Chapter 7

A Common Fixed Point Theorem for Suzuki-Meir-Keeler Contractions

In this chapter, The main result is an extension of a recent Meir-Keeler type fixed point theorem of Suzuki (2008) to a pair of maps on a metric space.
7.1 Introduction

The following important result due to Meir and Keeler [137] is a generalization of the classical Banach contraction theorem.

**Theorem 7.1.1.** A selfmap $T$ of a complete metric space $(X, d)$ satisfying the condition:

for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon,$$  

possesses a unique fixed point.

A map $T$ satisfying the condition (7.1.1) is popularly called Meir-Keeler contraction (see, for instance, [137] and [221]). The elegant technique employed to prove Theorem 7.1.1 attracted several authors to work along these lines and subsequently Theorem 7.1.1 was generalized and extended in various ways (see, for instance, [6], [9], [39], [68], [85], [95], [113], [120], [123], [157], [158], [159], [160], [164], [165], [170], [180], [181], [195], [202], [221], [246], and references of [85] and [221]).

Entirely different and an ingenious approach to generalize Theorem 7.1.1 was adopted by Suzuki [246] to obtain the following result.

**Theorem 7.1.2.** Let $T$ be a selfmap of a complete metric space $(X, d)$ such that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ and } d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) \leq \varepsilon; \quad (7.1.2)$$

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y). \quad (7.1.3)$$

Then $T$ has a unique fixed point in $X$.

The purpose of this chapter is to obtain an extension of Theorem 7.1.2 for a pair of maps on a metric space.

7.2 Main Results

Throughout the chapter we denote by $N$ the set of positive integers.

**Theorem 7.2.1.** Let $X$ be a complete metric space and let $S, T : X \to X$. Assume
that for each \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( x,y \in X \),

\[
\frac{1}{2} \min \{d(x,Sx),d(y,Ty)\} < d(x,y) \quad \text{and} \quad d(x,y) < \varepsilon + \delta \quad \text{implies} \quad d(Sx,Ty) \leq \varepsilon; \quad (7.2.1)
\]

and

\[
\frac{1}{2} \min \{d(x,Sx),d(y,Ty)\} < d(x,y) \quad \text{implies} \quad d(Sx,Ty) < d(x,y). \quad (7.2.2)
\]

Then there exists a unique element \( z \in X \) such that \( Sz = z = Tz \), that is, \( z \) is the unique common fixed point of \( S \) and \( T \).

**Proof.** We assert that
(i) \( d(x,Sx) > 0 \) implies \( d(Sx,TSx) < d(x,Sx) \), \( x \in X \), and
(ii) \( d(y,Ty) > 0 \) implies \( d(STy,Ty) < d(y,Ty) \), \( y \in X \).

Evidently it is enough to prove (i).

For any \( x \in X \), if \( d(x,Sx) > 0 \) then

\[
\frac{1}{2} d(x,Sx) < d(x,Sx). \quad (7.2.3)
\]

Now if

\[
d(x,Sx) < d(Sx,TSx) \quad (7.2.4)
\]

then by (7.2.2), \( d(Sx,TSx) < d(x,Sx) \), a contradiction to (7.2.4).

So \( d(Sx,TSx) < d(x,Sx) \). If

\[
d(Sx,TSx) = d(x,Sx) \quad (7.2.5)
\]

then by (7.2.2), \( d(Sx,TSx) < d(x,Sx) \), a contradiction to (7.2.5).

Thus

\[
d(Sx,TSx) < d(x,Sx) \quad (7.2.6)
\]

holds for all \( x \in X \) with \( Sx \neq x \).

Pick \( u_0 \in X \). Construct a sequence \( \{u_n\} \) in \( X \) such that

\[
u_1 = Tu_0, u_2 = Su_1, ..., u_{2n+1} = Tu_{2n}, u_{2n} = Su_{2n-1}, n \in N.
\]

If for any \( n, Su_{2n-1} = u_{2n-1} \), then \( u_{2n-1} \) is a fixed point of \( S \).

So we take \( Su_{2n-1} \neq u_{2n-1} \) for all \( n \in N \).
So by (i),
\[ d(Su_{2n-1}, Tu_{2n-1}) < d(u_{2n-1}, Su_{2n-1}), \quad \text{that is,} \]
\[ d(u_{2n}, u_{2n+1}) < d(u_{2n-1}, u_{2n}). \] (7.2.7)

If for any \( n, Tu_{2n} = u_{2n} \), then \( u_{2n} \) is a fixed point of \( T \). So we take \( Tu_{2n} \neq u_{2n} \) for all \( n \in N \).
Therefore by (ii),
\[ d(STu_{2n}, Tu_{2n}) < d(u_{2n}, Tu_{2n}), \quad \text{that is,} \]
\[ d(u_{2n+2}, u_{2n+1}) < d(u_{2n}, u_{2n+1}). \] (7.2.8)

Hence by (7.2.7) and (7.2.8), we have for all \( n \in N \),
\[ d(u_n, u_{n+1}) < d(u_{n-1}, u_n). \] (7.2.9)

Since the sequence \( \{d(u_n, u_{n+1})\} \) is strictly decreasing and bounded below by 0, \( \{d(u_n, u_{n+1})\} \) converges to some \( \alpha \geq 0 \). Assume \( \alpha > 0 \). Since \( \{d(u_n, u_{n+1})\} \) is strictly decreasing,
\[ d(u_n, u_{n+1}) > \alpha, n \in N. \] (7.2.10)

