Chapter 1

Introduction

1.1 Some fundamental results in fixed point theory

Fixed point theory is an exciting area of inquiry which started formally in the beginning of twentieth century as an important part of nonlinear analysis, although basic ideas of fixed points are found in the work of Augustin-Louis Cauchy (1789-1857), the origin of fixed point theory lies in the method of successive approximations for proving existence of solutions of differential equations introduced independently by Joseph Liouville (1809-1882) in 1837 and Charles Emile Picard (1856-1941) in 1890 (see also, Granas and Dugundji [73], Kirk and Sims [116] and Zeilder [264]).

Let $X$ be a nonempty set and $T$ a self-map of $X$. If there exists a point $x \in X$ such that $x = Tx$, then $x$ is called a fixed point (also called stationary point or a rest point or an invariant point or an equilibrium point) of $T$. A point $z \in Y$ is called a coincidence point of maps $S,T : Y \to X$ if $Sz = Tz$, where $Y$ and $X$ are any nonempty sets.

On historical point of view, the major classical result in fixed point theory is due to L. E. J. Brouwer (1912), which states that “a continuous map on a closed unit ball in $\mathbb{R}^n$ has a fixed point”.

This result is a special case of Schauder’s fixed point theorem (1930): “A continuous map on a convex compact subspace of a Banach space has a fixed point”.

However, the simplest and most used tool in nonlinear analysis is the classical Banach contraction theorem (cf. Theorem 1.1.1) (1912) due to Polish mathematician Stefan Banach (1882-1945).

There was a huge development on this line which has a tremendous impact on all branches of applicable mathematics and mathematical sciences. For details, one may refer to Agarwal et al. [5], Berinde [22], Bonsall [27], Geobel and Kirk [71], Gorniewicz [72], Granas and Dugundji [73], Istratescu [81], Kirk and Sims [116], Rhoades [191], [192], [193], [194], Rhoades et al. [195], [196], Rus et al. [201] and references thereof.

**Definition 1.1.1.** A map $T$ on a metric space $X$ is a **contraction** if there exists
$r \in [0, 1)$ such that
\begin{equation}
    d(Tx, Ty) \leq rd(x, y) \quad (1.1.1)
\end{equation}
for all $x, y \in X$.

Banach states the following theorem popularly known as Banach contraction theorem (Bct).

**Theorem 1.1.1.** A contraction on a complete metric space has a unique fixed point.

A map $T$ is called nonexpansive if $r = 1$ in (1.1.1) and Lipschitz if $r \geq 0$.

Notice that contraction $\Rightarrow$ nonexpansive $\Rightarrow$ Lipschitz, while its reverse implication is not true.

The Banach contraction theorem (Bct) is one of the most fascinating and classical result of the last century in the field of nonlinear analysis which provides a powerful technique for solving a variety of problems in mathematical sciences and engineering. The versatility of the applications of the Bct can be judged by the following quote as given by Peitgen et al. [174, p. 284]:

“If the works and achievements of mathematicians could be patented, then the contraction mapping theorem would probably be among those with the highest earnings up to now and the future”.

The literature of last sixty years abounds with papers which establish various generalizations of the Bct either by weakening the contractive properties of the map or by extending the structure of the ambient space (see, for instance, [32], [84], [88], [106], [107], [115], [134], [149], [188], [189], [237], [238], [239], [240], [241], [243], [251], [253], [254], [260] and others).

**Definition 1.1.2.** A selfmap $T$ of a metric space is strictly contractive if
\begin{equation}
    d(Tx, Ty) < d(x, y) \quad (1.1.2)
\end{equation}
for all distinct $x, y \in X$.

A map $T$ satisfying (1.1.2) need not have a fixed point on a complete metric space (see, for instance, [161] and [162]).

The following important result is due to Edelstein [64].

**Theorem 1.1.2.** A map $T$ satisfying (1.1.2) has a unique fixed point on a compact metric space $X$. 
We remark that the iterates of $T$ satisfying (1.1.2) need not converge, and if they converge then they converge to a unique fixed point. Moreover, $T$ satisfying (1.1.2) need not have a fixed point in a complete metric space. For example, let $X = [1, \infty)$ and $Tx = x + 1/x$. Then (1.1.2) holds for all distinct $x, y$ in $X$. Evidently, $T$ has no fixed points. So the requirement of compactness is essential.

