Chapter 5

Radio Labeling of Cacti
Chapter 5. Radio Labeling of Cacti

5.1 Introduction

The level of interference is two for \( L(2,1) \)-labeling while practically it has been observed that the interference might go beyond two levels. Radio \( k \)-labeling extends the level of interference considered in \( L(2,1) \)-labeling from two to the largest possible - the diameter of \( G \).

5.2 Radio Labeling

Motivated through the channel assignment problem for FM radio stations Chartrand et al. [7] have introduced the concept of radio \( k \)-labeling of a graph as follows.

Definition 5.2.1. A radio \( k \)-labeling \( f \) of \( G \) is an assignment of positive integers to the vertices of \( G \) such that

\[ d(u,v) + |f(u) - f(v)| \geq 1 + k, \text{ for all } u, v \in V(G) \text{ and positive integer } k. \]

The radio \( k \)-labeling number \( rc_k(f) \) of a radio \( k \)-labeling \( f \) of \( G \) is the maximum label assigned to a vertex of \( G \). The radio \( k \)-chromatic number \( rc_k(G) \) is \( \min \{ rc_k(f) \} \) over all radio \( k \)-labeling \( f \) of \( G \). A radio \( k \)-labeling \( f \) of \( G \) is a minimum radio \( k \)-labeling if \( rc_k(f) = rc_k(G) \).

Radio \( k \)-labelings generalize many graph labelings. It has been observed that for \( k = 1, rc_1(G) = \chi(G) \), the chromatic number of \( G \). For \( k = 2 \), the radio 2-labeling problem corresponds to the well studied \( L(2,1) \)-labeling problem and \( rc_2(G) = \lambda(G) \) which we already discussed in Chapter 4. For \( k = d - 1 \), the radio \((d - 1)\)-labeling is referred to as the radio antipodal labeling, because only antipodal vertices can have the same labels. In that case, \( rc_k(G) \) is called the radio antipodal number which is studied in detail by Chartrand et al. [7–10, 13] and Khennoufa and Togni [46, 47]. Finally, for the case \( k = d \), \( rc_k(G) \) is called the radio number. The precise definition of radio labeling is as follow.
**Definition 5.2.2.** A radio labeling $f$ of $G$ is an assignment of positive integers to the vertices of $G$ satisfying

$$d(u, v) + |f(u) - f(v)| \geq d + 1, \text{ for all } u, v \in V(G).$$

The span of $f$ is defined as $\max\{|f(u) - f(v) : u, v \in V(G)|\}$. The radio number, denoted by $rn(G)$, is the minimum span of a radio labeling for $G$. The radio labeling is studied by many researchers like Chartrand *et al.* [7], Heuvel *et al.* [40], Liu [53], Liu and Xie [54, 55], Liu and Zhu [56], Vaidya and Vihol [71], and Wang *et al.* [73]. A survey on radio labeling is also published by Chartrand and Zhang [18].

**Illustration 5.2.3.** In Figure 5.1, an optimal radio labeling for path $P_8$ is shown for which $rn(P_8) = 25$.

![Figure 5.1: $rn(P_8) = 25$.](image)

5.3 Some Known Results

In this Section, we present some known results about the radio number of graphs described as follows.

Chartrand *et al.* [7, 10] proved the following results.

**Theorem 5.3.1.** Let $G$ be a connected graph of order $n$ and diameter 2. Then

$$rn(G) = n + (t - 1)$$

if and only if the minimum number of components in a spanning linear forest of the complement $\overset{\sim}{G}$ of $G$ is $t$. ```
Theorem 5.3.2. Let $G$ be a connected graph of order $n$ and diameter 2. Then $rn(G) = n$ if and only if its complement $\bar{G}$ has a Hamiltonian path.

Theorem 5.3.3. If $G$ is a connected graph of order $n$ and diameter 2, then

$$n \leq rn(G) \leq 2n - 2$$

Moreover, for each integer $k$ with $n \leq k \leq 2n - 2$, there exists a connected graph $G$ of order $n$ and diameter 2 such that $rn(G) = k$.

Theorem 5.3.4. For $k \geq 3$

$$rn(C_n) \leq \begin{cases} k^2, & \text{if } n = 2k + 1 \\ k^2 - k + 1, & \text{if } n = 2k. \end{cases}$$

Moreover, $rn(C_n) \geq \lceil \frac{n}{2} - 1 \rceil - 1$ for $n \geq 6$.

Theorem 5.3.5. If $G$ is a connected graph of order $n$ and diameter 3, then

$$6 \leq rn(G) \leq 3n - 6.$$ 

Furthermore, for every integer $n \geq 4$, there exists a connected graph $G$ of order $n$ and diameter 3 with $rn(G) = 3n - 6$.

Theorem 5.3.6. If $G$ is a connected graph of order $d$ and maximum degree $\Delta$, then

$$rn(G) \geq 2 + \Delta(d - 1).$$

Theorem 5.3.7. If $G$ is a connected graph of diameter $d$ and clique number $\omega$, then

$$rn(G) \geq 1 + d(\omega - 1)$$

Theorem 5.3.8. If $G$ is a connected graph of order $n$ and diameter $d$, then

$$rn(P_{d+1}) \leq rn(G) \leq rn(P_{d+1}) + (n - d - 1)d.$$
Theorem 5.3.9. For any positive integer $n$,

$$rn(P_n) \leq \begin{cases} 
2k^2 + k, & \text{if } n = 2k + 1 \\
2(k^2 - k) + 1, & \text{if } n = 2k.
\end{cases}$$

Liu and Zhu [56] investigated the exact radio numbers for paths and cycles in the form of following results.

Theorem 5.3.10. For any $n \geq 3$,

$$rn(P_n) = \begin{cases} 
2k^2 + 2, & \text{if } n = 2k + 1 \\
2k(k - 1) + 1, & \text{if } n = 2k.
\end{cases}$$

Theorem 5.3.11. Let $C_n$ be the $n$-vertex cycle, $n \geq 3$. Then

$$rn(C_n) = \begin{cases} 
\frac{n-2}{2} \phi(n) + 1, & \text{if } n \equiv 0, 2 \pmod{4} \\
\frac{n-2}{2} \phi(n), & \text{if } n \equiv 1, 3 \pmod{4}
\end{cases}$$

where

$$\phi(n) = \begin{cases} 
k + 1, & \text{if } n = 4k + 1 \\
k + 2, & \text{if } n = 4k + r \text{ for some } r = 0, 2, 3.
\end{cases}$$

Khennoufa and Togni [46] investigated the radio number for hypercube $Q_n$ in the form of following result.

Theorem 5.3.12. For every positive integer $n$,

$$rn(Q_n) = (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1.$$ 

Liu and Xie [55] discuss radio labeling for square graph of paths and obtained the radio number for it.

Theorem 5.3.13. Let $P_n^2$ be a square path on $n$ vertices and let $k = \lfloor \frac{n}{2} \rfloor$. Then

$$rn(P_n^2) = \begin{cases} 
k^2 + 2, & \text{if } n \equiv 1 \pmod{4} \text{ and } n \geq 9 \\
k^2 + 1, & \text{otherwise.}
\end{cases}$$
Liu and Xie [54] discuss radio labeling for square of cycles. They investigated the exact radio number for square of any even cycles and some cases of odd cycles.

Wang et al. [73] investigated the radio number for ladder graphs in the form of following result.

**Theorem 5.3.14.** For any $n \geq 2$, \[rn(P_2 \Box P_n) = \begin{cases} 4, & n = 2 \\ 8, & n = 3 \\ 15, & n = 4 \\ n^2 - n + 3, & n \equiv 1(\text{mod} 2) \text{ and } n \geq 5 \\ n^2 - n + 4, & n \equiv 0(\text{mod} 2) \text{ and } n \geq 5. \end{cases}\]

In 2008, Liu [53] investigate lower bound for the radio number of trees and presented a class of graph families known as spiders achieving this lower bound.

### 5.4 Radio Number for Linear Cacti

In this section, we investigate the radio number for linear cacti.

**Lemma 5.4.1.** For $P_m(K_n)$, \[C(P_m(K_n)) = \begin{cases} K_n, & \text{if } m \text{ is odd} \\ K_1, & \text{if } m \text{ is even}. \end{cases}\]

**Proof.** Let $P_m(K_n)$ be the linear cactus of $m$ blocks and $k = \left\lfloor \frac{m}{2} \right\rfloor$. Let $S$ be the set of all vertices with minimum eccentricity in linear cactus $P_m(K_n)$. In $P_m(K_n)$, the eccentricity of each vertex $v^i_j$ of $v^i_{m+1}$, $1 \leq i \leq m$, $1 \leq j \leq n - 1$ is given as follows.

