Chapter 3

Some Cacti - Properties and Enumeration
3.1 Introduction

The previous chapter was intended to provide prerequisites needed for the advancement of topic while this chapter is aimed to introduce some new discrete structures. In the succeeding sections we will introduce a new graph family known as cactus (plural : cacti) and its various types. Some authors consider the cactus as blocks isomorphic to cycle $C_n$ and call it as $n$-ary cactus [30, 70] but we have used $K_n$ instead of $C_n$ in more general sense.

3.2 Cactus Graph

We first define $n$-complete cactus (simply cactus) and introduce some special types of cacti like linear cactus, spider cactus, caterpillar cactus and lobster cactus as follow.

Definition 3.2.1. An $n$-complete cactus $C(K_n)$ is a simple graph whose all the blocks are isomorphic to $K_n$.

Definition 3.2.2. A linear cactus $P_m(K_n)$ is a connected graph whose all the blocks are isomorphic to complete graph $K_n$ and block-cutpoint graph is a path $P_{2m-1}$. The vertex set and edge set of a linear cactus $P_m(K_n)$ is given by $V(P_m(K_n)) = \{v^i, v^1_{m+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and $E(P_m(K_n)) = \{v^i_{m+1}, v^j_{i+1}, v^k_{i+1} : 1 \leq i \leq m, 1 \leq j, k \leq n-1, j \neq k\}$.

Illustration 3.2.3. A linear cactus $P_7(K_4)$ is shown in Figure 3.1.

\[\text{Figure 3.1: } P_7(K_4)\]

Definition 3.2.4. A spider cactus $S_{n_1,n_2,...,n_k}(K_n)$ is a connected graph whose all the blocks are isomorphic to complete graph $K_n$ and block-cutpoint graph is a spider $S_{2n_1,2n_2}$.
The vertex set and edge set of a spider cactus are given by 

$$V(S_{n_1,n_2,...,n_k}(K_n)) = \{v_{i,j}^l, v_{n_i+1,1}^l : 1 \leq l \leq k, 1 \leq i \leq n_l, 1 \leq j \leq n - 1 \text{ where } v_{1,1}^1 = v_{1,1}^2 = ... = v_{1,1}^k = v_0 \}$$

and 

$$E(S_{n_1,n_2,...,n_k}(K_n)) = \{v_{i,j}^l v_{i,j+1,1}^l, v_{i,j}^l v_{i,m}^l : 1 \leq l \leq k, 1 \leq i \leq n_l, 1 \leq j, m \leq n - 1, j \neq m \}.$$ 

**Illustration 3.2.5.** A spider cactus $S_{4,3,2,1}(K_4)$ is shown in Figure 3.2.

**Definition 3.2.6.** A *caterpillar cactus* is a cactus obtained by replacing each edge of a caterpillar by a complete graph $K_n$.

**Illustration 3.2.7.** A caterpillar cactus is shown in Figure 3.3.
Definition 3.2.8. A lobster cactus is a cactus obtained by replacing each edge of a lobster by a complete graph $K_n$.

Illustration 3.2.9. A lobster cactus is shown in Figure 3.4.

![Figure 3.4: Lobster cactus.](image)

Definition 3.2.10. An $n$-complete $k$-regular cactus $C(K_n(k))$ is an $n$-complete cactus in which each cut vertex is exactly in $k$ blocks.

Illustration 3.2.11. A 3-complete 3-regular cactus is shown in Figure 3.5.

![Figure 3.5: 3-complete 3-regular cactus](image)
For \( n \)-complete \( k \)-regular cactus, readers should not be confused with the word \textit{regular} as the regularity is not in the sense of degree but it is the total number of blocks in which the given cut vertex lies. The block which contains only one cut vertex is called leaf block and that cut vertex is known as leaf block cut vertex. In \( n \)-complete \( k \)-regular cactus the vertices which are not cut vertices are known as leaf vertices.

### 3.3 Symmetric Regular Cactus

We define symmetric regular cactus as follows.

**Definition 3.3.1.** A symmetric \( n \)-complete \( k \)-regular cactus \( SC(K_n(k)) \) is an \( n \)-complete \( k \)-regular cactus in which the eccentricity of each leaf vertex is the same.