Kannan [105] (see also, [187], [209] and [242]) proved a fixed point theorem for maps on metric spaces, which is an extension of the Banach contraction. Indeed, he was the first to propose a fixed point theorem for a discontinuous map on a metric space.

Consider a map $T : X \to X$ such that

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \quad (1.1.3)$$

for all $x, y \in X$, where $0 \leq k < 1/2$.

The condition (1.1.3) is called Kannan contraction.

**Theorem 1.1.3.** A Kannan contraction on a complete metric space has a unique fixed point.

**Definition 1.1.3.** A map $T$ on a metric space $X$ is said to be a generalized contraction if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \quad (1.1.4)$$

for all $x, y \in X$.

Notice that the generalized contraction (1.1.4) is essentially due to Ćirić [40] which is listed as (21') in a comprehensive comparison of maps by Rhoades [191].

Kannan’s theorem motivated numerous extensions and generalizations of the Bct and his own fixed point theorem on various settings. One of the best generalizations, among contractions for single-valued maps is quasi-contraction given by Ćirić [42]:

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \quad (1.1.5)$$

for all $x, y \in X$, where $0 \leq k < 1$.

The following fixed point theorem due to Ćirić [42] is popularly called quasi contraction theorem in metric fixed point theory.

**Theorem 1.1.4.** A quasi contraction on a complete metric space has a unique fixed point.

For an excellent comparison of various contractive conditions for one and two maps, refer to Rhoades [191] (see also, [179]) in which he has listed 125 contractive
definitions and general fixed point theorems were either stated or proved. For some fundamental generalizations of the condition (1.1.1) and their comparison, one may refer to Boyd and Wong [32], Jachymski [85], [88], Kincs and Totik [115], Matkowski [134], Rakotch [179], Rhoades [191], [192], Suzuki [247] and references thereof.

A fixed point of a selfmap $T$ of $X$ can be considered as a common fixed point of $T$ and $I$, the identity map of $X$. In certain cases, $I$ can be replaced by a selfmap $S$ of $X$ to consider a common fixed point of $T$ and $S$. Motivated by this novel idea of Goebel [69] (see also Jungck [92]), we have the following definition.

**Definition 1.1.4.** Let $(X,d)$ be a metric space and $T,S : X \to X$ such that for every $x,y \in X$,

\[ d(Tx, Ty) \leq k \, d(Sx, Sy) \]  

where $0 \leq k < 1$.

Then $T, S$ satisfying (1.1.6) is generally called Goebel Jungck contraction on $X$.

Jungck [92] is credited for presenting a constructive proof regarding the existence of a common fixed point of commuting maps $S$ and $T$ satisfying (1.1.6) under some additional requirements.

Indeed, he proved that:

**Theorem 1.1.5.** Let $(X,d)$ be a complete metric space and $T,S : X \to X$ such that $T(X) \subset S(X)$ and (1.1.6). If $S$ and $T$ are commuting on $X$ and $S$ is continuous then $S$ and $T$ have a unique common fixed point, that is, there exists a unique point $z \in X$ such that $Sz = z = Tz$.

We remark that the continuity of the map $S$ in Theorem 1.1.5 implies that $T$ is also continuous. The above result was first generalized by Singh [210]. Thereafter, a new era of revolution in contractive fixed point theory started for commuting and noncommuting maps on metric spaces (see, for instance, [18], [34], [55], [131], [150], [151], [152], [153], [154], [155], [156], [172], [183], [184], [205], [250], [261], [262], [266] and others).

However, the following significantly improved version of Theorem 1.1.5 is essentially due to Singh [212] (see also Tivari and Singh [256], Singh et al. [214]) and suitable for applications while using constructive approach in solving a pair of equations. The same may be called Jungck-Singh contraction theorem for a pair of maps.

**Theorem 1.1.6.** Let $Y$ be an arbitrary nonempty set, $(X,d)$ a metric space and $T,S : Y \to X$ such that $T(Y) \subset S(Y)$ and (1.1.6). If $T(Y)$ or $S(Y)$ is a complete subspace of $X$, then $T$ and $S$ have a coincidence, i.e. there exists a point $z \in Y$ such that $Tz = Sz$. 