**Case - 1 : $m$ is odd.**

If $m$ is odd then $m = 2k + 1$ and in this case
\[ \varepsilon(v^j_i) = \begin{cases} 
  m + 1 - i, & 1 \leq i \leq k + 1, 1 \leq j \leq n - 1 \\
  i - 1, & k + 2 \leq i \leq m + 1, j = 1 \\
  i, & k + 2 \leq i \leq m, 2 \leq j \leq n - 1 
\end{cases} \]

\[ S = \min \{ \varepsilon(v^j_i) : 1 \leq i \leq m + 1, 1 \leq j \leq n - 1 \} \]

\[ = \{ v^j_{k+1}, v^1_{k+2} : 1 \leq j \leq n - 1 \} \]

The graph induced by \( S \) is isomorphic to the complete graph \( K_n \) and hence \( C(P_m(K_n)) = K_n \).

**Case - 2 : \( m \) is even.**

If \( m \) is even then \( m = 2k \) and in this case

\[ \varepsilon(v^j_i) = \begin{cases} 
  m + 1 - i, & 1 \leq i \leq k, 1 \leq j \leq n - 1 \\
  i - 1, & k + 1 \leq i \leq m + 1, j = 1 \\
  i, & k + 1 \leq i \leq m, 2 \leq j \leq n - 1 
\end{cases} \]

\[ S = \min \{ \varepsilon(v^j_i) : 1 \leq i \leq m + 1, 1 \leq j \leq n - 1 \} \]

\[ = \{ v^1_{k+1} \} \]

The graph induced by \( S \) is isomorphic to the complete graph \( K_1 \) and hence \( C(P_m(K_n)) = K_1 \).

Thus, from Case - 1 and Case - 2, we have

\[ C(P_m(K_n)) = \begin{cases} 
  K_n, & \text{if } m \text{ is odd} \\
  K_1, & \text{if } m \text{ is even.} 
\end{cases} \]

We define left and right side vertices of linear cactus according to its center. For that we rename the vertices of \( P_m(K_n) \) as follows.
For $m = 2k + 1$, rename the vertices $v^j_i$, $1 \leq i \leq m$, $1 \leq j \leq n - 1$ and $v^1_{m+1}$ by

$$v^j_i = \begin{cases} 
  v^j_{L(k+1-i)}, & 1 \leq i \leq k, 1 \leq j \leq n - 1 \\
  v^j_c, & i = k + 1, 1 \leq j \leq n - 1 \\
  v^j_c, & i = k + 2, j = 1 \\
  v^j_{R(i-(k+1))}, & k + 2 \leq i \leq 2k + 1, 2 \leq j \leq n - 1 \\
  v^1_{R(i-(k+2))}, & k + 3 \leq i \leq 2k + 2, j = 1.
\end{cases}$$

For $m = 2k$, rename the vertices $v^j_i$, $1 \leq i \leq m$, $1 \leq j \leq n - 1$ and $v^1_{m+1}$ by

$$v^j_i = \begin{cases} 
  v^j_{L(k+1-i)}, & 1 \leq i \leq k, 1 \leq j \leq n - 1 \\
  v^1_c, & i = k + 1, j = 1 \\
  v^j_{R(i-k)}, & k + 1 \leq i \leq 2k, 2 \leq j \leq n - 1 \\
  v^1_{R(i-(k+1))}, & k + 2 \leq i \leq 2k + 1, j = 1.
\end{cases}$$

**Illustration 5.4.2.** In Figure 5.2 and Figure 5.3, the linear cacti $P_5(K_6)$ and $P_6(K_6)$ are shown with new notation of vertices.
In a $P_m(K_n)$, two vertices $u$ and $v$ are called vertices of opposite side if $u = v^i_{Li}$ and $v = v^j_{Ri}$ and vice versa, for some $v^i_{Li}$ and $v^j_{Ri}$ except one of them is vertex of $C(P_m(K_n))$.

We define the level function on $V(P_m(K_n))$ to the set of whole numbers $W$ by

$$L(u) = \min\{d(u,w) : w \in V(C(P_m(K_n)))\}, \text{ for any } u \in V(P_m(K_n)).$$

In $P_m(K_n)$, the maximum level is $\lfloor \frac{m}{2} \rfloor$ and notice that the level of each $v^i_{Li}$ and $v^j_{Ri}$, $1 \leq i \leq k = \lfloor \frac{m}{2} \rfloor$, $1 \leq j \leq n-1$ is $i$.

The total level of graph $G$, denoted by $L(G)$, is defined as

$$L(G) = \sum_{u \in V(G)} L(u)$$

**Observation 5.4.3.** For $G = P_m(K_n)$,

(a) $d(u,v) \leq \begin{cases} L(u) + L(v) + 1, & \text{if } m \text{ is odd} \\ L(u) + L(v), & \text{if } m \text{ is even}. \end{cases}$

Moreover, equality hold if $u$ and $v$ are on opposite side.

(b) $L(G) = \begin{cases} \frac{1}{4}(m^2 - 1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{4}m(m+3)(n-1), & \text{if } m \text{ is even}. \end{cases}$

**Lemma 5.4.4.** Let $f$ be an assignment of non negative integers to $V(P_m(K_n))$ and $(u_1, u_2, u_3, ..., u_p)$ be the ordering of $V(P_m(K_n))$ such that $f(u_i) < f(u_{i+1})$ defined by $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$ and $d(u_i, u_{i+1}) \leq k + 1$, where $k = \lfloor \frac{m}{2} \rfloor$.

Then $f$ is radio labeling.

**Proof.** Let $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$ for $1 \leq i \leq p - 1$ and $d(u_i, u_{i+1}) \leq k + 1$, where $k = \lfloor \frac{m}{2} \rfloor$.

For each $i = 1, 2, ..., p - 1$, let $f_i = f(u_{i+1}) - f(u_i)$. Now, we want to prove that $f$ is a radio labeling. i.e. for any $i \neq j$, $|f(u_j) - f(u_i)| \geq d + 1 - d(u_i, u_j)$
Without loss of generality, let \( j \geq i + 2 \) then

\[
f(u_j) - f(u_i) = f_i + f_{i+1} + \ldots + f_{j-1}
\]

\[
= (j - i)(d + 1) - d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2}) - \ldots - d(u_{j-1}, u_j)
\]

**Case - 1 :** \( m \) is odd.

If \( m = 2k + 1 \) then \( d = 2k + 1 \) and hence

\[
f(u_j) - f(u_i) \geq (j - i)(d + 1) - (j - i)(k + 1)
\]

\[
= (j - i)(2k + 2) - (j - i)(k + 1)
\]

\[
= (j - i)(2k + 2 - k - 1)
\]

\[
= (j - i)(k + 1)
\]

\[
\geq 2(k + 1)
\]

\[
\geq d + 1 - d(u_i, u_j)
\]

**Case - 2 :** \( m \) is even.

If \( m = 2k \) then \( d = 2k \) and hence

\[
f(u_j) - f(u_i) \geq (j - i)(d + 1) - (j - i)(k + 1)
\]

\[
= (j - i)(2k + 1) - (j - i)(k + 1)
\]

\[
= (j - i)(2k + 1 - k - 1)
\]

\[
= (j - i)(k)
\]

\[
\geq 2k = 2k + 1 - 1
\]

\[
\geq d + 1 - d(u_i, u_j)
\]

Thus, in both cases, \( f \) is a radio labeling. \( \blacksquare \)
Theorem 5.4.5. For \( G = P_m(K_n) \),

\[
\text{rn}(P_m(K_n)) \geq \begin{cases} 
\frac{1}{2}(m^2 + 1)(n - 1), & \text{if } m \text{ is odd} \\
\frac{1}{2}m(m - 1)(n - 1) + 1, & \text{if } m \text{ is even.}
\end{cases}
\]

Proof. Let \( f \) be an optimal radio labeling for \( P_m(K_n) \) and \((u_1, u_2, u_3, \ldots, u_p)\) be the ordering of \( V(P_m(K_n)) \) such that \( 0 = f(u_1) < f(u_2) < f(u_3) < \ldots < f(u_p) \). Then \( f(u_{i+1}) - f(u_i) \geq d + 1 - d(u_i, u_{i+1}) \), for all \( 1 \leq i \leq p - 1 \). Summing these \( p - 1 \) inequalities we get

\[
\text{rn}(P_m(K_n)) = f(u_p) \geq (p - 1)(d + 1) - \sum_{i=1}^{p-1} d(u_i, u_{i+1})
\]

- - - (5.1)

Case - 1 : \( m \) is odd.

For \( P_m(K_n) \), we have

\[
\sum_{i=1}^{p-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1}) + 1]
\]

\[
= 2 \sum_{u \in V(G)} L(u) - L(u_1) - L(u_p) + (p - 1)
\]

\[
= 2 L(G) + (p - 1) \quad \text{(Choosing } u_1, u_p \in V(C(G)))
\]

- - - (5.2)

Substituting (5.2) in (5.1), we get

\[
\text{rn}(P_m(K_n)) = f(u_p) \geq (p - 1)(d + 1) - 2L(G) - (p - 1)
\]

\[
= (p - 1)d - 2L(G)
\]

\[
= m(n - 1)(m) - 2\frac{1}{4}(m^2 - 1)(n - 1)
\]

\[
= \frac{1}{2}(2m^2 - m^2 + 1)(n - 1)
\]

\[
= \frac{1}{2}(m^2 + 1)(n - 1)
\]

Case - 2 : \( m \) is even.

