Chapter 3

Common Fixed Point Theorems for Multivalued Maps in Metric Spaces

In this chapter we obtain common fixed point theorem that extends a recent result of Dorić and Lazović for a multivalued map on metric space satisfying Ćirić-Suzuki type generalized contraction. Further, we obtain a result which generalizes and extends classical fixed point theorems of Nadler, Reich, Rus and some recent Suzuki type fixed point theorems. Existence of a common solution for a class of functional equations arising in dynamic programming is also discussed.
3.1 Introduction

Let \((X, d)\) be a metric space and \(CL(X)\) the family of all nonempty closed subsets of \(X\). \((CL(X), H)\) equipped with the generalized Hausdorff metric \(H\) defined by

\[
H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},
\]

where \(A, B \in CL(X)\) and \(d(x, K) = \inf_{z \in K} d(x, z)\), is called the generalized hyperspace of \(X\).

For any nonempty subsets \(A, B\) of \(X\), \(d(A, B)\) denotes the gap between the subsets \(A\) and \(B\), while

\[
\rho(A, B) = \sup \{d(a, b) : a \in A, b \in B\},
\]

\(BN(X) = \{A : \emptyset \neq A \subseteq X\text{ and the diameter of } A\text{ is finite}\}\).

As usual, we write \(d(x, B)\) (resp. \(\rho(x, B)\)) for \(d(A, B)\) (resp. \(\rho(A, B)\)) when \(A = \{x\}\).

For the sake of brevity, we follow the following notations for \(x, y \in X\):

\[
m_1(Tx, Ty) = \max \{d(x, y), d(x, Tx), d(y, Ty)\},
\]

\[
m_2(Tx, Ty) = \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\},
\]

\[
m_3(Sx, Ty) = \max \left\{ d(x, y), \frac{d(x, Sx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Sx)}{2} \right\},
\]

\[
m_4(Tx, Ty) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\},
\]

\[
M(Sx, Ty) = \max \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\},
\]

\[
M'(Sx, Ty) = \max \{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}.
\]

Recently Suzuki [246] and Kikkawa and Suzuki [112] obtained interesting generalizations of the Banach’s classical fixed point theorem and other fixed point results by Jungck [92], and Nadler [141]. These results have important outcomes (see, for instance, [2], [4], [11], [12], [28], [29], [54], [57], [59], [65], [111], [112], [113], [114], [139], [148], [177], [228], [229], [230], [232], [247], [248], [249] and others).

The following result is essentially due to Kikkawa and Suzuki [113] (see also [230]) which generalizes the classical multivalued contraction theorem due to Nadler [141] (see also [1], [2], [7], [8], [13], [15], [29], [31], [36], [41], [43], [45], [46], [47], [50], [51], [53], [54], [56], [57], [59], [60], [66], [72], [73], [74], [75], [78], [79], [80], [82], [96], [99], [100], [101], [102], [103], [108], [109], [110], [113], [116], [117], [118], [119], [124], [133],...
Let \((X,d)\) be a complete metric space and let \(T : X \to \mathcal{CL}(X)\). Assume there exists \(r \in [0,1)\) such that for every \(x,y \in X\),
\[
d(x,Tx) \leq (1 + r)d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq rd(x,y).
\]
Then there exists \(z \in X\) such that \(z \in Tz\).

Theorem 3.1.2. Let \(X\) be a complete metric space and \(T : X \to \mathcal{CL}(X)\). Assume there exists \(r \in [0,1)\) such that for every \(x,y \in X\),
\[
d(x,Tx) \leq (1 + r)d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq rm_2(Tx,Ty).
\]
Then there exists \(z \in X\) such that \(z \in Tz\).

Theorem 3.1.3. Define a nonincreasing function \(\varphi\) from \([0,1)\) onto \((0,1]\) by
\[
\varphi(r) = \begin{cases} 
1 & \text{if } 0 \leq r < \frac{1}{2} \\
1 - r & \text{if } \frac{1}{2} \leq r < 1 
\end{cases} 
\]
(3.1.1)

Let \(X\) be a complete metric space and \(T : X \to \mathcal{CL}(X)\). Assume there exists \(r \in [0,1)\) such that for every \(x,y \in X\),
\[
\varphi(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq rm_4(Tx,Ty). 
\]
(3.1.2)
Then there exists \(z \in X\) such that \(z \in Tz\).

We remark that, for every \(x,y \in X\), the generalized contraction \(H(Tx,Ty) \leq r m_4(Tx,Ty), 0 \leq r < 1\), was first studied by Ćirić [41].

The following general common fixed point theorem is due to Sastry and Naidu [203].

Theorem 3.1.4. Let \(X\) be a complete metric space and \(S,T : X \to X\). Assume there exists \(r \in [0,1)\) such that for every \(x,y \in X\),
\[
d(Sx,Ty) \leq rM(Sx,Ty). 
\]
(3.1.3)
Then $S$ and $T$ have a unique common fixed point.

For an excellent discussion on several special cases and variants of Theorem 3.1.4, one may refer to Rus [200]. However, the generality of Theorem 3.1.4 may be appreciated from the fact that (3.1.3) in Theorem 3.1.4 cannot be replaced by

$$d(Sx, Ty) \leq rM'(Sx, Ty).$$

(3.1.4)

Indeed, Sastry and Naidu [203, Ex. 5], have shown that maps $S$ and $T$ satisfying (3.1.4) need not have a common fixed point on a complete metric space. Notice that the condition (3.1.4) with $S = T$ is Ćirić’s quasi-contraction [42]. We remark that, in Rhoades’ comprehensive comparison of contractive conditions [191], the condition (3.1.4) with $S = T$ is considered the most general contraction for a self-map of a metric space.

