Chapter 2

Basic Terminology and Preliminaries
Chapter 2. Basic Terminology and Preliminaries

2.1 Introduction

This chapter is intended to provide all the fundamental terminology and notations which are needed for the subsequent chapters.

2.2 Basic Definitions

Definition 2.2.1. A graph $G = (V(G), E(G))$ consists of two finite sets, the vertex set $V(G)$ which is a non-empty set of elements called vertices, and the edge set $E(G)$ which is a possibly empty set of elements called edges, such that each edge $e$ in $E(G)$ is assigned an unordered pair of vertices $(u, v)$ called the end vertices of $e$.

Definition 2.2.2. A trivial graph is a graph consisting of only one vertex and no edges.

Definition 2.2.3. The null graph is the graph whose edge set is empty.

Definition 2.2.4. Two vertices are adjacent if they are joined by an edge.

Definition 2.2.5. Two edges are adjacent if they have an endvertex in common.

Definition 2.2.6. If vertex $v$ is an endvertex of an edge $e$ then $e$ is incident on $v$.

Definition 2.2.7. Two adjacent vertices are called neighbours. The set of all neighbours of vertex $v$ is called the neighbourhood set of $v$. It is denoted by $N(v)$ or $N[v]$ and they are respectively known as open and closed neighbourhood sets.

$N(v) = \{u \in V(G)/u \text{ adjacent to } v \text{ and } u \neq v\}$

$N[v] = N(v) \cup \{v\}$

Definition 2.2.8. The degree of a vertex $v$ in a graph $G$, denoted as $d(v)$ or $d_G(v)$, is the number of edges incident on $v$, counting each loop twice.

Definition 2.2.9. A pendant vertex is a vertex of degree one.
Definition 2.2.10. A graph $G$ is $k$-regular graph if for some positive integer $k$, $d(v) = k$ for every vertex $v$ of the graph $G$.

Definition 2.2.11. A loop is an edge whose endpoints are equal. Multiple edges or parallel edges are edges having the same pair of endvertices.

Definition 2.2.12. A simple graph is a graph having no loops or multiple edges.

Definition 2.2.13. An isomorphism from a simple graph $G$ to a simple graph $H$ is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say $G$ is isomorphic to $H$, written $G \cong H$, if there is an isomorphism from $G$ to $H$.

Definition 2.2.14. A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, $G$ is known as a supergraph of $H$.

Definition 2.2.15. A complete graph is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on $n$ vertices is denoted as $K_n$.

Definition 2.2.16. A bipartite graph $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $U$ and $W$, such that each edge of $G$ has one endpoint in $U$ and one endpoint in $W$. The pair $U, W$ is called a (vertex) bipartition of $G$, and $U$ and $W$ are called the bipartition subsets.

Definition 2.2.17. A complete bipartite graph is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has $m$ vertices in one of its bipartition subsets and $n$ vertices in the other is denoted by $K_{m,n}$.

Definition 2.2.18. The graph $K_{1,n}$ is called a star.

Definition 2.2.19. A walk in a graph $G$ is an alternating sequence $W : v_0 e_1 v_1 ,..., e_p v_p$ of vertices and edges beginning and ending with vertices in which $v_{i−1}$ and $v_i$ are the ends of $e_i$; $v_0$ is the origin and $v_p$ is the terminus of $W$. A walk is closed if $v_0 = v_p$.

Definition 2.2.20. A walk is called a trail if all the edges appearing in the walk are distinct.
Definition 2.2.21. A trail in $G$ is called an Euler trail if it includes every edge of $G$.

Definition 2.2.22. A tour of $G$ is a closed walk of $G$ which includes every edge of $G$ at least once.

Definition 2.2.23. A Euler tour of $G$ is a tour which includes each edge of $G$ exactly once.

Definition 2.2.24. A graph $G$ is called Eulerian if it has an Euler tour.

Definition 2.2.25. A trail is called a path if all the vertices are distinct.

Definition 2.2.26. A Hamiltonian path in a graph $G$ is a path which contains every vertex of $G$.

Definition 2.2.27. A closed path is called a cycle.

Definition 2.2.28. A Hamiltonian cycle (or Hamiltonian circuit) in a graph $G$ is a cycle which contains every vertex of $G$.

Definition 2.2.29. A graph $G$ is called Hamiltonian if it has a Hamiltonian cycle.

Definition 2.2.30. A vertex $u$ is said to be connected to a vertex $v$ in a graph $G$ if there is a path in $G$ from $u$ to $v$ and a graph $G$ is called connected if every two of its vertices are connected otherwise it is called disconnected.

Definition 2.2.31. If $G$ has a $u, v$-path, then the distance from $u$ to $v$, denoted as $d(u, v)$, is the least length of a $u, v$-path.

Definition 2.2.32. The detour distance $D(u, v)$ between two vertices $u$ and $v$ is the length of a longest $u, v$-path between two vertices $u$ and $v$ in a connected graph $G$.

Definition 2.2.33. The diameter of $G$ denoted by $\text{diam}(G)$ or simply by $d$ is $\max\{d(u, v) : u, v \in V(G)\}$.

Definition 2.2.34. The eccentricity of a vertex $u$, denoted as $\varepsilon(u)$, is $\max\{d(u, v) : v \in V(G)\}$. 
Definition 2.2.35. The detour eccentricity of a vertex $u$, denoted as $\varepsilon_D(u)$, is $\max\{D(u,v) : v \in V(G)\}$.