Throughout this discussion by symmetric regular cactus we mean symmetric \( n \)-complete \( k \)-regular cactus with \( p \) vertices, \( b \) blocks and \( q \) edges. Occasionally we also use the notation \( SC(K_n(k))(d) \) for a symmetric \( n \)-complete \( k \)-regular cactus of diameter \( d \). A single block is recognized as a trivial cactus which is of no importance for us as it is isomorphic to \( K_n \) and will trivially satisfy all the properties of \( K_n \).

### 3.4 Construction of Symmetric Regular Cactus

We describe the procedure to construct symmetric regular cactus of odd and even diameter.

**3.4.1. Symmetric Regular Cactus of Odd Diameter**

Let \( K_n \) be the complete graph on \( n \) vertices say \( v_1, v_2, \ldots, v_n \) which is a trivial cactus. Take more \( n(k-1) \) copies of \( K_n \). For \( 1 \leq i \leq n \), identify one vertex of each copy of \( K_n \) from a bunch of \( (k-1) \) copies of \( K_n \) with each \( v_i \). Then the resultant graph
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is a symmetric \( n \)-complete \( k \)-regular cactus with diameter three. In this symmetric \( n \)-complete \( k \)-regular cactus, the vertices \( v_1, v_2, \ldots, v_n \) are now cut vertices and all other vertices are leaf vertices. Continuing this process, we can construct a symmetric regular cactus of odd diameter.

3.4.2. Symmetric Regular Cactus of Even Diameter

Let \( v_0 \) be any vertex. Take \( k \) copy of \( K_n \) and identify one vertex of each \( K_n \) with \( v_0 \). Then it is symmetric \( n \)-complete \( k \)-regular cactus of diameter two. Here \( v_0 \) is a cut vertex and all other vertices are leaf vertices. Now follow the procedure described in Section 3.4.1 to construct the desired symmetric \( n \)-complete \( k \)-regular cactus of even diameter.

We note that the center of a symmetric regular cactus is either a vertex or a block and accordingly it is called block centered symmetric regular cactus or vertex centered symmetric regular cactus. If diameter is odd then it is block centered and if diameter is even then it is vertex centered. In a symmetric \( n \)-complete \( k \)-regular cactus \( SC(K_n(k)) \), a vertex \( v \) is a terminal or leaf vertex if \( d(v) = n - 1 \) and the block which contains a terminal vertex is called terminal or leaf block. Also a vertex which is not leaf vertex is known as cut vertex or internal vertex and a block which is not leaf block is known as internal block.

Illustration 3.4.1. A symmetric 4-complete 2-regular cacti of diameter 5 and 6 are shown in Figure 3.6.

Observation 3.4.2. From the definition and construction of non trivial symmetric regular cactus, it is clear that

- \( 2 \leq n \leq p \)
- \( 2 \leq d \leq p - 1 \)
- \( 2 \leq k \leq p - 1 \)
3.5 Properties of Symmetric Regular Cactus

**Theorem 3.5.1.** For any $SC(K_n(k))$ of diameter $d$,

\[
P = \begin{cases} 
  n + \sum_{i=1}^{\frac{d-1}{2}} n(k-1)^i(n-1)^i, & \text{if } d \text{ is odd} \\
  1 + \sum_{i=1}^{\frac{d}{2}} k(k-1)^{i-1}(n-1)^i, & \text{if } d \text{ is even.}
\end{cases}
\]

\[
B = \begin{cases} 
  1 + \sum_{i=1}^{\frac{d-1}{2}} n(k-1)^i(n-1)^i-1, & \text{if } d \text{ is odd} \\
  k + \sum_{i=1}^{\frac{d}{2}} k(k-1)^{i-1}(n-1)^i, & \text{if } d \text{ is even.}
\end{cases}
\]

Moreover, in the formula of $p$ and $b$, the last term of the summation gives the total number of leaf vertices and leaf blocks respectively while the sum of remaining terms gives the total number of cut vertices and internal blocks respectively.

**Proof.** We prove this theorem using induction on $d$. If $d = 1$ then cactus is just one block which has $n$ leaf vertices as each vertex is of degree $n-1$. If $d = 2$ then regular cactus is a vertex centered and it is $k$ copies of complete graph $K_n$ each one sharing a common vertex. Therefore the number of vertices $p = kn - k + 1 = 1 + k(n-1)$ and the
common vertex is a cut vertex and the regular cactus has \(k(n - 1)\) leaf vertices while the number of blocks is \(k\).