For \( P_m(K_n) \), we have
\[
\sum_{i=1}^{p-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1})] \\
= 2 \sum_{u \in V(G)} L(u) - L(u_1) - L(u_p) \\
= 2 L(G) - 1 \quad \text{(Choosing } u_1 \in V(C(G)) \text{ and } u_p \in N(u_1)) 
\] - - - (5.3)

Substituting (5.3) in (5.1), we get

\[
\begin{align*}
\rho_n(P_m(K_n)) &= f(u_p) \\
&\geq (p - 1)(d + 1) - [2L(G) + 1] \\
&= (p - 1)(d + 1) - 2L(G) + 1 \\
&= m(n - 1)(m + 1) - 2\frac{1}{2}m(m + 3)(n - 1) + 1 \\
&= \frac{1}{2}(m^2 - m)(n - 1) + 1 \\
&= \frac{1}{2}m(m - 1)(n - 1) + 1
\end{align*}
\]

Thus, from Case - 1 and Case - 2, we have

\[
\rho_n(P_m(K_n)) \geq \begin{cases} \\
\frac{1}{2}(m^2 + 1)(n - 1), & \text{if } m \text{ is odd} \\
\frac{1}{2}m(m - 1)(n - 1) + 1, & \text{if } m \text{ is even.}
\end{cases}
\]

\[\text{Theorem 5.4.6.} \quad \text{Let } G = P_m(K_n) \text{ be a linear cactus and } k = \lfloor \frac{m}{2} \rfloor \text{ then}
\]

\[
\rho_n(P_m(K_n)) \leq \begin{cases} \\
\frac{1}{2}(m^2 + 1)(n - 1), & \text{if } m \text{ is odd} \\
\frac{1}{2}m(m - 1)(n - 1) + 1, & \text{if } m \text{ is even.}
\end{cases}
\]

\[\text{Proof.} \quad \text{Here we consider the following two cases.}
\]

\[\text{Case - 1 : } m \text{ is odd.}
\]

For \( G = P_m(K_n) \), define \( f : V(G) \to \{0, 1, 2, ..., \} \) by \( f(u_1) = 0, f(u_{i+1}) = f(u_i) + d - L(u_i) - L(u_{i+1}), 1 \leq i \leq p - 1 \) as per following ordering of vertices.
Chapter 5. Radio Labeling of Cacti

\[ v_2^C \rightarrow v_2^{Rk} \rightarrow v_2^{Lk} \rightarrow v_3^C \rightarrow v_3^{Rk} \rightarrow v_3^{Lk} \rightarrow \ldots \]

\[ v_2^{n-1} \rightarrow v_2^{Rk} \rightarrow v_2^{Lk} \rightarrow v_3^{n-1} \rightarrow v_3^{C} \rightarrow v_3^{Rk} \rightarrow v_3^{Lk} \rightarrow \ldots \]

\[ v_2^L \rightarrow v_2^{R(k-1)} \rightarrow v_3^L \rightarrow v_3^{R(k-1)} \rightarrow v_4^L \rightarrow v_4^{R(k-1)} \rightarrow \ldots \]

\[ v_2^{n-1} \rightarrow v_2^{R(k-1)} \rightarrow v_1^L \rightarrow v_1^{R(k-1)} \rightarrow \ldots \]

\[ v_2^L \rightarrow v_2^{R(k-2)} \rightarrow v_3^L \rightarrow v_3^{R(k-2)} \rightarrow v_4^L \rightarrow v_4^{R(k-2)} \rightarrow \ldots \]

\[ v_2^{n-1} \rightarrow v_2^{R(k-2)} \rightarrow v_1^L \rightarrow v_1^{R(k-2)} \rightarrow \ldots \]

\[ v_2^L \rightarrow v_2^{R} \rightarrow v_3^L \rightarrow v_3^{R} \rightarrow v_4^L \rightarrow v_4^{R} \rightarrow \ldots \]

\[ v_2^{n-1} \rightarrow v_2^{R} \rightarrow v_1^L \rightarrow v_1^{R} \rightarrow v_1^{C} \rightarrow \ldots \]

**Case - 2 :** \( m \) is even.

For \( G = P_m(K_n) \), define \( f : V(G) \rightarrow \{0, 1, 2, \ldots, \} \) by \( f(u_1) = 0, f(u_{i+1}) = f(u_i) + d + 1 - L(u_i) - L(u_{i+1}), 1 \leq i \leq p - 1 \) as per following ordering of vertices.

\[ v_2^L \rightarrow v_2^{Rk} \rightarrow v_3^L \rightarrow v_3^{Rk} \rightarrow v_4^L \rightarrow v_4^{Rk} \rightarrow \ldots \]

\[ v_2^L \rightarrow v_2^{R(k-1)} \rightarrow v_3^L \rightarrow v_3^{R(k-1)} \rightarrow v_4^L \rightarrow v_4^{R(k-1)} \rightarrow \ldots \]
Thus, it is possible to assign labeling to the vertices of $P_m(K_n)$ with span equal to the lower bound satisfying the condition of Lemma 5.4.4 and hence $f$ is a radio labeling. Thus, we have,

$$\text{rn}(P_m(K_n)) \leq \begin{cases} \frac{1}{2}(m^2 + 1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even}. \end{cases}$$

**Theorem 5.4.7.** For $G = P_m(K_n)$,

$$\text{rn}(P_m(K_n)) = \begin{cases} \frac{1}{2}(m^2 + 1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even}. \end{cases}$$

**Proof.** The proof follows from Theorem 5.4.5 and Theorem 5.4.6

**Corollary 5.4.8.** For $G = P_{an}(K_n)$, where $a \geq 1$

$$\text{rn}(P_{an}(K_n)) = \begin{cases} \frac{1}{2}(n-1)(a^2n^2+1), & \text{if } an \text{ is odd} \\ \frac{1}{2}an(n-1)(an-1)+1, & \text{if } an \text{ is even}. \end{cases}$$

**Proof.** Let $G = P_m(K_n)$ be a linear cactus then by Theorem 5.4.7,

$$\text{rn}(P_m(K_n)) = \begin{cases} \frac{1}{2}(m^2 + 1)(n-1), & \text{if } m \text{ is odd} \\ \frac{1}{2}m(m-1)(n-1)+1, & \text{if } m \text{ is even}. \end{cases}$$

We take $m = an$ and consider the following two cases.
Case - 1: \( an \) is odd.

\[
 rn(P_{an}(K_n)) = (p-1)d - 2L(G)
 = an(n-1)an - \frac{1}{2}(an - 1)(an + 1)(n-1)
 = \frac{1}{2}(n-1)(a^2n^2 + 1)
\]

Case - 2: \( an \) is even.

\[
 rn(P_{an}(K_n)) = (p-1)(d+1) - 2L(G) + 1
 = an(n-1)(an + 1) - \frac{1}{2}an(an + 3)(n-1) + 1
 = \frac{1}{2}an(n-1)(an - 1) + 1
\]

Thus, from Case - 1 and Case - 2, we have

\[
 rn(P_{an}(K_n)) = \begin{cases} 
 \frac{1}{2}(n-1)(a^2n^2 + 1), & \text{if } an \text{ is odd} \\
 \frac{1}{2}an(n-1)(an - 1) + 1, & \text{if } an \text{ is even.} 
\end{cases}
\]

Corollary 5.4.9. For \( G = P_m(K_{am}) \), where \( a \geq 1 \)

\[
 rn(P_m(K_{am})) = \begin{cases} 
 \frac{1}{2}(am - 1)(n^2 + 1), & \text{if } m \text{ is odd} \\
 \frac{1}{2}m(m-1)(am - 1) + 1, & \text{if } m \text{ is even.} 
\end{cases}
\]

Proof. Let \( G = P_m(K_n) \) be a linear cactus then by Theorem 5.4.7,

\[
 rn(P_m(K_n)) = \begin{cases} 
 \frac{1}{2}(m^2 + 1)(n-1), & \text{if } m \text{ is odd} \\
 \frac{1}{2}m(m-1)(n-1) + 1, & \text{if } m \text{ is even.} 
\end{cases}
\]

We take \( n = am \) and consider the following two cases

Case - 1: \( m \) is odd.

\[
 rn(P_m(K_{am})) = (p-1)d - 2L(G)
 = m(am - 1)m - \frac{1}{2}(m - 1)(m + 1)(am - 1)
\]
\[
= \frac{1}{2}(am - 1)(m^2 + 1)
\]

**Case - 2 : \(m\) is even.**

\[
\begin{align*}
\text{rn}(P_m(K_{am})) &= (p - 1)(d + 1) - 2L(G) + 1 \\
&= m(am - 1)(m + 1) - \frac{1}{2}m(m + 3)(am - 1) + 1 \\
&= \frac{1}{2}m(m - 1)(am - 1) + 1
\end{align*}
\]

Thus, from Case - 1 and Case - 2, we have

\[
\text{rn}(P_m(K_{am})) = \begin{cases} \\
\frac{1}{2}(am - 1)(n^2 + 1), & \text{if } m \text{ is odd} \\
\frac{1}{2}m(m - 1)(am - 1) + 1, & \text{if } m \text{ is even.} \\
\end{cases}
\]

In particular, if \(m = n\) then we have the following useful result in terms of \(n\) only.