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Indeed, for any \( x_0 \) in \( Y \), there exists a sequence \( x_n \) in \( Y \) such that \( Sx_{n+1} = Tx_n \), \( n = 0, 1, 2, \) \( Sx_n \) converges to \( Sz \) for some \( z \) in \( Y \), and \( Tz = Sz \), that is, \( T \) and \( S \) have a coincidence at \( z \). The coincidence value \( (Sz) \) is unique.

Further, if \( Y = X \) and \( T, S \) commute (just) at \( z \), then \( T \) and \( S \) have a unique common fixed point.

In 1969, Nadler [141] initiated the study of the multivalued contractions. He used the concept of the Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction theorem as a special case. Here, we recall that a Hausdorff metric \( H \) induced by a metric \( d \) on \( X \) is given by

\[
H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},
\]

where \( A, B \in CB(X) \), \( d(x, A) = \inf_{y \in A} d(x, y) \) and \( CB(X) \) is the collection of the closed bounded subsets of the space \( X \).

If \( T \) is multivalued map on \( X \), i.e., if \( T \) is a map from \( X \) to the collection of nonempty subsets of \( X \), then a point \( x \in X \) is a fixed point of \( T \) if \( x \in Tx \).

Let \( (X, d) \) be a metric space and \( T : X \to CB(X) \). Then \( T \) satisfying the following condition is called multivalued contraction or Nadler’s multivalued contraction if there exists \( r \in [0, 1) \) such that

\[
H(Tx, Ty) \leq k d(x, y)
\]

for all \( x, y \in X \).

Nadler [141] (see also, [72], [79], [107], [116], [142] and [264]) proved the following result.

**Theorem 1.1.7.** A multivalued contraction in the sense of (1.1.7) has a fixed point in a complete metric space.

**Definition 1.1.5.** A map \( T : X \to CL(X) \) is multivalued generalized contraction iff there exists a positive number \( k < 1 \) such that for every \( x, y \in X \),

\[
H(Tx, Ty) \leq k \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.
\]

Ćirić [41] generalized Theorem 1.1.1 to multivalued generalized contractions. In view of various applications of multivalued fixed point theorems, there is a massive development of multivalued contraction theorems (see, for instance, [1], [2], [5], [7], [8], [13], [15], [29], [36], [41], [44], [45], [46], [47], [50], [51], [53], [54], [56], [57], [59], [60], [61], [66], [70], [71], [74], [77], [78], [79], [80], [82], [83], [98], [99], [100], [101], [102], [103], [104], [108], [110], [113], [117], [121], [133], [138], [139], [140], [141], [142],
1.2 Recent development in fixed point theory

Definition 1.2.1. Define a nonincreasing function $\theta : [0, 1) \to (\frac{1}{2}, 1]$ by

$$\theta(r) = \begin{cases} 
1 & \text{if } 0 \leq r \leq \frac{1}{2}(\sqrt{5} - 1) \\
\frac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5} - 1) \leq r \leq \frac{1}{\sqrt{2}} \\
\frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1.
\end{cases}$$

A map $T$ on a metric space $X$ is said to be Suzuki contraction if there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rd(x, y) \quad (1.2.1)$$

for all $x, y \in X$.

Recently, Suzuki in [246] proved the following remarkable generalization of the Bct.

Theorem 1.2.1. A Suzuki contraction on a complete metric space has a unique fixed point and the sequence of Picard iterates $\{T^n x\}$ converges to the fixed point for any $x \in X$.

Forceful nature of the Suzuki contraction theorem has inspired many researchers to present some beautiful and interesting extensions and generalizations during a small span of five years (see, for instance, [2], [11], [12], [28], [29], [54], [57], [59], [65], [111], [112], [113], [114], [139], [148], [177], [228], [229], [230], [232], [247], [248], [249] and others).

In 2009, Suzuki [248] generalized Edelstein’s Theorem 1.1.2 as follows.

Theorem 1.2.2. Let $(X, d)$ be a compact metric space and let $T$ be a map on $X$. Assume that for every $x, y \in X$

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y). \quad (1.2.2)$$

Then $T$ has a unique fixed point.
Definition 1.2.2. Define a nonincreasing function $\psi : [0, 1) \to (\frac{1}{2}, 1]$ by

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{\sqrt{2}} \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

A map $T$ on a metric space $X$ is said to be Kikkawa Suzuki Kannan contraction if there exists $\alpha \in [0, 1/2)$ such that

$$\psi(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \quad (1.2.3)$$

for all $x, y \in X$.

