \( \Re f_0'(z) = -\frac{\pi - 2}{4\sqrt{2e}} = -0.1224 \cdots < 0 \). This justifies our claim.
Remark 2.3.1. Taking $\beta = \alpha$, it is obvious that Theorem 2.3.1 completely contains Theorem 1 proved by Singh, Singh and Gupta [126]. We claim that our result improves the result of Al-Amiri and Reade [3]. In fact, when we take $f(z) = f_1(z) =$
\(-z - 2 \log(1 - z)\) and \(\alpha = -1\) in Theorem 2.3.1, we observe that at \(z = i\),

\[
\Re \left[ (1 - \alpha) f_1'(z) + \alpha \left( 1 + \frac{zf_1''(z)}{f_1'(z)} \right) \right] = -1.
\]

Thus the function \(f_1\) does not satisfy the hypothesis of Theorem 1 of Al-Amiri and Reade [3] i.e. \(f_1 \notin \mathcal{H}_{-1}(0)\) although \(\Re(f_1'(z)) = \Re \left( \frac{1 + z}{1 - z} \right) > 0\) in \(E\).

The problem of univalence of functions in the class \(\mathcal{H}_\alpha(\beta)\) is still open for \(\alpha > 1\). This provides motivation for our next result.

**Theorem 2.3.2.** Let \(\alpha\) and \(\beta\) be real numbers such that \(\alpha \geq \beta > 1\). If \(f \in \mathcal{A}\) satisfies

\[
\Re \left[ (1 - \alpha) f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < \beta, \ z \in \mathbb{E},
\]

then \(\Re(f'(z)) > 0\) in \(E\). So, \(f\) is close-to-convex and hence univalent in \(E\).

**Proof.** Write \(f'(z) = p(z)\) as in Theorem 2.3.1 and noting that \(1 - \beta < 0\), condition (2.3) reduces to

\[
\Re \left[ \frac{1 - \alpha}{1 - \beta} p(z) + \frac{\alpha}{1 - \beta} \frac{zp'(z)}{p(z)} + \frac{\alpha - \beta}{1 - \beta} \right] > 0, \ z \in \mathbb{E}.
\]

The proof can now be completed on the same lines as in Theorem 2.3.1. \(\square\)

### 2.4 Extension of Region of Variability of \(I(\alpha; f)\)

Recall that a univalent function \(q\) is said to be dominant of differential subordination

\[
\psi(p(z), zp'(z); z) \prec h(z), \ \psi(p(0), 0; 0) = h(0),
\]

if \(h\) is univalent in \(E\), \(p(0) = q(0)\) and \(p \prec q\) for all \(p\) satisfying (2.4). A dominant \(\tilde{q}\) such that \(\tilde{q} \prec q\) for all dominants \(q\) of (2.4), is said to be the best dominant.

In Theorem 2.3.1, the differential operator \(I(\alpha; f)\) is allowed to vary in the right half plane \(\{w: \Re w > \beta\}\). By using the concept of differential subordination in this section, we extend the region of variability of differential operator \(I(\alpha; f)\) to get the best dominant for \(f'(z)\).
Theorem 2.4.1. Let $\alpha, \alpha \neq 0$, be a complex number. Let $q, q(z) \neq 0$, be a univalent function in $E$ satisfying therein the condition

$$\Re \left[ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right] > \max \left\{ 0, \Re \left( \frac{\alpha - 1}{\alpha} q(z) \right) \right\}. \quad (2.5)$$

If $p, p(z) \neq 0$ in $E$, satisfies the differential subordination

$$(1 - \alpha)(p(z) - 1) + \frac{zp'(z)}{p(z)} \prec (1 - \alpha)(q(z) - 1) + \frac{zq'(z)}{q(z)}, \quad (2.6)$$

then $p(z) \prec q(z)$ and $q$ is the best dominant.