**Corollary 5.4.10.** For \(G = P_n(K_n)\),

\[
\text{rn}(P_n(K_n)) = \begin{cases} \\
\frac{1}{2}(n - 1)(n^2 + 1), & \text{if } n \text{ is odd} \\
\frac{1}{2}n(n - 1)^2 + 1, & \text{if } n \text{ is even.} \\
\end{cases}
\]

**Illustration 5.4.11.** In Table 5.1 and Figure 5.4, the ordering of the vertices and optimal radio labeling for \(P_5(K_6)\) are shown respectively.

<table>
<thead>
<tr>
<th>Table 5.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_C^2 \rightarrow v_{R2}^2 \rightarrow v_{L2}^2 \rightarrow v_C^3 \rightarrow v_{R2}^3)</td>
</tr>
<tr>
<td>(v_{L2}^3 \rightarrow v_C^4 \rightarrow v_{R2}^4 \rightarrow v_{L2}^4 \rightarrow v_C^5)</td>
</tr>
<tr>
<td>(v_{R2}^5 \rightarrow v_{L2}^5 \rightarrow v_C^6 \rightarrow v_{L2}^1 \rightarrow v_{R1}^1)</td>
</tr>
<tr>
<td>(v_{L1}^2 \rightarrow v_{R1}^2 \rightarrow v_{L1}^3 \rightarrow v_{R1}^3 \rightarrow v_{L1}^4)</td>
</tr>
<tr>
<td>(v_{R1}^4 \rightarrow v_{L1}^5 \rightarrow v_{R1}^5 \rightarrow v_{L1}^1 \rightarrow v_{R2}^1)</td>
</tr>
<tr>
<td>(v_C^1)</td>
</tr>
</tbody>
</table>
Illustration 5.4.12. In Table 5.2 and Figure 5.5, the ordering of the vertices and optimal radio labeling for $P_6(K_6)$ are shown respectively.

**Figure 5.4:** $\lambda(P_3(K_6)) = 65$.

**Figure 5.5:** $\lambda(P_6(K_6)) = 91$. 

**Table 5.2**

<table>
<thead>
<tr>
<th>$v^2_{L1}$</th>
<th>$v^2_{R3}$</th>
<th>$v^3_{L1}$</th>
<th>$v^3_{R3}$</th>
<th>$v^4_{L1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v^4_{R3}$</td>
<td>$v^5_{L1}$</td>
<td>$v^5_{R3}$</td>
<td>$v^1_{L1}$</td>
<td>$v^1_{R3}$</td>
</tr>
<tr>
<td>$v^2_{L2}$</td>
<td>$v^2_{R2}$</td>
<td>$v^3_{L2}$</td>
<td>$v^3_{R2}$</td>
<td>$v^4_{L2}$</td>
</tr>
<tr>
<td>$v^4_{R2}$</td>
<td>$v^5_{L2}$</td>
<td>$v^5_{R2}$</td>
<td>$v^1_{L2}$</td>
<td>$v^1_{R2}$</td>
</tr>
<tr>
<td>$v^2_{L3}$</td>
<td>$v^2_{R1}$</td>
<td>$v^3_{L3}$</td>
<td>$v^3_{R1}$</td>
<td>$v^4_{L3}$</td>
</tr>
<tr>
<td>$v^4_{R1}$</td>
<td>$v^5_{L3}$</td>
<td>$v^5_{R1}$</td>
<td>$v^1_{L3}$</td>
<td>$v^1_{R3}$</td>
</tr>
<tr>
<td>$v^1_C$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.5 Lower Bound of Radio Number for Spider Cacti

In this section, we provide a lower bound of radio number for spider cactus and present a class of spider cactus achieving this lower bound.

In spider cactus a vertex having maximum degree is called a root of spider cactus denoted as a $w$. Throughout this section, by a graph $G$ we mean spider cactus $S_{n_1, n_2, ..., n_k} (K_n)$.

Define a level function $L$ from a root $w$ on vertex set of spider cactus by

$$L_w(u) = d(u, w), \text{ for any } u \in V(G)$$

The total level of a spider cactus from a fixed root $w$ is given by

$$L_w(G) = \sum_{u \in V(G)} L_w(u)$$

**Observation 5.5.1.** For $G = S_{n_1, n_2, ..., n_k} (K_n)$,

1. $p = (n_1 + n_2 + ... + n_k)(n - 1) + 1$
2. $d = n_1 + n_2$
3. $d(u, v) \leq L_w(u) + L_w(v)$
4. $L_w(G) = \sum_{i=1}^{k} \frac{n_i(n_i + 1)}{2}(n - 1)$

A radio labeling is a one-to-one integral function $f$ on $V(G)$ induces an ordering of $V(G)$, which is a line-up of the vertices with increasing images. We denote this ordering by $U(f)$, where $V(G) = U(f) = (u_0, u_1, u_2, ..., u_{p-1})$ with

$$0 = f(u_0) < f(u_1) < f(u_2) < ... < f(u_{p-1})$$

The span of $f$ is $f(u_{p-1})$

**Theorem 5.5.2.** Let $G = S_{n_1, n_2, ..., n_k} (K_n)$ be a spider cactus. Then

$$rn(G) \geq (n - 1) \sum_{i=1}^{k} n_i(n_1 + n_2 - n_i) + 1$$
Proof. Let \( f \) be an optimal radio labeling for \( G = S_{n_1,n_2,...,n_k}(K_n) \), where 0 = \( f(u_0) < f(u_1) < f(u_2) < ... < f(u_{p-1}) \). Then \( f(u_{i+1}) - f(u_i) \geq (d + 1) - d(u_i,u_{i+1}) \), for all \( 0 \leq i \leq p - 2 \). Summing these \((p - 1)\) inequalities, we get

\[
\begin{align*}
\text{rn}(G) = f(u_{p-1}) & \geq (p - 1)(d + 1) - \sum_{i=0}^{p-1} d(u_i,u_{i+1}) \\
\text{--- (5.4)}
\end{align*}
\]

Now, \( \sum_{i=0}^{p-1} d(u_i,u_{i+1}) \leq \sum_{i=0}^{p-1} [L_w(u_i) + L_w(u_{i+1})] \)

\[
= 2 \sum_{i=0}^{p-1} L_w(u_i) - L_w(u_0) - L_w(u_{p-1}) \\
= 2 L_w(G) - 1 \{ \text{Choosing } u_0 = w \text{ and } u_{p-1} \in N(u_0) \}
\]

\[
= 2 \sum_{i=1}^{k} n_i(n_i + 1) - 2(n - 1) - 1 = (n - 1) \sum_{i=1}^{k} n_i(n_i + 1) - 1 \quad \text{--- (5.5)}
\]

Substituting (5.5) in (5.4), we get

\[
\begin{align*}
\text{rn}(G) = f(u_{p-1}) & \geq (n_1 + n_2 + ... + n_k)(n - 1)(n_1 + n_2 + 1) - (n - 1) \sum_{i=1}^{k} n_i(n_i + 1) + 1 \\
& = (n - 1)[(n_1 + n_2 + ... + n_k)(n_1 + n_2 + 1) - \sum_{i=1}^{k} n_i(n_i + 1)] + 1 \\
& = (n - 1)\left[\sum_{i=1}^{k} n_i(n_1 + n_2 + 1) - \sum_{i=1}^{k} n_i(n_i + 1)\right] + 1 \\
& = (n - 1) \sum_{i=1}^{k} n_i(n_1 + n_2 - n_i) + 1
\end{align*}
\]

Thus, \( \text{rn}(G) = f(u_{p-1}) \geq (n - 1) \sum_{i=1}^{k} n_i(n_1 + n_2 - n_i) + 1 \)

Illustration 5.5.3. In Figure 5.6, an optimal radio labeling for spider cactus \( G = S_{3,3,3,3}(K_4) \) is shown for which \( \text{rn}(G) = 109 \).
5.6 Radio Number for Symmetric Regular Cacti

Let $G = SC(K_n(k))(d)$ be a symmetric $n$-complete $k$-regular cactus of diameter $d$.

For any vertex $w$ in a graph $G$, the weight of vertex $w$ is defined by

$$W_G(w) = \sum_{u \in V(G)} d(w, u).$$

The weight of a graph $G$ is the smallest weight among all possible vertices of $G$:

$$W(G) = \min\{W_G(w) : w \in V(G)\}.$$  

A vertex $w$ of a graph $G$ is called a weight center of $G$ if $W_G(w) = W(G)$. From the definition of symmetric regular cactus, we immediately observe the following.