The following result is due to Kikkawa and Suzuki [111].

**Theorem 1.2.3.** A map satisfying (1.2.3) has a unique fixed point in a complete metric space.

For some extensions and generalizations of the above theorem, we may refer to [59], [113], [139], [228] and others.

Definition 1.2.3. Define a strictly decreasing function $\eta : [0, 1) \to (\frac{1}{2}, 1]$ by

$$\eta(r) = \frac{1}{1 + r}.$$  

A map $T$ on a metric space $X$ is said to be Kikkawa Suzuki Nadler multivalued contraction if there exists $r \in [0, 1)$ such that

$$\eta(r)d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq r d(x, y) \quad (1.2.4)$$

for all $x, y \in X$.

The following result due to Kikkawa and Suzuki [113] has been extended and generalized by [59], [113], and others.

**Theorem 1.2.4.** A map satisfying (1.2.4) has a fixed point in a complete metric space.

Definition 1.2.4. Maps $T, S : X \to X$ are said to be Kikkawa Suzuki Jungck contraction if there exists $r \in [0, 1)$ such that

$$\theta(r)d(Sx, Tx) \leq d(Sx, Sy) \text{ implies } d(Tx, Ty) \leq r d(Sx, Sy) \quad (1.2.5)$$

for all $x, y \in X$. (For the definition of $\theta(r)$, refer to Definition 1.2.1)

The following result is due to Kikkawa and Suzuki [113].
Theorem 1.2.5. Let \((X,d)\) be a complete metric space and \(S,T : X \to X\) such that \(T(X) \subset S(X)\) and (1.2.5). If \(S\) and \(T\) are commuting and \(S\) is continuous then \(S\) and \(T\) have a unique common fixed point, that is, there exists a unique point \(z \in X\) such that \(Sz = z = Tz\).

One of the most important fixed point theorems for multivalued contractions appeared in [59]. To state this theorem, we need the following definition.

Definition 1.2.5. A map \(T : X \to CL(X)\) is said to be Dorić Lazović generalized multivalued contraction if there exists \(r \in [0, 1)\) such that

\[
\varphi(r)d(x, Tx) \leq d(x, y)
\]

implies

\[
H(Tx, Ty) \leq r \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}
\]

(1.2.6)

for all \(x, y \in X\). (For the definition of \(\varphi(r)\), refer to Definition 3.1.3)

Dorić Lazović [59] proved that:

Theorem 1.2.6. A map \(T\) satisfying (1.2.6) has a fixed point.

Definition 1.2.6. Maps \(T : Y \to CL(X)\) and \(S : Y \to X\) are said to be Singh Mishra Zamfirescu Jungck hybrid contraction if there exists \(r \in [0, 1)\) such that

\[
\eta(r)d(Sx, Tx) \leq d(Sx, Sy)
\]

implies

\[
H(Tx, Ty) \leq r \max \left\{ d(Sx, Sy), \frac{d(Sx, Tx) + d(Sy, Ty)}{2}, \frac{d(Sx, Ty) + d(Sy, Tx)}{2} \right\}
\]

(1.2.7)

for all \(x, y \in X\). (For the definition of \(\eta(r)\), refer to Definition 1.2.3)

Indeed, Singh and Mishra [228] proved that:

Theorem 1.2.7. Let \((X,d)\) be a metric space and \(T : Y \to CL(X), S : Y \to X\) such that \(T(Y) \subset S(Y)\) and (1.2.7). If one of \(T(Y)\) or \(S(Y)\) is a complete subspace of \(X\). Then \(T\) and \(S\) have a coincidence point, that is, there exists a point \(z \in Y\) such that \(Sz \in Tz\).

Further, if \(Y = X\), \(T\) and \(S\) have a common fixed point provided that \(T\) and \(S\) are IT-commuting at \(z\) and \(Sz\) is a fixed point of \(S\).
1.3 Weak Contraction

Weakly contractive maps introduced by Alber and Guerre-Delabriere [10], is a wider class of maps which contains the classical Banach contraction as a special case and is closely related to the nonlinear contractions of Boyd-Wong and Reich type.