Let the result is true for \(d = t\) then

**Case-1:** If \(t\) is odd then it has \(n + \sum_{i=1}^{t-1} n(k - 1)^{(n - 1)^i} + n(k - 1)^{(n - 1)^{t-1}}(n - 1)\) vertices and \(1 + \sum_{i=1}^{d-1} n(k - 1)^{(n - 1)^i}(n - 1)^{i-1}\) blocks. In the formula of \(p\) and \(b\), the last term of the summation which are \(n(k - 1)^{(n - 1)^{t-1}}\) and \(n(k - 1)^{(n - 1)^{t-3}}\) gives the total number of leaf vertices and leaf blocks respectively while the sum of remaining terms gives the total number of cut vertices and internal blocks respectively. Now again from a bunch of \(k - 1\) copies of \(K_n\) identify the one vertex of each copy of \(K_n\) with each leaf vertex then the total number of vertices \(p\)

\[
= n + \sum_{i=1}^{t-1} n(k - 1)^{(n - 1)^i} + n(k - 1)^{(n - 1)^{t-1}}(n - 1)
\]

\[
= n + \sum_{i=1}^{t+1} n(k - 1)^{(n - 1)^i};
\]

the total number of blocks \(b\)

\[
= 1 + \sum_{i=1}^{t-1} n(k - 1)^{(n - 1)^{i-1}} + n(k - 1)^{(n - 1)^{t-1}}(n - 1)
\]

\[
= 1 + \sum_{i=1}^{t+1} n(k - 1)^{(n - 1)^{i-1}}.
\]

**Case-2:** If \(t\) is even then it has \(1 + \sum_{i=1}^{t} k(k - 1)^{(n - 1)^i} + k(k - 1)^{n-1}(n - 1)\) vertices and \(1 + \sum_{i=1}^{t-1} k(k - 1)^{(n - 1)^i}(n - 1)^i\) blocks. In the formula of \(p\) and \(b\), the last term of the summation which are \(k(k - 1)^{n-1}(n - 1)^{t-1}\) and \(k(k - 1)^{(n - 1)^{t-1}}(n - 1)^{t-1}\) gives the total number of leaf vertices and leaf blocks respectively while the sum of remaining terms gives the total number of cut vertices and internal blocks respectively. Now again from a bunch of \(k - 1\) copies of \(K_n\) identify a vertex of each copy of \(K_n\) with each leaf vertex then the total number of vertices \(p\)

\[
= 1 + \sum_{i=1}^{t} k(k - 1)^{(n - 1)^i} + k(k - 1)^{(n - 1)^{t-1}}(n - 1)
\]
\[ = 1 + \sum_{i=1}^{\frac{t}{2}+1} k(k-1)^{i-1} (n-1)^i; \]

the total number of blocks \( b \)

\[ = k + \sum_{i=1}^{\frac{t}{2}-1} k(k-1)^i (n-1)^i + k(k-1)^{\frac{t}{2}} (n-1)^{\frac{t}{2}} (n-1) \]

\[ = 1 + \sum_{i=1}^{\frac{t}{2}} k(k-1)^i (n-1)^i. \]

Hence, the result is true for \( d = t + k \).

Therefore, the result is true for all \( d \). \[ \blacksquare \]

**Observation 3.5.2.** For symmetric regular cactus on \( p \) vertices, the trivial cactus \( K_n \) gives minimum value of \( d = 1 \) and a path on \( p \) vertices gives the maximum value of \( d = p - 1 \).

**Theorem 3.5.3.** If \( SC(K_n(k)) \) is symmetric regular cactus with \( p \) vertices then it has precisely \( \frac{1}{2} n(p-1) \) edges.

**Proof.** Let \( SC(K_n(k)) \) be a symmetric regular cactus with \( p \) vertices then it has \( 1 + \sum_{i=1}^{d-1} n(k-1)^i (n-1)^{i-1} \) blocks if \( d \) is odd and \( k + \sum_{i=1}^{\frac{t}{2}-1} k(k-1)^i (n-1)^i \) blocks if \( d \) is even. Now as each block is isomorphic to \( K_n \) it has \( \frac{1}{2} n(n-1) \) edges. Therefore the total number of edges \( q \)