**Observation 5.6.1.** For the graph $SC(K_n(k))(d)$, the weight center coincides with center of graph.
Let \( w \) be a fixed weight center of \( SC(K_n(k))(d) \) then if \( u \) is on the shortest \((w,v)\)-path, then \( u \) is an ancestor of \( v \), and \( v \) is a descendent of \( u \). A fixed weight center \( w \) is an ancestor of every vertex. It is obvious that every vertex is its own ancestor and descendent.

**Notation 5.6.2.** We use notation \( x − y − z \) if \( y \) is on the shortest path joining \( x \) and \( z \), where \( y \) is distinct from \( x \) and \( z \).

Define the level function \( L \) on \( V(G) \) from a fixed weight center \( w \) by

\[
L_w(u) = d(w,u), \text{ for any } u \in V(G).
\]

Note that, in \( SC(K_n(k))(d) \), the maximum level from a fixed weight center \( w \) is \( \left\lceil \frac{d}{2} \right\rceil \).

**Lemma 5.6.3.** For \( SC(K_n(k))(d) \), the number of vertices at level \( t \), where \( 1 \leq t \leq \left\lceil \frac{d}{2} \right\rceil \) from a fixed weight center \( w \) is \( k(k−1)^{t−1}(n−1)^{t} \) except for \( t = \left\lfloor \frac{d}{2} \right\rfloor \) when \( d \) is odd.

**Proof.** We prove by induction on \( t \). For \( t = 1 \), the graph induced by a fixed weight center and vertices of level one is \( k \) copies of complete graph \( K_n \) sharing a common vertex which is a weight center. Therefore, the number of vertices at level one is \( k(n−1) = k(k−1)^{1−1}(n−1)^{1} \). For \( t = 2 \), it is the leaf vertices of \((k−1)(n−1)\) blocks joined to each of these \( k(n−1) \) vertices which are \( k(n−1)(k−1)(n−1) = k(k−1)^{2−1}(n−1)^{2} \).

We assume the result is true for \( t = m \). Then in \( SC(K_n(k))(d) \), the total number of vertices at level \( m \) is \( k(k−1)^{m−1}(n−1)^{m} \). For \( t = m+1 \), it is the leaf vertices of \((k−1)(n−1)\) blocks joined to each of these \( k(k−1)^{m−1}(n−1)^{m} \) vertices which are \( k(k−1)^{m}(n−1)^{m+1} \). Hence the result is true for \( t = m+1 \).

**Remark 5.6.4.** In \( SC(K_n(k))(d) \), for odd \( d \), the number of vertices at level \( \left\lfloor \frac{d}{2} \right\rfloor \) from a fixed weight center \( w \) is \((k−1)^{d−1}(n−1)^{\frac{d−1}{2}} \). Because in this case, the center of \( SC(K_n(k))(d) \) is a block isomorphic to \( K_n \). Denoting the vertices of this block by \( v_0, v_1, ..., v_{n−1} \) then they are weight centers according to Observation 5.6.1. Without loss of generality if we designate \( v_0 \) as a fixed weight center then the vertices of level \( \left\lfloor \frac{d}{2} \right\rfloor \) occurs as descendent of \( v_1, v_2, ..., v_{n−1} \) at distance \( \left\lfloor \frac{d}{2} \right\rfloor \) apart which are precisely \((k−1)^{d−1}(n−1)^{\frac{d−1}{2}}(k−1)(n−1) = (k−1)^{\frac{d−1}{2}}(n−1)^{\frac{d+1}{2}} \).
Lemma 5.6.5. For $SC(K_n(k))(d)$, the weight of graph $G$ is given by

$$W(G) = \begin{cases} 
\sum_{i=1}^{d} ik(k-1)^{i-1}(n-1)^{i}, & d \text{ is even} \\
\sum_{i=1}^{d} ik(k-1)^{i-1}(n-1)^{i} + \frac{d+1}{2}(k-1)^{d-1}(n-1)^{\frac{d+1}{2}}, & d \text{ is odd}.
\end{cases}$$

Proof. The proof follows from Lemma 5.6.3 and Remark 5.6.4

We introduce the following two parameters for $SC(K_n(k))(d)$. For two distinct vertices $u, v \in V(SC(K_n(k))(d))$ and a fixed weight center $w$ of a graph $SC(K_n(k))(d)$ define

$$\phi_w(u,v) = \max\{L_w(t) : t \text{ is a common ancestor of } u \text{ and } v\}$$

and

$$\delta_w(u,v) = \begin{cases} 
0, & \text{if } u-w-v \text{ or } w-u-v \text{ or } w-v-u; \\
1, & \text{otherwise}.
\end{cases}$$

The distance between any two distinct vertices $u$ and $v$ of a graph $SC(K_n(k))$ in terms of $L_w(u), L_w(v), \phi_w(u,v)$ and $\delta_w(u,v)$ is given by

$$d(u,v) = L_w(u) + L_w(v) - 2\phi_w(u,v) - \delta_w(u,v).$$

Definition 5.6.6. A branch of $SC(K_n(k))(d)$ is a maximal subgraph of $SC(K_n(k))(d)$ which contains at most one weight center of $SC(K_n(k))(d)$ which is not a cut vertex.

Lemma 5.6.7. For $SC(K_n(k))(d)$, the number of branches is $k$ if $d$ is even and $n(k-1)$ if $d$ is odd.

Proof. We consider the following two cases.

Case - 1: $d$ is even.

In this case, $SC(K_n(k))(d)$ is a vertex centered symmetric regular cactus and the center of graph is a weight center which is exactly in $k$ blocks. Therefore, the number of branches is $k$. 
Case - 2: \(d\) is odd.

In this case, \(SC(K_n(k))(d)\) is a block centered symmetric regular cactus which is 
\(K_n\) with vertices say \(v_0, v_1, \ldots, v_{n-1}\). Moreover each vertex \(v_i, i = 0, 1, \ldots, n - 1\) is weight center and each vertex is exactly in \(k\) block. But a block with vertices \(v_i, i = 0, 1, \ldots, n - 1\) is not a branch as it contains more than one weight center. Therefore, the number of branches is \(n(k - 1)\).

\[\text{Observation 5.6.8.}\] For any two distinct vertices \(u, v\) and a fixed weight center \(w\) of graph \(SC(K_n(k))(d)\),

\(1) \phi_w(u, v) \geq 0,\) for any \(u, v \in V(SC(K_n(k))(d))\),

\(2) \phi_w(u, v) = \delta_w(u, v) = 0\) if \(u - w - v\),

\(3) \phi_w(u, v) = 0\) if \(u\) and \(v\) are in different branches,

\(4) \phi_w(u, v) \geq 1\) if \(w - u - v\) or \(w - v - u\).

We give minimum span of a radio labeling for a symmetric regular cactus. Without loss of generality, we always initiate with label 0 for any radio labeling \(f\). Then the span of \(f\) will be the maximum label assigned for radio labeling. A radio labeling for \(G\) with span equal to \(rn(G)\) is called an optimal labeling. A radio labeling is a one-to-one integral function \(f\) on \(V(G)\), with \(0 \in f(V(G))\), induces an ordering of \(V(G)\), which is a line-up of the vertices with increasing images. We denote this ordering by \(U(f)\), where \(V(G) = U(f) = (u_0, u_1, u_2, \ldots, u_{p-1})\) with

\(0 = f(u_0) < f(u_1) < f(u_2) < \ldots < f(u_{p-1})\).

Note that, if \(f\) is a radio labeling, then the span of \(f\) is \(f(u_{p-1})\).

For a radio labeling \(f\), with ordering \(0 = f(u_0) < f(u_1) < \ldots < f(u_{p-1})\) introduce

\[x_i = f(u_{i+1}) - f(u_i) + L_w(u_{i+1}) + L_w(u_i) - d - 1,\] for any \(0 < i < p - 2\)

We claim that \(x_i \geq 2\phi_w(u_i, u_{i+1}) + \delta_w(u_i, u_{i+1})\) and hence \(x_i \geq 0\). Because

\[x_i = f(u_{i+1}) - f(u_i) + L_w(u_{i+1}) + L_w(u_i) - d - 1\]
\[ \geq d + 1 - d(u_i, u_{i+1}) + L_w(u_{i+1}) + L_w(u_i) - d - 1 \]

\[ = L_w(u_{i+1}) + L_w(u_i) - d(u_i, u_{i+1}) \]

\[ = L_w(u_{i+1}) + L_w(u_i) - [L_w(u_{i+1}) + L_w(u_i) - 2\phi_w(u_i, u_{i+1}) - \delta_w(u_i, u_{i+1})] \quad (\because \text{by (5.6)}) \]

\[ = 2\phi_w(u_i, u_{i+1}) + \delta_w(u_i, u_{i+1}) \]

\[ \geq 0 \quad (\because \text{Observation 5.6.8}) \]

Moreover, \( x_i = 0 \) if \( u_i \) and \( u_{i+1} \) are according to \( u_i - w - u_{i+1} \) and \( x_i \geq \delta_w(u_i, u_{i+1}) \) if \( u_i \) and \( u_{i+1} \) are in different branches.