A particular case of our first main result (cf. Theorem 3.2.2) generalizes Theorems 3.1.1 and 3.1.2. Some other special cases are also discussed. The second main result of this chapter (cf. Theorem 3.2.9) generalizes Theorems 3.1.3 and 3.1.4. Further, a corollary of Theorem 3.2.9 is used to obtain a unique common fixed point theorem for multivalued maps on a metric space with values in $BN(X)$. As another application, we deduce the existence of a common solution for a general class of functional equations under much weaker conditions than those in [17], [20], [21], [25], [228], [232] and others.

### 3.2 Main Results and Examples

We shall need the following lemma essentially due to Nadler [141], (see also [13], [41], [197, p. 4], [199], [200, p. 76]).

**Lemma 3.2.1.** If $A, B \in CL(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.

**Theorem 3.2.2.** Let $X$ be a complete metric space and let $S, T : X \to CL(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\min\{d(x, Sx), d(y, Ty)\} \leq (1 + r)d(x, y) \text{ implies } H(Sx, Ty) \leq rm_3(Sx, Ty).$$

Then there exists an element $u \in X$ such that $u \in Su \cap Tu$.

**Proof.** Obviously $m_3(Sx, Ty) = 0$ iff $x = y$ is a common fixed point of $S$ and $T$. So we may assume that $m_3(Sx, Ty) > 0$.

Let $\varepsilon > 0$ be such that $\beta = r + \varepsilon < 1$. Let $u_0 \in X$ and $u_1 \in Tu_0$. By Lemma 3.2.1, there exists $u_2 \in Su_1$ such that

$$d(u_2, u_1) \leq H(Su_1, Tu_0) + m_3(Su_1, Tu_0).$$
Similarly, there exists $u_3 \in T_u 2$ such that
\[ d(u_3, u_2) \leq H(T_u 2, S_u 1) + \varepsilon m(T_u 2, S_u 1). \]

Continuing in this manner, we find a sequence \( \{u_n\} \) in \( X \) such that
\[ u_{2n+1} \in T_u 2n, u_{2n+2} \in S_u 2n+1 \]
and
\[ d(u_{2n+1}, u_{2n}) \leq H(T_u 2n, S_u 2n-1) + m(T_u 2n, S_u 2n-1), \]
\[ d(u_{2n+2}, u_{2n+1}) \leq H(S_u 2n+1, T_u 2n) + \varepsilon m(S_u 2n+1, T_u 2n). \]

Now, we show that for any \( n \in \mathbb{N} \),
\[ d(u_{2n+1}, u_{2n}) \leq \beta d(u_{2n-1}, u_{2n}). \] (3.2.1)

Suppose if \( d(u_{2n-1}, S_u 2n-1) \geq d(u_{2n}, T_u 2n) \), then
\[ \min\{d(u_{2n-1}, S_u 2n-1), d(u_{2n}, T_u 2n)\} \leq (1 + r)d(u_{2n-1}, u_{2n}). \]

Therefore by the assumption,
\[ d(u_{2n+1}, u_{2n}) \leq H(S_u 2n-1, T_u 2n) \]
\[ \leq r m(S_u 2n-1, T_u 2n) \]
\[ \leq r m(S_u 2n-1, T_u 2n) + \varepsilon m(S_u 2n-1, T_u 2n) \]
\[ = \beta m(S_u 2n-1, T_u 2n) \]
\[ = \beta \max\left\{d(u_{2n-1}, u_{2n}), \frac{d(u_{2n-1}, S_u 2n-1) + d(u_{2n}, T_u 2n)}{2}, \right. \]
\[ \left. \frac{d(u_{2n-1}, T_u 2n) + d(u_{2n}, S_u 2n-1)}{2} \right\} \]
\[ \leq \beta \max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}. \]

This yields (3.2.1).

Suppose, if \( d(u_{2n}, T_u 2n) \geq d(u_{2n-1}, S_u 2n-1) \), then
\[ \min\{d(u_{2n-1}, S_u 2n-1), d(u_{2n}, T_u 2n)\} \leq (1 + r)d(u_{2n-1}, u_{2n}). \]
Therefore by the assumption,
\[
\begin{align*}
    d(u_{2n+1}, u_{2n}) & \leq H(Su_{2n-1}, Tu_{2n}) \\
    & \leq r \, m_3(Su_{2n-1}, Tu_{2n}) \\
    & \leq r \, m_3(Su_{2n-1}, Tu_{2n}) + \varepsilon \, m_3(Su_{2n-1}, Tu_{2n}) \\
    & = \beta \, m_3(Su_{2n-1}, Tu_{2n}) \\
    & = \beta \max \left\{ d(u_{2n-1}, u_{2n}), \frac{d(u_{2n-1}, Su_{2n-1}) + d(u_{2n}, Tu_{2n})}{2}, \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2} \right\} \\
    & \leq \beta \max \{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}.
\end{align*}
\]
This proves (3.2.1). In an analogous manner, we show that
\[
    d(u_{2n+2}, u_{2n+1}) \leq \beta d(u_{2n+1}, u_{2n}). \tag{3.2.2}
\]
We conclude from (3.2.1) and (3.2.2) that for any \( n \in \mathbb{N} \),
\[
    d(u_{n+1}, u_n) \leq \beta d(u_n, u_{n-1}).
\]
Therefore \( \{u_n\} \) is a Cauchy sequence and has a limit in \( X \). Call it \( u \).
Since \( u_n \to u \), there exists \( n_0 \in \mathbb{N} \) (natural numbers) such that
\[
    d(u, u_n) \leq \frac{1}{3} d(u, y) \quad \text{for } y \neq u \text{ and all } n \geq n_0.
\]
Then as in [228, p. 3376] and [247, p. 1862],
\[
(1 + r)^{-1} d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, u_{2n}) \leq d(u_{2n-1}, u) + d(u, u_{2n}) \leq \frac{2}{3} d(y, u) = d(y, u) - \frac{1}{3} d(y, u) \leq d(y, u) - d(u_{2n-1}, u) \leq d(u_{2n-1}, y).
\]
Therefore

\[ d(u_{2n-1}, Su_{2n-1}) \leq (1 + r)d(u_{2n-1}, y). \]  \hspace{1cm} (3.2.3)