\[ = \begin{cases} 
\{1 + \sum_{i=1}^{\frac{d-1}{2}} n(k-1)^i (n-1)^{i-1}\} \frac{1}{2} n(n-1), & \text{if } d \text{ is odd} \\
\{k + \sum_{i=1}^{\frac{t}{2}-1} k(k-1)^i (n-1)^i\} \frac{1}{2} n(n-1), & \text{if } d \text{ is even.} 
\end{cases} \]

\[ = \begin{cases} 
\frac{1}{2} n \{ (n-1) + \sum_{i=1}^{\frac{d-1}{2}} n(k-1)^i (n-1)^i \}, & \text{if } d \text{ is odd} \\
\frac{1}{2} n \{ k(n-1) + \sum_{i=2}^{\frac{t}{2}} k(k-1)^{i-1} (n-1)^i \}, & \text{if } d \text{ is even.} 
\end{cases} \]
= \left\{ \begin{array}{ll}
\frac{1}{2}n \{ n + \sum_{i=1}^{d-1} n(k-1)^i(n-1)^i - 1 \}, & \text{if } d \text{ is odd} \\
\frac{1}{2}n \{ 1 + \sum_{i=1}^{d} k(k-1)^{i-1}(n-1)^i - 1 \}, & \text{if } d \text{ is even.} 
\end{array} \right.

= \frac{1}{2}n(p - 1). \text{ Hence the result.} \hspace{1cm} \blacksquare

**Corollary 3.5.4.** There does not exist symmetric regular cacti on even number of vertices with odd completeness of blocks.

**Proof.** We prove by contradiction. Suppose that there exists a symmetric regular cactus on even number of vertices with odd completeness of blocks then \( p \) is even and \( n \) is odd and hence \( \frac{1}{2}n(p - 1) \) is not integer. But by Theorem 3.5.3, the total number of edges in a symmetric regular cactus is \( q = \frac{1}{2}n(p - 1) \) which is not an integer, a contradiction. Hence the result. \( \blacksquare \)

**Theorem 3.5.5.** A symmetric \( n \)-complete \( k \)-regular cactus is Eulerian if and only if \( n \) is odd.

**Proof.** We use the fundamental result which states that a connected graph \( G \) is Eulerian if and only if each vertex in \( G \) has even degree. The degree of each vertex in symmetric \( n \)-complete \( k \)-regular cactus is even if and only if \( n \) is odd and hence the result. \( \blacksquare \)

**Observation 3.5.6.** A symmetric \( n \)-ary \( k \)-regular cactus is always an Eulerian graph.

### 3.6 Enumeration of Symmetric Regular Cactus

In this section, we enumerate all the symmetric regular cactus on given \( p \) vertices as follows.

**Theorem 3.6.1.** The total number of symmetric \( n \)-complete \( k \)-regular cactus of diameter \( d \) on \( p \) vertices is the total number of positive integral solutions of equations
\[ n(\Phi_{\frac{d+1}{2}}((n-1)(k-1))) = p \text{ for odd } d \text{ and } 1+k(n-1)\Phi_{\frac{d}{2}}((n-1)(k-1)) = p \text{ for even } d, \]

where \( \Phi_p(x) = 1 + x + x^2 + \ldots + x^{p-1} \).

**Proof.** Let \( SC(K_n(k))(d) \) be the symmetric \( n \)-complete \( k \)-regular on \( p \) vertices and diameter \( d \). Then by Theorem 3.5.1, the positive integers \( n, k \) and \( d \) satisfy the equations

\[
p = n + \sum_{i=1}^{d-1} n(k-1)^i(n-1)^i \text{ if } d \text{ is odd and } p = 1 + \sum_{i=1}^{d} k(k-1)^{i-1}(n-1)^i \text{ if } d \text{ is even.}
\]

i.e. \( n(\Phi_{\frac{d+1}{2}}((n-1)(k-1))) = p \) if \( d \) is odd and \( 1+k(n-1)\Phi_{\frac{d}{2}}((n-1)(k-1)) = p \) if \( d \) is even, where \( \Phi_p(x) = 1 + x + x^2 + \ldots + x^{p-1} \).

On the other hand, for \( p \) there exist positive integers \( n, k \) and \( d \) which satisfy the equations \( n(\Phi_{\frac{d+1}{2}}((n-1)(k-1))) = p \) for odd \( d \) and \( 1+k(n-1)\Phi_{\frac{d}{2}}((n-1)(k-1)) = p \) for even \( d \), where \( \Phi_p(x) = 1 + x + x^2 + \ldots + x^{p-1} \).