**Lemma 5.6.9.** Let \( G = SC(K_n(k))(d) \). Suppose \( f \) is a non-negative integral one-to-one function on \( V(G) \) with the ordering of \( V(G) = U(f) = (u_0, u_1, u_2, \ldots, u_{p-1}) \).

Then \( f \) is a radio labeling for \( G \) if and only if for any set of consecutive vertices \( \{u_i, u_{i+1}, u_{i+2}, \ldots, u_j\}, 0 \leq i < j \leq p - 1; \) the following inequality hold.

\[ \sum_{t=i}^{j-1} x_t \geq 2( \sum_{t=i+1}^{j-1} L_w(u_t)) - (j - i - 1)(d + 1) + 2\phi_w(u_i, u_j) + \delta_w(u_i, u_{i+1}) \]

**Proof.** Suppose \( f \) is a radio labeling for \( G = SC(K_n(k))(d) \).

Then \( \sum_{t=i}^{j-1} x_t = \sum_{t=i}^{j-1} [f(u_{t+1}) - f(u_t) + L_w(u_{t+1}) + L_w(u_t) - d - 1] \)

\[ = f(u_j) - f(u_i) - (j - i)(d + 1) + 2( \sum_{t=i+1}^{j-1} L_w(u_t)) + L_w(u_i) + L_w(u_j) \]

Using equation (5.6) in the definition of radio labeling, we have

\[ f(u_j) - f(u_i) \geq d + 1 - d(u_i, u_j) \]

\[ = d + 1 - L_w(u_i) - L_w(u_j) + 2\phi_w(u_i, u_j) + \delta_w(u_i, u_j) \]

Substituting this in above expression, we have

\[ \sum_{t=i}^{j-1} x_t \geq d + 1 - L_w(u_i) - L_w(u_j) + 2\phi_w(u_i, u_j) + \delta_w(u_i, u_j) - (j - i)(d + 1) + 2( \sum_{t=i+1}^{j-1} L_w(u_t)) + L_w(u_i) + L_w(u_j) \]
\[ f = d \]

Let \( \gamma \) denote the number of vertices in a branch.

Proof. Let \( f \) be an optimal radio labeling for \( G = SC(K_n(k))(d) \), where \( f(u_0) = 0 < f(u_1) < f(u_2) < \ldots < f(u_{p-1}) \). Then \( f(u_{i+1}) - f(u_i) \geq (d + 1) - d(u_{i+1}, u_i) \) for all \( 0 \leq i \leq p - 2 \). Summing these \( p - 1 \) inequalities, we get

\[ rn(SC(K_n(k))(d)) = f(u_{p-1}) \geq (p - 1)(d + 1) - \sum_{i=0}^{p-2} d(u_{i+1}, u_i). \]  

- - - (5.7)

Theorem 5.6.10. Let \( G = SC(K_n(k))(d) \) be a symmetric regular cactus of order \( p \). Then

\[ rn(G) = \begin{cases} 
(p - 1)(d + 1) + 1 - 2W(G), & \text{if } d \text{ is even} \\
(p - 1)(d + 1) + 1 - 2W(G) + (n - 2)(k - 1)\gamma, & \text{if } d \text{ is odd,}
\end{cases} \]

where \( \gamma \) denotes the number of vertices in a branch.
Let \( w \) be a fixed weight center then in the last term of the equation (5.7), each vertex occurs exactly twice except \( u_0 \) and \( u_{p-1} \) which occurs exactly once. Hence by (5.6) we have

\[
\sum_{i=0}^{p-2} d(u_{i+1}, u_i) = 2 \left( \sum_{u \in V(G)} L_w(u) \right) - L_w(u_0) - L_w(u_{p-1}) - \\
2 \sum_{i=1}^{p-2} \phi_w(u_{i+1}, u_i) - \sum_{i=1}^{p-2} \delta_w(u_{i+1}, u_i).
\] - - - (5.8)

We consider the following two cases.

**Case - 1: \( d \) is even.**

In this case, \( G = SC(K_n(k))(d) \) has one weight center say \( w \) and \( k \) branches. In \( SC(K_n(k))(d) \), if we choose \( u_0 = w \) and \( u_i, u_{i+1} \) in different branches then \( u_i - w - u_{i+1} \), for all \( 0 \leq i \leq p - 2 \), according to Observation 5.6.8, we have \( \phi_w(u_{i+1}, u_i) = \delta_w(u_{i+1}, u_i) = 0 \) and choosing \( u_{p-1} \) adjacent to \( u_0 \) then equation (5.8) becomes,

\[
\sum_{i=0}^{p-2} d(u_{i+1}, u_i) = 2 \left( \sum_{u \in V(G)} L_w(u) \right) - 1 = 2W(G) - 1.
\] - - - (5.9)

By substituting equation (5.9) in equation (5.7), we get

\[
\text{rn}(SC(K_n(k))(d)) = f(u_{p-1}) \geq (p - 1)(d + 1) + 1 - 2W(G)
\] - - - (5.10)

which is a lower bound of the radio number for symmetric regular cacti of even diameter.

Now, we give radio labeling of \( G \) with span equal to the lower bound stated in (5.10). First we give ordering for the vertices of \( G \). Let \( v_{i,j}^{m,l} \) be the \( l^{th} \) vertex of \( m^{th} \) block at level \( j \) in \( i^{th} \) branch. Notice that \( v_{i,j}^{m,l} = v_0 \) when \( i = j = m = l = 0 \). In this ordering between two vertices \( u \) and \( v \), we put sign \( \rightarrow \) that mean first define radio labeling for \( u \) and then define radio labeling for \( v \). For ordering of vertices, an important task is to determine block order which contains vertices of particular level. In any block notice that the level of \((n - 1)\) vertices is same and the level of a vertex is one less than of \((n - 1)\) vertices. We give ordering to blocks with respect to the vertices which are at
maximum level. In a branch, to give ordering to blocks which contains vertices of level \( l \), we first choose a vertex \( u \) at level \( l \) and give order one to this block. Next find a vertex \( v \) such that \( d(u, v) = 2L(u) - 1 \) and give order two to the block which contains \( v \). Continue this process till all the blocks receive labels which contains vertices of the same level. The blocks of other branches can be labeled in like way. Define \( f : V(G) \rightarrow \{0, 1, 2, \ldots\} \) by \( f(u_{i+1}) = f(u_i) + d + 1 - L_w(u_i) - L_w(u_{i+1}) \) as per following ordering of vertices:

\[
f(v_0) = 0 \quad \rightarrow
\]

\[
\begin{align*}
v^{1, 1}_{1, \frac{d}{2}} & \rightarrow v^{1, 1}_{2, \frac{d}{2}} & \rightarrow v^{1, 1}_{3, \frac{d}{2}} & \rightarrow \ldots & \rightarrow v^{1, 1}_{k, \frac{d}{2}} \\
v^{2, 1}_{1, \frac{d}{2}} & \rightarrow v^{2, 1}_{2, \frac{d}{2}} & \rightarrow v^{2, 1}_{3, \frac{d}{2}} & \rightarrow \ldots & \rightarrow v^{2, 1}_{k, \frac{d}{2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v^{m, 1}_{1, \frac{d}{2}} & \rightarrow v^{m, 1}_{2, \frac{d}{2}} & \rightarrow v^{m, 1}_{3, \frac{d}{2}} & \rightarrow \ldots & \rightarrow v^{m, 1}_{k, \frac{d}{2}} \\
v^{1, 2}_{1, \frac{d}{2}} & \rightarrow v^{1, 2}_{2, \frac{d}{2}} & \rightarrow v^{1, 2}_{3, \frac{d}{2}} & \rightarrow \ldots & \rightarrow v^{1, 2}_{k, \frac{d}{2}} \\
v^{2, 2}_{1, \frac{d}{2}} & \rightarrow v^{2, 2}_{2, \frac{d}{2}} & \rightarrow v^{2, 2}_{3, \frac{d}{2}} & \rightarrow \ldots & \rightarrow v^{2, 2}_{k, \frac{d}{2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v^{m, 2}_{1, \frac{d}{2}} & \rightarrow v^{m, 2}_{2, \frac{d}{2}} & \rightarrow v^{m, 2}_{3, \frac{d}{2}} & \rightarrow \ldots & \rightarrow v^{m, 2}_{k, \frac{d}{2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v^{1, n-1}_{1, \frac{d}{2}} & \rightarrow v^{1, n-1}_{2, \frac{d}{2}} & \rightarrow v^{1, n-1}_{3, \frac{d}{2}} & \rightarrow \ldots & \rightarrow v^{1, n-1}_{k, \frac{d}{2}}
\end{align*}
\]
Case - 2: $d$ is odd.