Now either \( d(u_{2n-1}, Su_{2n-1}) \leq d(y, Ty) \) or \( d(y, Ty) \leq d(u_{2n-1}, Su_{2n-1}) \).

In either case, by (3.2.3) and the assumption,

\[ d(u_{2n}, Ty) \leq H(Su_{2n-1}, Ty) \]
\[ \leq m_3(Su_{2n-1}, Ty). \]
\[ \leq r \max \left\{ d(u_{2n-1}, y), \frac{d(u_{2n-1}, Su_{2n-1}) + d(y, Ty)}{2}, \frac{d(u_{2n-1}, Ty) + d(y, Su_{2n-1})}{2} \right\}. \]

Making \( n \to \infty \),

\[ d(u, Ty) \leq r \max \left\{ d(u, y), \frac{d(u, u) + d(y, Ty)}{2}, \frac{d(u, Ty) + d(y, u)}{2} \right\} \]
\[ \leq r \max \left\{ d(u, y), \frac{d(u, Ty) + d(u, y)}{2} \right\}. \]  \hspace{1cm} (3.2.4)

It is clear from (3.2.4) that

\[ d(u, Ty) \leq rd(u, y). \]  \hspace{1cm} (3.2.5)

Now we show that

\[ H(Su, Ty) \leq r \max \left\{ d(u, y), \frac{d(u, Su) + d(y, Ty)}{2}, \frac{d(u, Ty) + d(y, Su)}{2} \right\} \]  \hspace{1cm} (3.2.6)

Assume that \( y \neq u \). Then for every \( n \in \mathbb{N} \), there exists \( z_n \in Ty \) such that

\[ d(u, z_n) \leq d(u, Ty) + \frac{1}{n}d(y, u). \]
So we have by (3.2.5),
\[
\begin{align*}
d(y, Ty) &\leq d(y, z_n) \\
&\leq d(y, u) + d(u, z_n) \\
&\leq d(y, u) + d(u, Ty) + \frac{1}{n}d(y, u) \\
&\leq d(y, u) + rd(u, y) + \frac{1}{n}d(u, y) \\
&= \left(1 + r + \frac{1}{n}\right)d(y, u).
\end{align*}
\]

Hence
\[
d(y, Ty) \leq (1 + r)d(y, u). \tag{3.2.7}
\]

Now either \(d(u, Su) \leq d(y, Ty)\) or \(d(y, Ty) \leq d(u, Su)\).

So in either case by (3.2.7) and the assumption, \(H(Su, Ty) \leq rm_3(Su, Ty)\), which is (3.2.6).

Now taking \(y = u_{2n}\) in (3.2.6), we have
\[
d(Su, u_{2n+1}) \leq H(Su, Tu_{2n}) \\
\leq r \max \left\{ \frac{d(u, u_{2n})}{2}, \frac{d(u, Su) + d(u_{2n}, u_{2n+1})}{2}, \frac{d(u, u_{2n+1}) + d(u_{2n}, Su)}{2} \right\}.
\]

Passing to the limit this obtains \(d(Su, u) \leq \frac{r}{2}d(Su, u)\). So \(u \in Su\), as \(Su\) is closed.

In an analogous manner, we can show that \(u \in Tu\).

Corollary 3.2.3. Let \(X\) be a complete metric space and \(S, T : X \to X\). Assume there exists \(r \in [0, 1)\) such that for every \(x, y \in X\),
\[
\min\{d(x, Sx), d(y, Ty)\} \leq (1 + r)d(x, y) \implies d(Sx, Ty) \leq rm_3(Sx, Ty).
\]

Then \(S\) and \(T\) have a unique common fixed point.

Proof. It comes from Theorem 3.2.2 that \(S\) and \(T\) have a common fixed point. The uniqueness of the common fixed point follows easily.

Corollary 3.2.4. Theorem 3.1.2.

Corollary 3.2.5 ([228]). Let \(X\) be a complete metric space and \(T : X \to X\). Assume there exists \(r \in [0, 1)\) such that for every \(x, y \in X\),
\[
d(x, Tx) \leq (1 + r)d(x, y) \implies d(Tx, Ty) \leq rm_2(Tx, Ty).
\]
Then $T$ has a unique fixed point.

**Proof.** It comes from Corollary 3.2.3 when $S = T$. $\square$

Now we give two examples to show the generality of our results.

**Example 3.2.6.** Let $X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$ and $d$ be defined by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$ 

Let $S$ and $T$ be such that

$$S(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2 \end{cases} \quad \text{and} \quad T(x_1, x_2) = \begin{cases} (0, x_1) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2 \end{cases}$$

Then maps $S$ and $T$ do not satisfy (3.1.1) of Theorem 3.1.3 (e.g. $(x, y) = ((4, 5), (5, 4))$). However, $S$ and $T$ satisfy all the hypotheses of Corollary 3.2.3.