Proof. Let us define the functions $\theta$ and $\phi$ as follows:

$$\theta(w) = (1 - \alpha)(w - 1),$$

and

$$\phi(w) = \frac{\alpha}{w}.$$

Obviously, the functions $\theta$ and $\phi$ are analytic in domain $D = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $D$. Now, define the functions $Q_1$ and $h$ as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \alpha \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = (1 - \alpha)(q(z) - 1) + \frac{zq'(z)}{q(z)}.$$  

Then in view of condition (2.5), we have

(1) $Q_1$ is starlike in $E$ and

(2) $\Re \frac{z h'(z)}{Q_1(z)} > 0, \ z \in E.$

Thus conditions (ii) and (iii) of Lemma 2.2.2, are satisfied.

In view of (2.6), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 2.2.2. \qed
We now study some applications of Theorem 2.4.1 to univalent functions.

**Theorem 2.4.2.** Let \( q \) be as in Theorem 2.4.1. If \( f \in \mathcal{A}, f'(z) \neq 0, z \in \mathbb{E}, \) satisfies the differential subordination

\[
(1 - \alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec (1 - \alpha)q(z) + \alpha \left( 1 + \frac{q'(z)}{q(z)} \right),
\]

then \( f'(z) \prec q(z) \) in \( \mathbb{E} \) and \( q \) is the best dominant, where \( \alpha \) is a non-zero complex number.

**Proof.** Proof follows by writing \( p(z) = f'(z) \) in Theorem 2.4.1. \( \square \)

**Remark 2.4.1.** When we select the dominant \( q(z) = \frac{1 + z}{1 - z} \), then

\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0, \quad z \in \mathbb{E},
\]

and for all real number \( \alpha, \quad 0 < \alpha \leq 2, \) we have

\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1 - \alpha}{\alpha} \frac{1}{q(z)} \right) > 0, \quad z \in \mathbb{E}.
\]

Thus \( q \) satisfies the condition (2.5) of Theorem 2.4.1. Moreover

\[
(1 - \alpha)q(z) + \alpha \left( 1 + \frac{zq'(z)}{q(z)} \right) = (1 - \alpha) \frac{1 + z}{1 - z} + \alpha \left( 1 + \frac{2z}{1 - z} \right) = F_1(z).
\]

For \( 0 < \alpha \leq 2, \) we see that \( F_1 \) is a conformal mapping of the unit disk \( \mathbb{E} \) with \( F_1(0) = 1 \) and \( F_1(\mathbb{E}) = \mathbb{C} \setminus \left\{ w \in \mathbb{C} : \Re(w) = \alpha, \quad |\Im(w)| \geq \sqrt{\alpha(2 - \alpha)} \right\} \).

In view of the above, we have the following result.

**Corollary 2.4.1.** Let \( \alpha, \quad 0 < \alpha \leq 2, \) be a real number. Suppose that \( f \in \mathcal{A}, f'(z) \neq 0 \) in \( \mathbb{E}, \) satisfies the condition

\[
(1 - \alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec F_1(z), \quad z \in \mathbb{E},
\]

then \( f \) is close-to-convex and hence univalent in \( \mathbb{E}, \) where \( F_1 \) is as in Remark 2.4.1.
Remark 2.4.2. The result in Corollary 2.4.1 shows that for \(0 < \alpha \leq 2\), the operator \(I(\alpha; f)\) can vary in the whole complex plane except two slits \(\Re(w) = \alpha, \quad |\Im(w)| \geq \sqrt{\alpha(2 - \alpha)}\) parallel to imaginary axis. The cases for \(\alpha = \frac{1}{10}, \alpha = \frac{1}{2}\) and \(\alpha = \frac{3}{2}\) are shown below.

Figure 2.4.1 (when \(\alpha = 1/10\))

Figure 2.4.2 (when \(\alpha = 1/2\))
Remark 2.4.3. As $f \in \mathcal{A}$, the differential operator

$$(1 - \alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)$$

takes value 1 at $z = 0$. So, we immediately get the next two results from Corollary 2.4.1.

**Corollary 2.4.2.** Let $0 < \alpha < 1$, be a real number. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies the differential inequality

$$\Re \left[ (1 - \alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > \alpha,$$

then $\Re(f'(z)) > 0$, $z \in \mathbb{E}$, therefore, $f$ is close-to-convex and hence $f$ is univalent in $\mathbb{E}$.