In this case, $G = SC(K_n(k))(d)$ has $n$ weight centers say $v_0, v_1, \ldots v_{n-1}$ and $n(k-1)$ branches. Without loss of generality, fix $v_0$ as a weight center $w$. For $SC(K_n(k))(d)$, if we choose $u_0 = w$ and $u_i, u_{i+1}$ according to $u_i - w - u_{i+1}$ then by Observation 5.6.8,
\[ \phi_w(u_{i+1}, u_i) = \delta_w(u_{i+1}, u_i) = 0 \]

But in \( SC(K_n(k))(d) \), it is possible only for \( 2(k-1)\gamma \) vertices, where \( \gamma \) denotes the number of vertices in a branch. Therefore, by Observation 5.6.8, for \( (n-2)(k-1)\gamma \) number of vertices either \( \phi_w(u_{i+1}, u_i) = 0 \) or \( \delta_w(u_{i+1}, u_i) = 0 \). Note that if \( \phi_w(u_{i+1}, u_i) = 0 \) then \( \delta_w(u_{i+1}, u_i) = 1 \) and \( \delta_w(u_{i+1}, u_i) = 0 \) then \( \phi_w(u_{i+1}, u_i) \geq 1 \). To maximize the left hand side of (5.8) choose vertices \( u_i \) and \( u_{i+1} \) in different branches and \( u_{p-1} \) adjacent to \( u_0 \) then we obtain

\[
\sum_{i=0}^{p-2} d(u_{i+1}, u_i) = 2 \left( \sum_{u \in V(G)} L_w(u) \right) - 1 - (n-2)(k-1)\gamma
\]

\[
= 2 W(G) - 1 - (n-2)(k-1)\gamma.
\]

By substituting (5.11) in (5.7), we get

\[ rn(SC(K_n(k))(d)) = f(u_{p-1}) \geq (p-1)(d+1) + 1 - 2W(G) + (n-2)(k-1)\gamma \] - - (5.12)

Now, we give radio labeling of \( G \) with span equal to the stated lower bound in (5.12). First we give ordering for the vertices of \( G \). Let the first branch is ordered with respect to \( v_0 \) and the second branch with respect to \( v_1 \), and so on. Let \( v_{i,j}^{m,l} \) be as per case-1. Define \( f : V(G) \to \{0,1,2,...\} \) by \( f(u_{i+1}) = f(u_i) + d + 1 - L_w(u_i) - L_w(u_{i+1}) + \delta_w(u_i, u_{i+1}) \) as per following ordering of vertices.

\[
f(v_0) = 0 \to \]

\[
v_{1,1}^{1,1} \to ... \to v_{n,1}^{1,1} \to v_{n+1,1}^{1,1} \to ... \to v_{n(k-1),1}^{1,1} \rightarrow v_{1,1}^{1,1} \]

\[
v_{2,1}^{2,1} \to ... \to v_{n,1}^{2,1} \to v_{n+1,1}^{2,1} \to ... \to v_{n(k-1),1}^{2,1} \rightarrow v_{1,1}^{2,1} \]

\[
... \to ... \to ... \to ... \to ... \to ... \to \]

\[
v_{m,1}^{m,1} \to ... \to v_{n,1}^{m,1} \to v_{n+1,1}^{m,1} \to ... \to v_{n(k-1),1}^{m,1} \rightarrow v_{1,1}^{m,1} \]

\[
v_{1,2}^{1,2} \to ... \to v_{n,1}^{1,2} \to v_{n+1,1}^{1,2} \to ... \to v_{n(k-1),1}^{1,2} \rightarrow v_{1,1}^{1,2} \]
Thus, in case - (1) and case - (2), it is possible to assign label to vertices of $SC(K_n(k))(d)$ with span equal to the lower bound which satisfies the condition of Lemma 5.6.9 of radio labeling and hence the result. ■
We demonstrate the result of Theorem 5.6.10 by means of following two examples.

**Illustration 5.6.11.** In Figure 5.7, an ordering of vertices and optimal radio labeling of $SC(K_4(2))(6)$ is shown.

For $SC(K_4(2))(6)$, the number of vertices $p$ is 79. The number of branches is 2 and the number of vertices at level 1, 2 and 3 are 6, 18 and 54 respectively. The weight $W(SC(K_4(2))(6))$ of a graph $SC(K_4(2))(6)$ is 204. Hence, $rn(SC(K_4(2))(6)) = (p - 1)(d + 1) + 1 - 2W(SC(K_4(2))(6)) = (79 - 1)(6 + 1) + 1 - 2(204) = 139$. An assignment of label can be performed in the following manner.

$$f(v_0) = 0$$

$$f(v_{1,3}^{1,1}) = 4 \quad \rightarrow \quad f(v_{2,3}^{1,1}) = 5 \quad \rightarrow \quad f(v_{1,3}^{2,1}) = 6 \quad \rightarrow \quad f(v_{2,3}^{2,1}) = 7$$

$$f(v_{1,3}^{3,1}) = 8 \quad \rightarrow \quad f(v_{2,3}^{3,1}) = 9 \quad \rightarrow \quad f(v_{1,3}^{4,1}) = 10 \quad \rightarrow \quad f(v_{2,3}^{4,1}) = 11$$

$$f(v_{1,3}^{5,1}) = 12 \quad \rightarrow \quad f(v_{2,3}^{5,1}) = 13 \quad \rightarrow \quad f(v_{1,3}^{6,1}) = 14 \quad \rightarrow \quad f(v_{2,3}^{6,1}) = 15$$

$$f(v_{1,3}^{7,1}) = 16 \quad \rightarrow \quad f(v_{2,3}^{7,1}) = 17 \quad \rightarrow \quad f(v_{1,3}^{8,1}) = 18 \quad \rightarrow \quad f(v_{2,3}^{8,1}) = 19$$

$$f(v_{1,3}^{9,1}) = 20 \quad \rightarrow \quad f(v_{2,3}^{9,1}) = 21 \quad \rightarrow \quad f(v_{1,3}^{10,1}) = 22 \quad \rightarrow \quad f(v_{2,3}^{10,1}) = 23$$

$$f(v_{1,3}^{2,2}) = 24 \quad \rightarrow \quad f(v_{2,3}^{2,2}) = 25 \quad \rightarrow \quad f(v_{1,3}^{3,2}) = 26 \quad \rightarrow \quad f(v_{2,3}^{3,2}) = 27$$

$$f(v_{1,3}^{4,2}) = 28 \quad \rightarrow \quad f(v_{2,3}^{4,2}) = 29 \quad \rightarrow \quad f(v_{1,3}^{5,2}) = 30 \quad \rightarrow \quad f(v_{2,3}^{5,2}) = 31$$

$$f(v_{1,3}^{6,2}) = 32 \quad \rightarrow \quad f(v_{2,3}^{6,2}) = 33 \quad \rightarrow \quad f(v_{1,3}^{7,2}) = 34 \quad \rightarrow \quad f(v_{2,3}^{7,2}) = 35$$

$$f(v_{1,3}^{8,2}) = 36 \quad \rightarrow \quad f(v_{2,3}^{8,2}) = 37 \quad \rightarrow \quad f(v_{1,3}^{9,2}) = 38 \quad \rightarrow \quad f(v_{2,3}^{9,2}) = 39$$
\[
f(v_{1,3}^{1,3}) = 40 \quad \rightarrow \quad f(v_{2,3}^{1,3}) = 41 \quad \rightarrow \quad f(v_{1,3}^{2,3}) = 42 \quad \rightarrow \quad f(v_{2,3}^{2,3}) = 43
\]

\[
f(v_{1,3}^{3,3}) = 44 \quad \rightarrow \quad f(v_{2,3}^{3,3}) = 45 \quad \rightarrow \quad f(v_{1,3}^{4,3}) = 46 \quad \rightarrow \quad f(v_{2,3}^{4,3}) = 47
\]

\[
f(v_{1,3}^{5,3}) = 48 \quad \rightarrow \quad f(v_{2,3}^{5,3}) = 49 \quad \rightarrow \quad f(v_{1,3}^{6,3}) = 50 \quad \rightarrow \quad f(v_{2,3}^{6,3}) = 51
\]

\[
f(v_{1,3}^{7,3}) = 52 \quad \rightarrow \quad f(v_{2,3}^{7,3}) = 53 \quad \rightarrow \quad f(v_{1,3}^{8,3}) = 54 \quad \rightarrow \quad f(v_{2,3}^{8,3}) = 55
\]

\[
f(v_{1,3}^{9,3}) = 56 \quad \rightarrow \quad f(v_{2,3}^{9,3}) = 57 \quad \rightarrow \quad f(v_{1,3}^{1,1}) = 59 \quad \rightarrow \quad f(v_{2,3}^{1,1}) = 62
\]

\[
f(v_{1,2}^{2,1}) = 65 \quad \rightarrow \quad f(v_{2,2}^{2,1}) = 68 \quad \rightarrow \quad f(v_{1,2}^{3,1}) = 71 \quad \rightarrow \quad f(v_{2,2}^{3,1}) = 74
\]

\[
f(v_{1,2}^{1,2}) = 77 \quad \rightarrow \quad f(v_{2,2}^{1,2}) = 80 \quad \rightarrow \quad f(v_{1,2}^{2,2}) = 83 \quad \rightarrow \quad f(v_{2,2}^{2,2}) = 86
\]

\[
f(v_{1,2}^{3,2}) = 89 \quad \rightarrow \quad f(v_{2,2}^{3,2}) = 92 \quad \rightarrow \quad f(v_{1,2}^{4,2}) = 95 \quad \rightarrow \quad f(v_{2,2}^{4,2}) = 98
\]

\[
f(v_{1,2}^{2,3}) = 101 \quad \rightarrow \quad f(v_{2,2}^{2,3}) = 104 \quad \rightarrow \quad f(v_{1,2}^{3,3}) = 107 \quad \rightarrow \quad f(v_{2,2}^{3,3}) = 110
\]

\[
f(v_{1,1}^{1,1}) = 114 \quad \rightarrow \quad f(v_{2,1}^{1,1}) = 119 \quad \rightarrow \quad f(v_{1,1}^{2,1}) = 124 \quad \rightarrow \quad f(v_{2,1}^{2,1}) = 129
\]

\[
f(v_{1,1}^{1,3}) = 134 \quad \rightarrow \quad f(v_{2,1}^{1,3}) = 139 = rn(SC(K_4(2))(5))
\]

**Illustration 5.6.12.** In Figure 5.8, an ordering of vertices and optimal radio labeling of \(SC(K_4(2))(5)\) is shown.