**Example 3.2.7.** Let $X = \{(1, 1), (4, 1), (1, 4), (4, 5), (5, 4)\}$ and $d$ be defined by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

Let $T$ be such that

$$T(x_1, x_2) = \begin{cases} (x_1, 1) & \text{if } x_1 \leq x_2 \\ (1, x_2) & \text{if } x_1 > x_2 \end{cases}$$

Then $T$ satisfies all the hypotheses of Corollary 3.2.5, but does not satisfy Ciric’s quasi-contraction, viz. (3.1.2) with $S = T$ (e.g. $x = (4, 5), y = (5, 4)$).

**Question 3.2.8.** Can we replace “$H(Sx, Ty) \leq r m_2(Sx, Ty)$” in Theorem 3.2.2 by the following:

$$H(Sx, Ty) \leq r M(Sx, Ty).$$

(3.2.8)

We remark that (3.2.8) with $S = T$ is the Ciric’s generalized contraction [41] for $T : X \to CL(X)$.

The following Theorem provides the answer to the Question 3.2.8.

**Theorem 3.2.9.** Let $X$ be a complete metric space and $S, T : X \to CL(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\phi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \implies H(Sx, Ty) \leq r M(Sx, Ty).$$

(3.2.9)

Then there exists an element $u \in X$ such that $u \in Su \cap Tu$. 31
Proof. Obviously $M(Sx, Ty) = 0$ iff $x = y$ is a common fixed point of $S$ and $T$. So, we may assume $M(Sx, Ty) > 0$ for distinct $x, y \in X$. Let $\varepsilon > 0$ be such that $\beta = r + \varepsilon < 1$. Let $u_0 \in X$ and $u_1 \in Tu_0$. Then by Lemma 3.2.1, there exists $u_2 \in Su_1$ such that

$$d(u_2, u_1) \leq H(Su_1, Tu_0) + \varepsilon M(Su_1, Tu_0).$$

Similarly, there exists $u_3 \in Tu_2$ such that

$$d(u_3, u_2) \leq H(Tu_2, Su_1) + \varepsilon M(Tu_2, Su_1).$$

Continuing in this manner, we find a sequence $\{u_n\}$ in $X$ such that

$$u_{2n+1} \in Tu_{2n} \quad \text{and} \quad u_{2n+2} \in Su_{2n+1}$$

such that

$$d(u_{2n+1}, u_{2n}) \leq H(Tu_{2n}, Su_{2n-1}) + \varepsilon M(Tu_{2n}, Su_{2n-1}),$$

$$d(u_{2n+2}, u_{2n+1}) \leq H(Su_{2n+1}, Tu_{2n}) + \varepsilon M(Su_{2n+1}, Tu_{2n}).$$

Now, we consider two cases and show that for any $n \in N$,

$$d(u_{2n+1}, u_{2n}) \leq \beta d(u_{2n-1}, u_{2n}). \quad (3.2.10)$$

Case (i): If $d(u_{2n-1}, Su_{2n-1}) \geq d(u_{2n}, Tu_{2n})$, then

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}).$$

Therefore by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq r M(Su_{2n-1}, Tu_{2n}) \quad (3.2.11)$$

Case (ii): If $d(u_{2n}, Tu_{2n}) \geq d(u_{2n-1}, Su_{2n-1})$, then

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}).$$

So by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq r M(Su_{2n-1}, Tu_{2n}). \quad (3.2.12)$$
Hence in either case we obtain by (3.2.11) and (3.2.12),
\[
d(u_{2n}, u_{2n+1}) \leq H(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n})
\leq r M(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n})
= \beta M(Su_{2n-1}, Tu_{2n})
= \beta \max\left\{d(u_{2n-1}, u_{2n}), d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n}), \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2}\right\}
\leq \beta \max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}.
\]
This yields (3.2.10). Analogously, we obtain
\[
d(u_{2n+2}, u_{2n+1}) \leq \beta d(u_{2n+1}, u_{2n}),
\]
and conclude that, for any \(n \in \mathbb{N}\),
\[
d(u_{n+1}, u_n) \leq \beta d(u_n, u_{n-1}).
\]
Therefore \(\{u_n\}\) is a Cauchy sequence and has a limit in \(X\). Call it \(u\).

Now we show that for any \(y \in X - \{u\}\),
\[
d(u, Ty) \leq r \max\{d(u, y), d(y, Ty)\} \tag{3.2.13}
\]
and
\[
d(u, Sy) \leq r \max\{d(u, y), d(y, Sy)\} \tag{3.2.14}
\]
Since \(u_n \to u\), there exists \(n_0 \in \mathbb{N}\) (natural numbers) such that
\[
d(u, u_n) \leq \frac{1}{3} d(u, y) \quad \text{for } y \neq u \text{ and all } n \geq n_0.
\]
Then as in [228, p. 3376] and [247, p. 1862],
\[
\varphi(r)d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, Su_{2n-1})
\leq d(u_{2n-1}, u_{2n})
\leq d(u_{2n-1}, u) + d(u, u_{2n})
\leq \frac{2}{3} d(y, u)
= d(y, u) - \frac{1}{3} d(y, u)
\leq d(y, u) - d(u_{2n-1}, u)
= d(u_{2n-1}, y).
\]
Therefore
\[ \varphi(r)d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, y) \]  
(3.2.15)

Now either \( d(u_{2n-1}, Su_{2n-1}) \leq d(y, Ty) \) or \( d(y, Ty) \leq d(u_{2n-1}, Su_{2n-1}) \).
So in either case by (3.2.15),
\[ \varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(y, Ty)\} \leq d(u_{2n-1}, y). \]