**Corollary 2.4.3.** Let $1 < \alpha \leq 2$, be a real number. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies the differential inequality

$$\Re \left[ (1 - \alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < \alpha,$$

then $\Re(f'(z)) > 0$, $z \in \mathbb{E}$, therefore, $f$ is close-to-convex and hence $f$ is univalent in $\mathbb{E}$.
Result in Corollary 2.4.2 was proved by Singh, Singh and Gupta [126].

On writing \( p(z) = \frac{zf'(z)}{f(z)} \) in Theorem 2.4.1, we obtain the following result.

**Theorem 2.4.3.** Let \( q \) be as in Theorem 2.4.1. If \( f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}, \) satisfies the differential subordination

\[
(1 - 2\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec (1 - \alpha)q(z) + \alpha \frac{zq'(z)}{q(z)},
\]

then \( \frac{zf'(z)}{f(z)} \prec q(z) \) and \( q \) is the best dominant, where \( \alpha \) is a non-zero complex number.

When we select \( q(z) = \frac{1+z}{1-z} \) in Theorem 2.4.3, we obtain the following result.

**Corollary 2.4.4.** If \( f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}, \) satisfies the differential subordination

\[
(1 - 2\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec (1 - \alpha) \frac{1+z}{1-z} + 2\alpha \frac{z}{1-z^2} = F_2(z),
\]

where \( 0 < \alpha \leq 2 \), is a real number, then \( f \in S^* \).

Note that \( F_2 \) is a conformal mapping of the unit disk \( \mathbb{E} \) with \( F_2(0) = 1 - \alpha \) and

\[
F_2(\mathbb{E}) = \mathbb{C} \setminus \left\{ w \in \mathbb{C} : \Re(w) = 0, \left| \Im(w) \right| \geq \sqrt{\alpha(2-\alpha)} \right\}.
\]

**Remark 2.4.4.** Note that for \( f \in \mathcal{A}, \) the differential operator

\[
(1 - 2\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)
\]

takes value \( 1 - \alpha \) at \( z = 0 \). We immediately deduce the next two results.

**Corollary 2.4.5.** Let \( \alpha, 0 < \alpha < 1, \) be a real number. If \( f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in \mathbb{E}, \) satisfies the differential inequality

\[
\Re \left[ (1 - 2\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0,
\]

then \( f \in S^* \).
Corollary 2.4.6. Let \( \alpha, \; 1 < \alpha \leq 2 \), be a real number. If \( f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, \; z \in \mathbb{E}, \) satisfies the differential inequality
\[
\Re \left[ (1 - 2\alpha) \frac{zf''(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < 0,
\]
then \( f \in S^* \).

2.5 A subclass of \( \mathcal{H}_\alpha (\beta) \)

As mentioned earlier, nothing is known about the mapping properties of the functions in the class \( \mathcal{H}_\alpha (\beta) \) except that the functions in this class are close-to-convex univalent in \( \mathbb{E} \). Therefore, it will be of interest to identify some subclasses of \( \mathcal{H}_\alpha (\beta) \) which have some well-known mapping properties.

Define the class \( \mathcal{R}(\alpha, \mu) \) consisting of functions \( f \in \mathcal{A} \) which satisfy
\[
\left| (1 - \alpha) f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| < \mu, \; z \in \mathbb{E}, \tag{2.7}
\]
where \( \alpha \) and \( \mu \) are pre-assigned real numbers. We claim that functions in the class \( \mathcal{R}(\alpha, \mu) \) are starlike / strongly starlike for suitable values of \( \alpha \) and \( \mu \).

To prove our claim, we shall need the following lemmas.