For \(SC(K_4(2))(5)\), the number of vertices \(p\) is 52 and \(\gamma\) is 13. The number of branches is 4 and the number of vertices at level 1, 2 and 3 are 6, 18 and 27 respectively. The weight \(W(SC(K_4(2))(5))\) of a graph \(SC(K_4(2))(5)\) is 123. Hence, \(rn(SC(K_4(2))(5)) = (p - 1)(d + 1) + 1 − 2W(SC(K_4(2))(5)) + (n - 2)(k - 1)\gamma = (52 - \ldots\)
1)(5 + 1) + 1 − 2(123) + (4 − 2)(2 − 1)13 = 87. An assignment of label can be perform in following manner.

\[ f(v_0) = 0 \]

\[ f(v_{2,3}^{1,1}) = 3 \quad \rightarrow \quad f(v_{3,3}^{1,1}) = 4 \quad \rightarrow \quad f(v_{4,3}^{1,1}) = 5 \quad \rightarrow \quad f(v_{1,2}^{1,1}) = 6 \]

\[ f(v_{2,3}^{2,1}) = 7 \quad \rightarrow \quad f(v_{3,3}^{2,1}) = 8 \quad \rightarrow \quad f(v_{4,3}^{2,1}) = 9 \quad \rightarrow \quad f(v_{1,2}^{2,1}) = 10 \]

\[ f(v_{2,3}^{3,1}) = 11 \quad \rightarrow \quad f(v_{3,3}^{3,1}) = 12 \quad \rightarrow \quad f(v_{4,3}^{3,1}) = 13 \quad \rightarrow \quad f(v_{1,2}^{3,1}) = 14 \]

\[ f(v_{2,3}^{1,2}) = 15 \quad \rightarrow \quad f(v_{3,3}^{1,2}) = 16 \quad \rightarrow \quad f(v_{4,3}^{1,2}) = 17 \quad \rightarrow \quad f(v_{1,2}^{1,2}) = 18 \]

\[ f(v_{2,3}^{2,2}) = 19 \quad \rightarrow \quad f(v_{3,3}^{2,2}) = 20 \quad \rightarrow \quad f(v_{4,3}^{2,2}) = 21 \quad \rightarrow \quad f(v_{1,2}^{2,2}) = 22 \]

\[ f(v_{2,3}^{3,2}) = 23 \quad \rightarrow \quad f(v_{3,3}^{3,2}) = 24 \quad \rightarrow \quad f(v_{4,3}^{3,2}) = 25 \quad \rightarrow \quad f(v_{1,2}^{3,2}) = 26 \]

\[ f(v_{2,3}^{1,3}) = 27 \quad \rightarrow \quad f(v_{3,3}^{1,3}) = 28 \quad \rightarrow \quad f(v_{4,3}^{1,3}) = 29 \quad \rightarrow \quad f(v_{1,2}^{1,3}) = 30 \]

\[ f(v_{2,3}^{2,3}) = 31 \quad \rightarrow \quad f(v_{3,3}^{2,3}) = 32 \quad \rightarrow \quad f(v_{4,3}^{2,3}) = 33 \quad \rightarrow \quad f(v_{1,2}^{2,3}) = 34 \]

\[ f(v_{2,3}^{3,3}) = 35 \quad \rightarrow \quad f(v_{3,3}^{3,3}) = 36 \quad \rightarrow \quad f(v_{4,3}^{3,3}) = 37 \quad \rightarrow \quad f(v_{1,2}^{3,3}) = 38 \]

\[ f(v_{2,2}^{1,1}) = 40 \quad \rightarrow \quad f(v_{3,2}^{1,1}) = 43 \quad \rightarrow \quad f(v_{4,2}^{1,1}) = 46 \quad \rightarrow \quad f(v_{1,1}^{1,1}) = 49 \]

\[ f(v_{2,2}^{2,1}) = 52 \quad \rightarrow \quad f(v_{3,2}^{2,1}) = 55 \quad \rightarrow \quad f(v_{4,2}^{2,1}) = 58 \quad \rightarrow \quad f(v_{1,1}^{2,1}) = 61 \]

\[ f(v_{2,2}^{3,1}) = 64 \quad \rightarrow \quad f(v_{3,2}^{3,1}) = 67 \quad \rightarrow \quad f(v_{4,2}^{3,1}) = 70 \quad \rightarrow \quad f(v_{1,1}^{3,1}) = 73 \]

\[ f(v_1) = 77 \quad \rightarrow \quad f(v_2) = 82 \quad \rightarrow \quad f(v_3) = 87 = rn(SC(K_4(2))(5)) \]
Chapter 5. Radio Labeling of Cacti

Figure 5.7: \( rn(SC(K_4(2))(6)) = 139. \)

Figure 5.8: \( rn(SC(K_4(2))(5)) = 87. \)
5.7 Lower and Upper Bounds of Radio Number for Arbitrary Cacti

We continue with notations and terminology introduced in previous sections.

**Theorem 5.7.1.** Let $G = C(K_n)$ be an $n$-complete cactus of diameter $d$ and order $p$. Then

$$(p - 1)(d + 1) + 1 - 2W(C(K_n)) \leq \text{rn}(C(K_n)) \leq \text{rn}(SC(K_n(k))(d)).$$

**Proof.** Let $f$ be an optimal radio labeling for $G = C(K_n)$, where $0 = f(u_0) < f(u_1) < f(u_2) < \ldots < f(u_{p-1})$. Then $f(u_{i+1}) - f(u_i) \geq (d + 1) - d(u_{i+1}, u_i)$ for all $0 \leq i \leq p - 2$. Summing up these $p - 1$ inequalities, we get

$$\text{rn}(C(K_n)) = f(u_{p-1}) \geq (p - 1)(d + 1) - \sum_{i=0}^{p-2} d(u_{i+1}, u_i). \quad - - - (5.13)$$

Let $w$ be a fixed weight center then in the last term of the equation of (5.13), each vertex occurs exactly twice except $u_0$ and $u_{p-1}$, for which each occurs exactly once. Hence we get

$$\sum_{i=0}^{p-2} d(u_{i+1}, u_i) \geq 2 \left( \sum_{u \in V(G)} L_w(u) \right) - L_w(u_0) - L_w(u_{p-1}) - 2 \sum_{i=0}^{p-2} \phi_w(u_{i+1}, u_i) \quad - - - (5.14)$$

In $C(K_n)$, if we choose $u_0 = w$ and $u_i, u_{i+1}$ according to $u_i - w - u_{i+1}$ then $\phi_w(u_{i+1}, u_i) = 0$ and choosing $u_{p-1}$ adjacent to $u_0$ then by equation (5.14) we have,

$$\sum_{i=0}^{p-2} d(u_{i+1}, u_i) \geq 2 W(C(K_n)) - 1 \quad - - - (5.15)$$

By substituting equation (5.15) in equation (5.13), we obtain lower bound for the radio number for $n$-complete cactus.

For upper bound, observe that a graph $C(K_n)$ of diameter $d$ is a subgraph of $SC(K_n(k))(d)$, where $k = \frac{\Delta}{n-1}$ in which $\Delta$ denotes the maximum degree of vertex in $C(K_n)$.

Thus, $(p - 1)(d + 1) + 1 - 2W(C(K_n)) \leq \text{rn}(C(K_n)) \leq \text{rn}(SC(K_n(k))(d)).$
5.8 Further Scope of Research

For spider cacti, to investigate exact radio number is an open area of research.

5.9 Concluding Remarks

The channel assignment problem is of prime importance for an interference free transmitter network. The study of radio labeling is an attempt in this direction. We have investigated radio number for various graph families. The pattern prescribed in the investigations are useful to locate the geographic location of any transmitter in the network. Thus detection of fault will be faster and easily manageable.

The next chapter is aimed to report some more results on radio labeling.