Hence by the assumption (3.2.9),
\[ d(u_{2n}, Ty) \leq H(Su_{2n-1}, Ty) \]
\[ \leq rM(Su_{2n-1}, Ty) \]
\[ \leq r \max \left\{ d(u_{2n-1}, y), d(u_{2n-1}, Su_{2n-1}), d(y, Ty), \frac{d(u_{2n-1}, Ty) + d(y, Su_{2n-1})}{2} \right\}. \]

Making \( n \to \infty \),
\[ d(u, Ty) \leq r \max \left\{ d(u, y), d(u, u), d(y, Ty), \frac{d(u, Ty) + d(y, u)}{2} \right\} \]
\[ \leq r \max \{d(u, y), d(y, Ty), d(u, Ty)\}. \]

This yields (3.2.13). Similarly, we can show (3.2.14). Now, we show that \( u \in Su \cap Tu \).
For \( 0 \leq r < \frac{1}{2} \), the following cases arise.

Case I: Suppose \( u \not\in Su \) and \( u \not\in Tu \). Then as in [57, p. 6], let \( a \in Tu \) be such that
\[ 2rd(a, u) < d(u, Tu), \]
and \( a \in Su \) be such that \( 2rd(a, u) < d(u, Su) \). Since \( a \in Tu \) implies \( a \neq u \), we have from (3.2.13) and (3.2.14),
\[ d(u, Ta) \leq r \max\{d(u, a), d(a, Ta)\} \]
(3.2.16)
and
\[ d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}. \]
(3.2.17)

On the other hand, since \( \varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u) \),
\[ \varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u). \]
Therefore by the assumption (3.2.9),
\[ d(Sa, a) \leq H(Sa, Tu) \]
\[ \leq r \max \left\{ d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2} \right\} \]
\[ = r \max \left\{ d(a, u), d(a, Sa), \frac{1}{2} d(u, Sa) \right\}. \]

This gives \( d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u). \)
So by (3.2.17), \( d(Sa, u) \leq rd(a, u). \) Thus
\[ d(u, Tu) \leq d(u, Sa) + H(Sa, Tu) \]
\[ \leq rd(a, u) + rd(a, u) \]
\[ = 2rd(a, u) < d(u, Tu) \] (by the assumption of Case I).

This contradicts \( u \not\in Tu. \) Consequently \( u \in Tu. \) Similarly \( u \in Su. \)

Case II: Let \( u \in Su \) and \( u \not\in Tu. \) Then as in the previous case, let \( a \in Tu \) be such that
\[ 2rd(a, u) < d(u, Tu). \]

Since \( a \neq u, \) we have from (3.2.14),
\[ d(u, Sa) \leq r \max \{d(u, a), d(a, Sa)\}. \] (3.2.18)

On the other hand, Since \( \varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u), \)
\[ \varphi(r) \min \{d(a, Sa), d(u, Tu)\} \leq d(a, u). \]

Therefore by the assumption (3.2.9),
\[ d(Sa, a) \leq H(Sa, Tu) \]
\[ \leq r \max \left\{ d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2} \right\} \]
\[ = r \max \left\{ d(a, u), d(a, Sa), \frac{1}{2} d(u, Sa) \right\}. \]

This gives \( d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u). \)
So by (3.2.17), \( d(Sa, u) \leq rd(a, u) \). Thus
\[
d(u, Tu) \leq d(u, Sa) + H(Sa, Tu) \\
\leq rd(a, u) + rd(a, u) \\
= 2rd(a, u) < d(u, Tu) \quad \text{(by the assumption of Case II)}.
\]
This contradicts \( u \notin Tu \). Consequently \( u \in Tu \).

Case III: \( u \in Tu \) and \( u \notin Su \).

As in the previous case, it follows that \( u \in Su \). Now we consider the case \( \frac{1}{2} \leq r < 1 \).

First we show that
\[
H(Sx, Tu) \leq r \max \left\{ d(x, u), d(x, Sx), d(u, Tu), \frac{d(x, Tu) + d(u, Sx)}{2} \right\}.
\]
Assume that \( x \neq u \). Then for every \( n \in \mathbb{N} \), there exists \( z_n \in Sx \) such that
\[
d(u, z_n) \leq d(u, Sx) + \frac{1}{n} d(x, u).
\]
Therefore
\[
d(x, Sx) \leq d(x, z_n) \\
\leq d(x, u) + d(u, z_n) \\
\leq d(x, u) + d(u, Sx) + \frac{1}{n} d(x, u).
\] (3.2.19)

Using (3.2.14) with \( y = x \), (3.2.19) implies
\[
d(x, Sx) \leq d(x, u) + r \max \{d(x, u), d(x, Sx)\} + \frac{1}{n} d(u, x).
\] (3.2.20)

If \( d(x, u) \geq d(x, Sx) \), then (3.2.20) gives
\[
d(x, Sx) \leq d(x, u) + rd(x, u) + \frac{1}{n} d(u, x) \\
= \left( 1 + r + \frac{1}{n} \right) d(x, u).
\]

Making \( n \to \infty \),
\[
d(x, Sx) \leq (1 + r) d(x, u).
\]
Thus \( \varphi(r) d(x, Sx) = (1 - r) d(x, Sx) \leq \left( \frac{1}{1 + r} \right) d(x, Sx) \leq d(x, u) \).
Then $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$, and by the assumption (3.2.9),

$$H(Sx, Tu) \leq r \max \left\{ d(x, u), d(x, Sx), d(u, Tu), \frac{d(x, Tu) + d(u, Sx)}{2} \right\}$$  \hspace{1cm} (3.2.21)

If $d(x, u) < d(x, Sx)$, then (3.2.20) gives

$$d(x, Sx) \leq d(x, u) + rd(x, Sx) + \frac{1}{n}d(u, x),$$

that is, $(1 - r)d(x, Sx) \leq \left(1 + \frac{1}{n}\right)d(x, u)$. Making $n \to \infty$,

$$\varphi(r)d(x, Sx) \leq d(x, u).$$

Then $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$, and by the assumption, we get (3.2.21).