Alber and Guerre-Delabriere [10] generalized the condition (1.1.1) and coined the concept of weakly contractive maps (see also, [48], [222], [252] and others).

Definition 1.3.1. [194]. A map $T : X \to X$ is weakly contractive if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \varphi_1(d(x, y)),$$

where $\varphi_1 : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that $\varphi_1$ is positive on $(0, \infty)$ and $\varphi_1(0) = 0$.

Alber and Guerre-Delabriere [10] used the concept of weakly contractive maps for obtaining their results in the setting of Hilbert spaces. However, they remarked that their results are true at least for uniformly smooth and uniformly convex Banach spaces (see [194, p. 2684]). Rhoades [194] extended some of the work of Alber and Guerre-Delabriere [10] to arbitrary Banach spaces and obtained an interesting generalization of the Bct in the framework of a metric space for weakly contractive maps. Precisely, he obtained the following result.

Theorem 1.3.1. Let $T : X \to X$ be a weakly contractive map. Then $T$ has a unique fixed point provided that $X$ is complete.

Notice that weakly contractive maps contain contractions, as the special case $(\varphi_1(t) = (1 - k)t)$.

1.4 Meir-Keeler Contraction

The Meir-Keeler contraction, an important generalization of the classical Banach contraction has received enormous attention during the last four decades.

Rakotch [179] generalized the Bct by replacing the constant $k$ in (1.1.1) by a real-valued function. Indeed, he considered

$$d(Tx, Ty) \leq \phi(d(x, y))d(x, y), x, y \in X,$$

where $\phi : R^+ \to [0, 1)$ is a monotonically decreasing function, where $R^+$ denotes the set of non-negative real numbers.

Given a function $\phi : R^+ \to R^+$ such that $\phi(t) < t$ for $t > 0$, and a self map $T$ of $X$. Then we say that $T$ is $\phi$-contractive (see, for example, Jachymski [87]) if

$$d(Tx, Ty) \leq \phi d(x, y), x, y \in X.$$
In general, $\phi$ is called a contractive gauge function (see [84]). Various classes of gauge functions have been considered to generalize the result of Rakotch [179]. Browder [33] obtained a result for a complete bounded metric space satisfying the condition (1.4.2), where $\phi : R^+ \to R^+$ is non-decreasing and continuous from the right.

Browder’s result was immediately generalized by Boyd and Wong [32]. They relaxed the requirement of boundedness of the space and, instead, assumed $\phi : R^+ \to R^+$ to be upper-semi continuous from the right (not necessarily nondecreasing) such that $T$ is $\phi$-contractive (see also Kirk and Sims [116] and Lim [123]). On the other hand, Matkowski [135] generalized Browder’s result by taking $\phi$ to be non-decreasing (not necessarily upper semicontinuous) such that (1.4.2) and the following condition (1.4.3) are satisfied:

$$\phi^n(t) = 0 \quad \text{as} \quad n \to \infty, \quad t \in R^+. \quad (1.4.3)$$

We remark that the classes of contractive gauge functions studied by Boyd and Wong [32] and Matkowski [135] are independent (see Jachymski [86, p. 2328, p. 2334] and Jachymski [87, p.151]. For an excellent discussion on this aspect, one may refer to Singh et al.

A somewhat different approach to generalize the Bct which received substantial attention was adopted by Meir-Keeler [137]. Precisely, they obtained the following impressive result.

**Theorem 1.4.1.** Assume $T : X \to X$ satisfying the following condition:

For a given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\epsilon \leq d(x, y) < \epsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \epsilon. \quad (1.4.4)$$

Then $T$ possesses a unique fixed point provided that $X$ is complete.

We remark that Theorem 1.4.1 significantly generalizes the results of Browder [33] and Boyd and Wong [32]. However, it is interesting to note that the result of Matkowski [135, Th. 1.2] is independent of Theorem 1.4.1 (see [84]). Further, the Meir-Keeler contraction (1.4.3) is equivalent to the contractive gauge function $\phi$ defined as $\phi(d(Tx, Ty)) \leq d(x, y)$ for all $x, y \in X$, where $\phi$ is lower semicontinuous from the right on $[0, \infty)$ such that $\phi(t) > t$ for $t > 0$ (see Wong [263] and Jachymski [84]).