Lemma 2.5.1. Suppose \( f \in \mathcal{A} \) is such that \( f'(z) < 1 + az \) in \( \mathbb{E} \), where \( 0 < a \leq 1 \), then
\[
\frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^\mu, \; z \in \mathbb{E},
\]
where \( 0 < a \leq \frac{2 \sin \left( \frac{\mu \pi}{2} \right)}{\sqrt{5 + 4 \cos \left( \frac{\mu \pi}{2} \right)}}, \; 0 < \mu < 1. \)

Lemma 2.5.2. Let \( f \in \mathcal{A} \) be such that \( f'(z) < 1 + az \) in \( \mathbb{E} \), where \( 0 < a \leq \frac{1}{2} \), then
\[
\frac{zf'(z)}{f(z)} \prec 1 + \left( \frac{3a}{2 - a} \right) z, \; z \in \mathbb{E}.
\]
Lemma 2.5.1 is due to Ponnusamy and Singh [88], Lemma 2.5.2 is due to Ponnusamy [86]

**Theorem 2.5.1.** Let $\alpha$ and $\mu$ be real numbers such that $0 < \alpha < 1$ and $0 < \mu \leq \frac{\alpha}{2}$.

If $f \in \mathcal{R}(\alpha, \mu)$, then

$$|f'(z) - 1| < \frac{2\mu}{\alpha}, \quad z \in \mathbb{E}.$$

**Proof.** Let us write

$$f'(z) = 1 + \frac{2\mu}{\alpha} w(z),$$

where $w$ is analytic in $\mathbb{E}$ with $w(0) = 0$. Now we will show that $|w(z)| < 1, z \in \mathbb{E}$. If $|w(z)| \not< 1$, by Lemma 2.2.3, there exists $z_0, \ |z_0| < 1$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$ where $k \geq 1$. Writing $w(z_0) = e^{i\theta}$, we have

$$\left|(1 - \alpha) f'(z_0) + \alpha \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) - 1\right|$$

$$= \left|(1 - \alpha) \left(1 + \frac{2\mu}{\alpha} w(z_0)\right) + \alpha \left(1 + \frac{2\mu z_0 w'(z_0)}{1 + \frac{2\mu}{\alpha} w(z_0)}\right) - 1\right|$$

$$\geq 2\mu \left|1 - \frac{z_0 w'(z_0)}{1 + \frac{2\mu}{\alpha} w(z_0)}\right|$$

$$\geq 2\mu \left|1 - \frac{\alpha(1 - \alpha)}{\alpha} \right| \left(\because \mu \leq \frac{\alpha}{2}\right)$$

$$= \frac{2\mu}{\alpha + 2\mu}.$$
\[ \geq \mu, \]
which is a contradiction to (2.7). Therefore, we must have \(|w(z)| < 1, z \in \mathbb{E}\). Hence
\[ |f'(z) - 1| < \frac{2\mu}{\alpha}, z \in \mathbb{E}. \]

\[ \square \]

**Theorem 2.5.2.** Let \(\alpha\) and \(\mu\) be real numbers such that \(1 \leq \alpha < 2\) and \(0 < \mu \leq \frac{2 - \alpha}{2}\). If \(f \in \mathcal{R}(\alpha, \mu)\), then
\[ |f'(z) - 1| < \frac{2\mu}{2 - \alpha}, z \in \mathbb{E}. \]

**Proof.** Let us write
\[ f'(z) = 1 + \frac{2\mu}{2 - \alpha}w(z), \]
where \(w\) be analytic in \(\mathbb{E}\) with \(w(0) = 0\). Now we will show that \(|w(z)| < 1, z \in \mathbb{E}\). If \(|w(z)| \not< 1\), by Lemma 2.2.3, there exists \(z_0, |z_0| < 1\) such that \(|w(z_0)| = 1\) and \(z_0w'(z_0) = kw(z_0)\) where \(k \geq 1\). Writing \(w(z_0) = e^{i\theta}\), we have
\[
\left| (1 - \alpha)f'(z_0) + \alpha \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) - 1 \right|
\]
\[ = \left| (1 - \alpha) \left( 1 + \frac{2\mu}{2 - \alpha} w(z_0) \right) + \alpha \left( 1 + \frac{2\mu}{2 - \alpha} z_0 w'(z_0) \right) - 1 \right| \]
\[ = \left| \frac{2(1 - \alpha)\mu}{2 - \alpha} e^{i\theta} + \frac{2\mu k e^{i\theta}}{1 + \frac{2\mu}{2 - \alpha} e^{i\theta}} \right| \]
\[ \geq \frac{2\mu}{2 - \alpha} \left| 1 - \frac{2\mu(\alpha - 1)}{2 - \alpha} e^{i\theta} \right| \]
\[ \geq \frac{2\mu}{2 - \alpha} \left| 1 - \frac{2\mu(\alpha - 1)}{2 - \alpha} e^{i\theta} \right| \]