Taking $x = u_{2n+1}$ in (3.2.21) and passing to the limit, we obtain

$$d(u, Tu) \leq rd(u, Tu).$$

This gives $u \in Tu$. Analogously, $u \in Su$. \hfill \Box

The following result generalizes Theorem 3.1.4.

**Corollary 3.2.10.** Let $X$ be a complete metric space and $S, T : X \to X$. Suppose there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \implies d(Sx, Ty) \leq rM(Sx, Ty),$$

Then $S$ and $T$ have a unique common fixed point.

**Proof.** For single-valued maps $S$ and $T$, it comes from Theorem 3.2.9 that they have a common fixed point. The uniqueness of the common fixed point follows easily. \hfill \Box

**Remark 3.2.1.** Theorem 3.1.3 is obtained as a particular case of Theorem 3.2.9 when $S = T$.

Now we derive the following result due to Dorić and Lazović [59, Corollary 2.3].

**Corollary 3.2.11.** Let $X$ be a complete metric space and $T : X \to X$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\varphi(r)d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq rm_4(Tx, Ty).$$  \hspace{1cm} (3.2.22)

Then there exists a unique $z \in X$ such that $z = Tz$.

**Proof.** It comes from Corollary 3.2.10 when $S = T$. \hfill \Box
Now we derive the following result essentially due to Dhompongsa and Yingtaweesthitkul [57] which extends the Sct (see also [230]) and its extension by Moț and Petruşel [139, Remark 6.6(b)].

**Corollary 3.2.12.** Let $X$ be a complete metric space and $T : X \to X$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$d(x, Tx) \leq (1 + r)d(x, y) \text{ implies } d(Tx, Ty) \leq rm_1(Tx, Ty).$$

(3.2.23)

Then there exists a unique $z \in X$ such that $z = Tz$.

The following example show the generality of our results.

**Example 3.2.13.** Let $X = \{(0, 0), (0, 1), (1, 0), (1, 2), (2, 1)\}$ be endowed with the metric $d$ defined by

$$d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|.$$

Let $S$ and $T$ be such that

$$S(x_1, x_2) = \begin{cases} (0, 0) & \text{if } (x_1, x_2) \neq (1, 2), (2, 1) \\ (1, 0) & \text{if } (x_1, x_2) = (1, 2) \\ (0, 1) & \text{if } (x_1, x_2) = (2, 1), \end{cases}$$

and

$$T(x_1, x_2) = \begin{cases} (0, 0) & \text{if } (x_1, x_2) \neq (1, 2), (2, 1) \\ (0, 1) & \text{if } (x_1, x_2) = (1, 2) \\ (1, 0) & \text{if } (x_1, x_2) = (2, 1). \end{cases}$$

Then $S$ and $T$ do not satisfy the condition (3.1.3) of Theorem 3.1.4 at $x = (1, 2), y = (1, 2)$ and at $x = (2, 1), y = (2, 1)$. However, this is readily verified that all the hypotheses of corollary 3.2.10 are satisfied for the maps $S$ and $T$.

### 3.3 Fixed point theorems for multivalued maps

**Theorem 3.3.1.** Let $X$ be a complete metric space and $P, Q : X \to BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\min\{\rho(x, Px), \rho(y, Qy)\} \leq (1 + r)d(x, y)$$

(3.3.1)
implies
\[
\rho(Px, Qy) \leq r \max \left\{ d(x, y), \frac{\rho(x, Px) + \rho(x, Qy)}{2}, \frac{d(x, Qy) + d(y, Px)}{2} \right\} \tag{3.3.2}
\]

Then there exists a unique point \( z \in X \) such that \( z \in Px \cap Qy \).

**Proof.** Choose \( \lambda \in (0, 1) \). Define single-valued maps \( S, T : X \to X \) as follows.
For each \( x \in X \), let \( Sx \) be a point of \( Px \) which satisfies
\[
d(x, Sx) \geq r^\lambda \rho(x, Px).
\]

Similarly, for each \( y \in X \), let \( Ty \) be a point of \( Qy \) such that
\[
d(y, Ty) \geq r^\lambda \rho(y, Qy).
\]

Since \( Sx \in Px \) and \( Ty \in Qy \),
\[
d(x, Sx) \leq \rho(x, Px) \quad \text{and} \quad d(y, Ty) \leq \rho(y, Qy).
\]

So (3.3.1) gives
\[
\min\{d(x, Sx), d(y, Ty)\} \leq \min\{\rho(x, Px), \rho(y, Qy)\} \leq (1 + r)d(x, y), \tag{3.3.3}
\]
and this implies (3.3.2). Therefore
\[
d(Sx, Ty) \leq \rho(Px, Qy)
\]
\[
\leq r.r^{-\lambda} \max \left\{ r^\lambda d(x, y), \frac{r^\lambda \rho(x, Px) + r^\lambda \rho(y, Qy)}{2}, \frac{r^\lambda d(x, Qy) + r^\lambda d(y, Px)}{2} \right\}
\]
\[
\leq r^{1-\lambda} \max \left\{ d(x, y), \frac{d(x, Sx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Sx)}{2} \right\}.
\]