Now, construct symmetric \( n \)-complete \( k \)-regular cactus on given number of \( p \) vertices by employing the construction procedure described in Sections 3.4.1 and 3.4.2 respectively.

**Corollary 3.6.2.** A trivial cactus is the only symmetric regular cactus of odd diameter on prime number of vertices.

**Proof.** Suppose that there exists a nontrivial symmetric regular cactus, where \( d \) odd, \( p \) is prime and \( 3 \leq d \leq p-1, 1 < n < p \) and \( 1 < k < p-1 \) such that

\[ n(\Phi_{\frac{d+1}{2}}((n-1)(k-1))) = p, \]

i.e. \( n + n(k-1)(n-1) + n(k-1)^2(n-1)^2 + \ldots + n(k-1)^{\frac{d-1}{2}}(n-1)^{\frac{d-1}{2}} = p \)

i.e. \( n(1 + (k-1)(n-1) + (k-1)^2(n-1)^2 + \ldots + (k-1)^{\frac{d-1}{2}}(n-1)^{\frac{d-1}{2}}) = p \)

i.e. \( n|p \) which contradicts the fact that \( p \) is prime. Hence the result.

**Corollary 3.6.3.** \( SC(K_2(p)) \) is the only symmetric regular cactus of even diameter on \( p+1 \) vertices, where \( p \) is prime.
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Proof. Suppose that there exists a symmetric regular cactus other than $SC(K_2(p))$ with

d is even and $2 < d \leq p - 1$, $2 < n \leq p$ and $1 < k < p$ such that

$$1 + k(n - 1)\Phi_{\frac{d}{2}}((n - 1)(k - 1)) = p + 1$$

i.e. $1 + k(n - 1) + k(k - 1)(n - 1)^2 + k(k - 1)^2(n - 1)^3 + ... + k(k - 1)^{\frac{d}{2}-1}(n - 1)^{\frac{d}{2}} = p + 1$.

i.e. $k(n - 1)[1 + (k - 1)(n - 1) + (k - 1)^2(n - 1)^2 + ... + (k - 1)^{\frac{d}{2}-1}(n - 1)^{\frac{d}{2}-1}] = p$.

i.e. $k|p$ which contradicts the fact that $p$ is prime. Hence the result.

Illustration 3.6.4. In the following Table 3.1, all symmetric regular cacti with number of vertices $\leq 10$ is given.

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<th>Vertices $p$</th>
<th>Total No. of cactus</th>
<th>Cactus Type</th>
<th>Diameter $d$</th>
<th>Regularity $k$</th>
<th>Completeness $n$</th>
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### 3.7 Drawing of Symmetric Regular Cactus

For more clarity of the concept, all the symmetric regular cacti on 9 vertices are shown.

![Figure 3.7: Trivial Cactus on 9 vertices.](image-url)
Chapter 3. *Some cacti - properties and enumeration*  

**Figure 3.8:** Symmetric 3-complete 2-regular cactus of diameter 3.

**Figure 3.9:** Symmetric 5-complete 2-regular cactus of diameter 2.

**Figure 3.10:** Symmetric 3-complete 4-regular cactus of diameter 2.
3.8 An Application of Symmetric Regular Cactus

The formation of committees in any administration often give rise to a symmetric regular cactus provided the number of persons in each committee is equal to say $n$. Then a single person or a committee which is formed for any purpose will can form other committees under them. Any member(s) of committee pass over a specific task to other committee under own leadership. Such arrangement will create either vertex centered symmetric regular cactus or block centered symmetric regular cactus. Again each member of this committee make other committees to make their work easy. Continuing in this way the formation of committees will give rise to a symmetric regular cactus. The main advantage of such model is that the work is evenly distributed and the whole system will work effectively. This model is more fruitful only if the graph remains connected.

3.9 Further Scope of Research

- Investigate some more properties of symmetric regular cacti.
• Find some more applications of symmetric regular cacti.

• Discussion on different labeling problems in the context of symmetric regular cacti.

3.10 Concluding Remarks

We have introduced a concept of symmetric $n$-complete $k$-regular cacti and also obtained the exact number of vertices, blocks and edges for the same. The symmetric $n$-complete $k$-regular cacti on given number of vertices are also enumerated.

The next chapter is targeted to discuss distance two labeling ($L(2, 1)$-labeling).