35
\[ \geq \frac{2\mu}{2-\alpha} \left| 1 - \frac{2\mu(\alpha-1)}{2-\alpha} \right| \]

\[ \geq \frac{2\mu}{2-\alpha} \left| 1 - \frac{(2-\alpha)(\alpha-1)}{2-\alpha} \right| \quad (\because \, 2\mu \leq 2 - \alpha \text{ and } \alpha \geq 1) \]

\[ \geq \mu, \]

which is a contradiction to (2.7). Therefore, we must have \(|w(z)| < 1, \, z \in \mathbb{E}\). Hence

\[ |f'(z) - 1| < \frac{2\mu}{2-\alpha}, \, z \in \mathbb{E}. \]

\[ \square \]

Recall that a function \( f \in \mathcal{A} \) is said to be strongly starlike of order \( \delta \), \( 0 < \delta \leq 1 \), if

\[ \left| \frac{zf'(z)}{f(z)} \right| < \frac{\delta \pi}{2}, \, z \in \mathbb{E}, \]

or, equivalently

\[ zf'(z) < \left( \frac{1+z}{1-z} \right)^{\delta}, \, z \in \mathbb{E}. \]

Using Lemma 2.5.1 in next two results, we show that functions in \( \mathcal{R}(\alpha, \mu) \) are strongly starlike of order \( \delta \) for suitable values of \( \alpha \) and \( \mu \).

**Corollary 2.5.1.** If \( f \in \mathcal{R}(\alpha, \mu) \) for some \( \alpha \in \mathbb{R} \) with \( 0 < \alpha < 1 \) and \( 0 < \mu \leq \frac{\alpha}{2} \), then

\[ \frac{zf''(z)}{f'(z)} < \left( \frac{1+z}{1-z} \right)^{\delta}, \, z \in \mathbb{E}, \]

where \( 0 < \frac{2\mu}{\alpha} \leq \frac{2\sin \left( \frac{\pi \delta}{2} \right)}{\sqrt{5 + 4\cos \left( \frac{\pi \delta}{2} \right)}} \leq 1 \), \( 0 < \delta < 1 \), and hence \( f \) is strongly starlike of order \( \delta \) in \( \mathbb{E} \).

Using Theorem 2.5.2 along with Lemma 2.5.1, we get the following result.
Corollary 2.5.2. If \( f \in \mathcal{R}(\alpha, \mu) \) for some \( \alpha \in \mathbb{R} \) with \( 1 \leq \alpha < 2 \) and \( 0 < \mu \leq \frac{2 - \alpha}{2} \), then
\[
\frac{zf'(z)}{f(z)} < \left(\frac{1+z}{1-z}\right)^\delta, \quad z \in \mathbb{E},
\]
where \( 0 < \frac{2\mu}{2-\alpha} \leq \frac{2\sin\left(\frac{\pi\delta}{2}\right)}{\sqrt{5+4\cos\left(\frac{\pi\delta}{2}\right)}} \leq 1, \quad 0 < \delta < 1, \) and hence \( f \) is strongly starlike of order \( \delta \) in \( \mathbb{E} \).

Using Lemma 2.5.2 along with Theorem 2.5.1 and Theorem 2.5.2, following two results show that functions in \( \mathcal{R}(\alpha, \mu) \) are starlike (bounded) in \( \mathbb{E} \) for suitable values of \( \alpha \) and \( \mu \).

Corollary 2.5.3. If \( f \in \mathcal{R}(\alpha, \mu) \) for some \( \alpha \in \mathbb{R} \) with \( 0 < \alpha < 1 \) and \( 0 < \mu \leq \frac{\alpha}{4} \), then
\[
\frac{zf'(z)}{f(z)} < 1 + \frac{3\mu}{\alpha - \mu} z, \quad z \in \mathbb{E},
\]
and hence \( f \) is starlike in \( \mathbb{E} \).