So (3.3.3), viz., \( \min\{d(x, Sx), d(y, Ty)\} \leq (1 + r^')d(x, y) \) implies
\[
d(Sx, Ty) \leq r' \max \left\{ d(x, y), \frac{d(x, Sx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Sx)}{2} \right\},
\]
where \( r' = r^{1-\lambda} < 1 \). Hence by Corollary 3.2.3, \( S \) and \( T \) have a unique point \( z \in X \) such that \( Sz = Tz = z \). This implies \( z \in Px \cap Qz \). \( \Box \)

The following result show that Theorem 3.2.1 is a generalization of the result of Singh and Mishra [228, Theorem 3.6].
**Corollary 3.3.2.** Let $X$ be a complete metric space and $P : X \to BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\rho(x, Px) \leq (1 + r)d(x, y)$$

implies

$$\rho(Px, Py) \leq r \max \left\{ d(x, y), \frac{\rho(x, Px) + \rho(y, Py)}{2}, \frac{d(x, Py) + d(y, Px)}{2} \right\}.$$

Then there exists a unique point $z \in X$ such that $z \in Pz$.

**Proof.** It comes from Theorem 3.3.1 when $Q = P$. □

We remark that Corollaries 3.2.5 and 3.3.2 generalize fixed point theorems from [141], [186], [200] and others.

**Theorem 3.3.3.** Let $X$ be a complete metric space and $P, Q : X \to BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y) \quad (3.3.4)$$

implies

$$\rho(Px, Qy) \leq r \max \left\{ d(x, y), \frac{\rho(x, Px), \rho(y, Qy)}{2}, \frac{d(x, Qy) + d(y, Px)}{2} \right\} \quad (3.3.5)$$

Then there exists a unique point $z \in X$ such that $z \in Pz \cap Qz$.

**Proof.** Choose $\lambda \in (0, 1)$. Define single-valued maps $S, T : X \to X$ as follows. For each $x \in X$, let $Sx$ be a point of $Px$ which satisfies

$$d(x, Sx) \geq r^\lambda \rho(x, Px).$$

Similarly, for each $y \in X$, let $Ty$ be a point of $Qy$ such that

$$d(y, Ty) \geq r^\lambda \rho(y, Qy).$$

Since $Sx \in Px$ and $Ty \in Qy$,

$$d(x, Sx) \leq \rho(x, Px) \quad \text{and} \quad d(y, Ty) \leq \rho(y, Qy).$$

So, (3.3.4) gives

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq \varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y), \quad (3.3.6)$$

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and this implies (3.3.5). Therefore
\[ d(Sx, Ty) \leq \rho(Px, Qy) \]
\[ \leq r \cdot r^{-\lambda} \max \left\{ r^\lambda d(x, y), r^\lambda \rho(x, Px), r^\lambda \rho(y, Qy), \right. \]
\[ \left. \frac{r^\lambda d(x, Qy) + r^\lambda d(y, Px)}{2} \right\} \]
\[ \leq r^{1-\lambda} \max \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}. \]

So (3.3.6), viz., \( \varphi(r') \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \) implies
\[ d(Sx, Ty) \leq r' \max \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}, \]
where \( r' = r^{1-\lambda} < 1. \)

Hence by Corollary 3.2.10, \( S \) and \( T \) have a unique point \( z \in X \) such that \( Sz = Tz = z \). This implies \( z \in Pz \cap Qz \). \( \square \)

**Corollary 3.3.4.** Let \( X \) be a complete metric space and \( P : X \to BN(X) \). Assume there exists \( r \in [0, 1) \) such that for every \( x, y \in X \),
\[ \rho(x, Px) \leq (1 + r)d(x, y) \]
implies
\[ \rho(Px, Py) \leq r \max \left\{ d(x, y), \rho(x, Px), \rho(y, Py), \frac{d(x, Py) + d(y, Px)}{2} \right\} \]
Then there exists a unique point \( z \in X \) such that \( z \in Pz \).

**Proof.** It comes from Theorem 3.3.3 when \( Q = P \). \( \square \)

### 3.4 Applications to dynamic programming

Throughout this section, we assume that \( Y \) and \( Z \) are Banach spaces, \( W \subseteq Y \) and \( D \subseteq Z \). Let \( R \) denote the field of reals, \( g_1, g_2 : W \times D \to R \) and \( G_1, G_2 : W \times D \times R \to R \). Taking \( W \) and \( D \) as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving functional equations:
\[ p_i = \sup_{y \in D} \{g_i(x, y) + G_i(x, y, p_i(x, y))\}, \quad x \in W, \ i = 1, 2. \quad (3.4.1) \]
In the multistage process, some functional equations arise in a natural way (cf. Bellman [20] and Bellman and Lee [21]; see also [17], [19], [24], [25], [26], [27], [35], [52], [59], [91], [125], [126], [127], [128], [129], [130], [167], [168], [169], [173], [227] and others). In this section, we study the existence of common solution of the functional equations (3.4.1) arising in dynamic programming.