Definition 2.2.36. The radius of a graph $G$, denoted as $\text{rad}(G)$, is $\min\{\varepsilon(u) : u \in V(G)\}$.

Definition 2.2.37. The detour radius of a graph $G$, denoted as $\text{rad}_D(G)$, is $\min\{\varepsilon_D(u) : u \in V(G)\}$.

Definition 2.2.38. The center of a graph $G$, denoted as $C(G)$, is the subgraph induced by the vertices of minimum eccentricity.

Definition 2.2.39. The detour center of a graph $G$, denoted as $C_D(G)$, is the subgraph induced by the vertices of minimum detour eccentricity.

Definition 2.2.40. A vertex $v$ of a graph $G$ is called a cut vertex if deletion of $v$ leaves a graph disconnected. A block or a component of a graph $G$ is a maximal connected subgraph of $G$ that has no cut vertices.

Definition 2.2.41. The block-cutpoint graph of a graph $G$ is a bipartite graph $H$ in which one partite set consists of the cut vertices of $G$, and the other partite set has a vertex $b_i$ for each block $B_i$ of $G$. We include $vb_i$ as an edge of $H$ if and only if $v \in B_i$.

Definition 2.2.42. A graph $G$ is called acyclic if it contains no cycle.

Definition 2.2.43. A graph $G$ is called a tree if it is a connected acyclic graph.

Definition 2.2.44. A spider $S_{n_1,n_2,...,n_k}$ with $n_1 \geq n_2 \geq ... \geq n_k$, $k \geq 3$, where $n_i \in \mathbb{Z}^+$ is the length of the $i$th leg. Hence, $|V(S_{n_1,n_2,...,n_k})| = n_1 + n_2 + ... + n_k + 1$. The vertex set of spider is denoted by $V(S_{n_1,n_2,...,n_k}) = V_1 \cup V_2 \cup ... \cup V_k$, where each $V_i$ is the vertex set of the $i$th leg; that is assuming $v_{i,0} = v_{0,0}$. $V_i = \{v_{i,j} : 0 \leq j \leq n_k\}$, where $v_{i,j}v_{i,j+1} \in E(S_{n_1,n_2,...,n_k})$, $0 \leq j \leq n_k - 1$.

Definition 2.2.45. A caterpillar is a tree in which a single path (the spine) is incident to (or contains) every edge.
Definition 2.2.46. A lobster is a tree having a path (of maximum length) from which every vertex has distance at most \( k \), where \( k \) is an integer. The maximum distance of the vertex from the path is called the diameter of the lobster.

Definition 2.2.47. The graph obtained by joining a single pendant edge to each vertex of a path is called a comb graph denoted by \( P_n \odot K_1 \).

Definition 2.2.48. The middle graph of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and in which two vertices are adjacent if and only if either they are adjacent edges of \( G \) or one is a vertex of \( G \) and the other is an edge incident on it. The middle graph of \( G \) is denoted by \( M(G) \).

Definition 2.2.49. The total graph of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and two vertices are adjacent whenever they are either adjacent or incident in \( G \). The total graph of \( G \) is denoted by \( T(G) \).

Definition 2.2.50. The square of a simple connected graph \( G \), denoted by \( G^2 \), is defined to be the graph with the same vertex set as \( G \) and in which two vertices \( u \) and \( v \) are joined by an edge if and only if in \( G \) we have \( 1 \leq d(u, v) \leq 2 \).

Definition 2.2.51. A one point union of regular graph \( G \) denoted by \( G' \) is the graph obtained by taking \( v \) as a common vertex such that any two copy of \( G \) are edge disjoint and do not have any vertex in common except \( v \).

Definition 2.2.52. A Friendship graph \( F_n \) is a one point union of \( n \) copies of cycle \( C_3 \).

Definition 2.2.53. The join of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 + G_2 \), to be the graph with vertex set and edge set given as follows:

\[
V(G_1 + G_2) = V(G_1) \cup V(G_2),
\]

\[
E(G_1 + G_2) = E(G_1) \cup E(G_2) + J,
\]

where \( J = \{uv : u \in V(G_1), v \in V(G_2)\} \). Thus \( J \) consists of edges which join every vertex of \( G_1 \) to every vertex of \( G_2 \).

Definition 2.2.54. The wheel graph \( W_n \) is defined to be the join \( K_1 + C_n \), of an isolated vertex with a cycle of length \( n \).
**Definition 2.2.55.** The cartesian product of $G$ and $H$, written as $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting $(u,v)$ adjacent to $(u',v')$ if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

**Definition 2.2.56.** The strong product $G \boxtimes H$ of $G$ and $H$ is the graph in which the vertex $(u,v)$ is adjacent to the vertex $(u',v')$ if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$, or $uu' \in E(G)$ and $vv' \in E(H)$.

**Definition 2.2.57.** A cactus (plural: cacti) is a connected graph whose blocks are isomorphic to cycles or edges. An $n$-ary cactus is a connected graph whose all blocks are isomorphic to $C_n$. If $n = 3$ then it is known as a triangular cactus.

### 2.3 Concluding Remarks

This chapter provides basic definitions and terminology required for the advancement of the topic. For all other standard terminology and notations we refer to Harary [39], Clark and Holton [20], Wilson [77], Gross and Yellen [37], Chartrand and Lesniak [11], West [75], Narsingh Deo [25], Bondy and Murty [2], Balakrishnan and Ranganathan [1]. For the brief account of detour distance we refer to Chartrand and Zhang [17].

The next chapter is focused on a new graph family known as cacti.