Corollary 2.5.4. If \( f \in \mathcal{R}(\alpha, \mu) \) for some \( \alpha \in \mathbb{R} \) with \( 1 \leq \alpha < 2 \) and \( 0 < \mu \leq \frac{2 - \alpha}{4} \), then
\[
\frac{zf'(z)}{f(z)} < 1 + \frac{3\mu}{2 - \alpha - \mu} z, \quad z \in \mathbb{E},
\]
and hence \( f \) is starlike in \( \mathbb{E} \).

2.6 A Radius Problem for the Class \( \mathcal{H}_\alpha(\beta) \)

In this section, we shall be dealing with the problem of finding largest real number \( r_{\alpha,\beta} < 1 \) such that when \( \Re f'(z) > 0, \) \( z \in \mathbb{E} \), then \( f \in \mathcal{H}_\alpha(\beta) \) in \( |z| = r < r_{\alpha,\beta} \). The same problem was studied by Al-Amiri and Reade [3] for the class \( \mathcal{H}_\alpha(0) \) and they proved the following result:
Theorem 2.6.1. Let \( f \) be in the class of normalized regular functions with \( \Re(f'(z)) > 0 \) for \( z \in \mathbb{E} \). Then \( f \in \mathcal{H}_\alpha(0) \) in \( |z| = r < r_{\alpha,0} \), where

\[(i) \quad r_{\alpha,0} = \frac{1}{1 + \sqrt{2\alpha}}, \quad \alpha \geq 0, \text{ and} \]

\[(ii) \quad r_{\alpha,0} = \sqrt{\frac{1 - \alpha - \sqrt{\alpha(\alpha - 1)}}{1 - \alpha}}, \quad \alpha < 0.\]

All the results are sharp.

Note that \( f \in \mathcal{H}_\alpha(\beta) \) if

\[
\Re \left[ (1 - \alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > \beta, \quad z \in \mathbb{E},
\]

or equivalently,

\[
\Re \left[ \frac{1 - \alpha}{1 - \beta} f'(z) + \frac{\alpha}{1 - \beta} \frac{zf''(z)}{f'(z)} + \frac{\alpha - \beta}{1 - \beta} \right] > 0, \quad z \in \mathbb{E} \tag{2.8}
\]

If \( \Re f'(z) > 0 \) in \( \mathbb{E} \), then the problem of finding largest \( r_{\alpha,\beta} \) for fixed real numbers \( \alpha \) and \( \beta \) for which \( f \) satisfies (2.8) in \( |z| = r < r_{\alpha,\beta} \), is equivalent to finding the smallest positive root of \( Q_{\alpha,\beta}(r) = 0 \) where

\[
Q_{\alpha,\beta}(r) = \min_p \min_{|z|=r<1} \Re \left[ \frac{1 - \alpha}{1 - \beta} p(z) + \frac{\alpha}{1 - \beta} \frac{zp'(z)}{p(z)} + \frac{\alpha - \beta}{1 - \beta} \right],
\]

for \( p(z) = f'(z) \), a regular function in \( \mathbb{E} \) with \( p(0) = 1 \) and \( \Re p(z) > 0 \).

Corresponding to Theorem 2.6.1 of Al-Amiri and Reade [3], we present the following one for the class \( \mathcal{H}_\alpha(\beta) \).

Theorem 2.6.2. Let \( f \) be in the class of normalized regular functions with \( \Re(f'(z)) > 0 \) for \( z \in \mathbb{E} \). Then \( f \in \mathcal{H}_\alpha(\beta) \) in \( |z| = r < r_{\alpha,\beta} \), where

\[(i) \quad r_{\alpha,\beta} = \frac{1 - \sqrt{2\alpha(1 - \beta) + \beta^2}}{1 - 2\alpha + \beta}, \quad \alpha \geq 0 \text{ and } \beta < 1, \text{ and}\]
(ii) $r_{\alpha, \beta} = \sqrt{1 + \frac{\alpha}{\sqrt{\alpha(\alpha - 1)}}}$, $\alpha < 0$ and $\beta \leq 0$.

All the results are sharp.

Proof. The proof runs on the same lines as in Theorem 2.6.1 of Al-Amiri and Reade [3], so we omit the details here. \hfill $\square$