Let $B(W)$ denote the set of all bounded real-valued functions on $W$. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

(DP-1) $G_1, G_2, g_1$ and $g_2$ are bounded.
(DP-2) There exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\varphi(r) \min\{|h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|\} \leq |h(t) - k(t)|$$

implies

$$|G_1(x, y, h(t)) - G_2(x, y, k(t))| \leq rM(A_1 h(t), A_2 k(t))$$

where

$$M(A_1 h(t), A_2 k(t)) = \max \left\{ |h(t) - k(t)|, |h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|, \frac{|h(t) - A_2 k(t)| + |k(t) - A_1 h(t)|}{2} \right\}$$

where $A_1, A_2$ are defined as follows:

$$A_i h(x) = \sup_{y \in D} G_i(x, y, h(x, y)), \quad x \in W, \ h \in B(W), \ i = 1, 2.$$

**Theorem 3.4.1.** Assume the conditions (DP-1) and (DP-2). Then the functional equations (3.4.1), $i = 1, 2$, have a unique common solution in $B(W)$.

**Proof.** For any $h, k \in B(W)$, let $d(h, k) = \sup\{|h(x) - k(x)| : x \in W\}$. Then $(B(W), d)$ is a complete metric space.

Let $\lambda$ be any arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$ and choose $y_1, y_2 \in D$ such that

$$A_i h_i < G_i(x, y_i, h_i(x_i)) + \lambda, \quad (3.4.2)$$

where $x_i = (x, y_i), \ i = 1, 2$.

Further,

$$A_1 h_1 \geq G_1(x, y_2, h_1(x_2)), \quad (3.4.3)$$

$$A_2 h_2 \geq G_2(x, y_1, h_2(x_1)). \quad (3.4.4)$$
Therefore, the first inequality in (DP-2) becomes
\[
\varphi(r) \min\{|h_1(x) - A_1 h_1(x)|, |h_2(x) - A_2 h_2(x)|\} \leq |h_1(x) - h_2(x)|, \tag{3.4.5}
\]
and this together with (3.4.2) and (3.4.4) implies
\[
A_1 h_1 - A_2 h_2 < G_1(x, y_1, h_1(x_1)) - G_2(x, y_1, h_2(x_1)) + \lambda \\
\leq |G_1(x, y_1, h_1(x_1)) - G_2(x, y_1, h_2(x_1))| + \lambda \\
\leq r M(A_1 h_1, A_2 h_2) + \lambda. \tag{3.4.6}
\]
Similarly, (3.4.2), (3.4.3) and (3.4.5) imply
\[
A_2 h_2(x) - A_1 h_1(x) \leq r M(A_1 h_1, A_2 h_2) + \lambda. \tag{3.4.7}
\]
So, from (3.4.6) and (3.4.7), we obtain
\[
|A_1 h_1(x) - A_2 h_2(x)| \leq r M(A_1 h_1, A_2 h_2) + \lambda. \tag{3.4.8}
\]
Since this inequality is true for any \( x \in W \), and \( \lambda > 0 \) is arbitrary, on taking supremum, we find from (3.4.5) and (3.4.8) that
\[
\varphi(r) \min\{d(h_1, A_1 h_1), d(h_2, A_2 h_2)\} \leq d(h_1, h_2)
\]
implies
\[
d(A_1 h_1, A_2 h_2) \leq r M(A_1 h_1, A_2 h_2).
\]
Therefore, Corollary 3.2.10 applies, wherein \( A_1 \) and \( A_2 \) correspond respectively to the maps \( S \) and \( T \). So \( A_1 \) and \( A_2 \) have a unique common fixed point \( h^* \) that is, \( h^*(x) \) is the unique bounded common solution of the functional equations (3.4.1), \( i = 1, 2 \).

**Corollary 3.4.2.** Suppose that the following conditions hold:

(i) \( G_1, G_2, g_1 \) and \( g_2 \) are bounded.

(ii) There exists \( r \in [0, 1) \) such that for every \( (x, y) \in W \times D \), \( h, k \in B(W) \) and \( t \in W \),
\[
\min\{|h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|\} \leq |h(t) - k(t)|
\]
implies
\[
|G_1(x, y, h(t)) - G_2(x, y, k(t))| \leq r m_3(A_1 h(t), A_2 k(t))
\]
where $A_1, A_2$ are defined as follows:

$$A_i h(x) = \sup_{y \in D} G_i(x, y, h(x, y)), \quad x \in W, \ h \in B(W), \ i = 1, 2.$$ 

Then the functional equation (3.4.1) possesses a unique bounded solution in $W$.

The following result generalizes a recent result of Singh and Mishra [228, Cor. 4.2] which in turn extends certain results from [17], [21] and [25].

**Corollary 3.4.3.** Suppose that the following conditions hold.

(i) $G$ and $g$ are bounded.

(ii) There exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\varphi(r) |h(t) - Kh(t)| \leq |h(t) - k(t)|$$

implies

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq r m_4(K, h(t), k(t)),$$

where $K$ is defined as

$$Kh(t) = \sup_{y \in D} \{g(t, y) + G(t, y, h(t, y))\}, \quad t \in W, \ h \in B(W).$$

Then the functional equation (3.4.1) with $G_1 = G_2 = G$ and $g_1 = g_2 = g$ possesses a unique bounded solution in $W$.

**Proof.** It comes from Theorem 3.4.1 when $g_1 = g_2 = g$ and $G_1 = G_2 = G$. □

**Corollary 3.4.4.** Suppose that the following conditions hold.

(i) $G$ and $g$ are bounded.

(ii) There exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$|h(t) - Kh(t)| \leq |h(t) - k(t)|$$

implies

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq r m_2(K, h(t), k(t)),$$

where $K$ is defined as

$$Kh(t) = \sup_{y \in D} \{g(t, y) + G(t, y, h(t, y))\}, \quad t \in W, \ h \in B(W).$$

Then the functional equation (3.4.1) with $H_1 = H_2 = G$ and $g_1 = g_2 = g$ possesses a unique bounded solution in $W$. 

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