Theorem 1.4.1 was subsequently generalized, among others, by Chung [39], Hegedus and Sizilagyi [76], Jachymski [84], [85], Leader [122], Maiti and Pal [132], Matkowski and Wegrzyk [136], Rao and Rao [182], Suzuki [244], [245], [246], Tan and Minh [254], and Tomar [258] for a selfmap (see also, [123], [165], [170], [221], and [244]).

For the comparison of various Meir-Keeler type conditions, one may refer to Jachymski [85] (see also Park [163]). Some interesting variants of the condition (1.4.3)
have also been discussed by Jachymski [85]. For a detailed study of Meir-Keeler multivalued contractions, one may refer to Singh.

In due course of time, a number of Meir-Keeler type fixed point theorems have been obtained with various weaker forms of commuting maps (see [38], [89], [90], [93], [95], [150], [151], [156], [157], [160], [170], [181], [193], [195], [215], [221], [253] and [258]). However, using entirely a different approach, recently Suzuki [246] obtained the following important result which is indeed an extension of a recent generalization of the Bct by Jungck [92] and Theorem 1.4.1.

**Theorem 1.4.2.** Let \((X, d)\) be a complete metric space. Assume \(T : X \to X\) satisfying the following condition:

for each \(\varepsilon > 0\), there exists a \(\delta > 0\) such that for all \(x, y \in X\),

\[
\frac{1}{2} d(x, Tx) < d(x, y) \quad \text{and} \quad d(x, y) < \varepsilon + \delta \quad \text{imply} \quad d(Tx, Ty) \leq \varepsilon;
\]

\[
\frac{1}{2} d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y).
\]

Then \(T\) has a unique fixed point in \(X\).

Along these lines, Kikkawa Suzuki [113] and others have generalized above theorem for a pair of maps.

### 1.5 A brief summary

The main work in this thesis centers around the development of Suzuki type various contractions for single-valued and multivalued maps on metric spaces. We investigate existence of coincidence and common fixed points of maps for such contractions. Our results extend, unify and generalize several known results as well.

The work of this thesis is organized in seven chapters.

The first chapter is introductory and present a background material needed for the rest of the chapters.

The work of second chapter is largely based on Suzuki type Hardy Roger contraction for a pair of single-valued maps, generalizing (Bct) and some other results by Chatterjea [37], Dhompongsa and Yingtaweesittikul [57], Hardy and Roger [75], Kannan [106], Reich [188], Rus [198], Singh and Mishra [229], Suzuki [246] and Wong [261]. Also, we apply our results to obtain fixed points theorems for multivalued maps.

The intent of the third chapter is to present common fixed point theorems generalizing recent results of Kikkawa and Suzuki [113], Singh and Mishra [228], Đorić
and Lazović [59] and others. The work of this chapter is largely based on Suzuki-Zamfirescu contraction and Ćirić Suzuki generalized contraction for a pair of multi-valued maps.

In Chapter IV, we use Ćirić-Suzuki generalized contractions for two pairs of hybrid maps. Our results extend and generalize various results of [59], Kikkawa and Suzuki [113], Singh and Mishra [223], Singh and Mishra [230], Singh et al. [231] and others.

Chapter Vth reveals coincidence and common fixed point theorems for a multivalued and two single-valued maps satisfying Fisher type Ćirić-Suzuki generalized contraction are obtained. Our results improve and extend some known results.

Chapter VI is devoted to the study of generalized weak contractions. By combining the idea of Suzuki contraction [246] and generalized weak contraction [58], we obtain a new type of fixed point theorem generalizing the results of Ćirić [42], Dutta and Choudhary [62], Rhoades [194] and others.

The study of fixed points obtained by combining the idea of Suzuki contraction and Meir-Keeler contraction is relatively a new area of study. Suzuki [246] has initiated work along these lines for a single-valued map. In seventh chapter, we extend his result for a pair of maps in a more general way to obtain a unique common fixed point.

Besides examples and illustrations, applications regarding the existence of solutions of functional equations arising in dynamic programming, are discussed. Certain applications presented in this thesis are also demonstrated by examples.