Now by (7.2.7),
\[ \frac{1}{2} d(u_{2n}, u_{2n+1}) < d(u_{2n-1}, u_{2n}), \]
that is,
\[ \frac{1}{2} \min\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\} < d(u_{2n-1}, u_{2n}), \]
that is,
\[ \frac{1}{2} \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} < d(u_{2n-1}, u_{2n}). \] (7.2.11)

By the definition of \( \alpha \), there exists \( 2n - 1 \in N \) such that
\[ d(u_{2n-1}, u_{2n}) < \alpha + \delta. \] (7.2.12)

Then in view of (7.2.1), (7.2.11) and (7.2.12) yield \( d(u_{2n}, u_{2n+1}) \leq \alpha \). This is a contradiction to (7.2.10). So our assumption \( \alpha > 0 \) is wrong. Hence \( \alpha = 0 \), and we
have proved that
\[ \lim_{n \to \infty} d(u_n, u_{n+1}) = 0. \]

Now fix \( \varepsilon > 0 \). Then there exists
\[ \delta \in (0, \varepsilon) \] (\( i.e. \delta > 0 \)) (7.2.13)

such that by (7.2.1),
\[ \frac{1}{2} \min\{d(x, Sx), d(y, Ty)\} < d(x, y) \text{ and } d(x, y) < \varepsilon + \delta \implies d(Sx, Ty) \leq \varepsilon. \]

Since \( \delta > 0 \) and \( \lim_{n \to \infty} d(u_n, u_{n+1}) = 0 \), there exists \( p \in N \) such that
\[ d(u_n, u_{n+1}) < \delta \text{ for all } n \geq p. \] (7.2.14)

Now we show by induction that
\[ d(u_p, u_{p+q}) < \varepsilon + \delta, q \in N. \] (7.2.15)

Notice that from (7.2.14), \( d(u_p, u_{p+1}) < \delta \).

Since \( \varepsilon > 0 \), we obtain \( d(u_p, u_{p+1}) < \delta < \varepsilon + \delta \). Hence (7.2.15) holds for \( q = 1 \).

Now we consider the following two cases:
(a) \( d(u_p, u_{p+q}) \leq \varepsilon \), and
(b) \( d(u_p, u_{p+q}) > \varepsilon \).

In the case (a), we have by triangle inequality,
\[ d(u_p, u_{p+q+1}) \leq d(u_p, u_{p+q}) + d(u_{p+q}, u_{p+q+1}) \leq \varepsilon + d(u_{p+q}, u_{p+q+1}). \]

So by (7.2.14), \( d(u_p, u_{p+q+1}) < \varepsilon + \delta \).

Thus in the case (a), (7.2.15) holds for all \( q \in N \).

In the case (b), we have
\[ \varepsilon < d(u_p, u_{p+q}) < \varepsilon + \delta. \] (7.2.16)

Combining (7.2.13) and (7.2.14), we have
\[ d(u_p, u_{p+1}) < \delta < \varepsilon. \] (7.2.17)

Combining (7.2.16) and (7.2.17), we obtain
\[ d(u_p, u_{p+1}) < \delta < \varepsilon < d(u_p, u_{p+q}). \] (7.2.18)
Clearly from (7.2.16), $d(u_p, u_p + q) > 0$. So

$$d(u_p, u_{p+q}) < 2d(u_p, u_{p+q}).$$  \hfill (7.2.19)

Combining (7.2.18) and (7.2.19), we obtain

$$d(u_p, u_{p+1}) < 2d(u_p, u_{p+q}).$$

Therefore

$$\frac{1}{2} \min \{d(u_p, u_{p+1}), d(u_{p+q}, u_{p+q+1})\} < d(u_p, u_{p+q}) \quad \text{and} \quad d(u_p, u_{p+q}) < \varepsilon + \delta.$$ 

Hence by (7.2.1),

$$d(u_{p+1}, u_{p+q+1}) \leq \varepsilon. \hfill (7.2.20)$$

Using (7.2.14) and (7.2.20) in the triangle inequality

$$d(u_p, u_{p+q+1}) \leq d(u_p, u_{p+1}) + d(u_{p+1}, u_{p+q+1}),$$

we obtain

$$d(u_p, u_{p+q+1}) < \varepsilon + \delta.$$ 

Therefore by induction, (7.2.15) holds for every $q \in N$ in case (b) as well, and we conclude that $d(u_p, u_{p+q}) < \varepsilon + \delta$ for every $q \in N$.

Consequently $\lim_{n \to \infty} \sup_{q>n} d(u_n, u_q) = 0$.

Therefore $\{u_n\}$ is a Cauchy sequence. Since $X$ is complete, the sequence $\{u_n\}$ has a limit in $X$. Call it $z$.

Now we show that $z$ is a common fixed point of $S$ and $T$.

Since $u_{2n} \neq Tu_{2n}$ for all $n \in N$, the sequence $\{d(u_{2n}, u_{2n+1})\}$ is strictly decreasing.

If we assume that

$$d(u_{2n}, u_{2n+1}) \geq 2d(u_{2n}, z) \quad \text{and} \quad d(u_{2n+1}, u_{2n+2}) \geq 2d(u_{2n+1}, z)$$

holds for some $n \in N$, then we have

$$d(u_{2n}, u_{2n+1}) \leq d(u_{2n}, z) + d(z, u_{2n+1}) \leq d(u_{2n}, u_{2n+1}) + d(u_{2n+1}, u_{2n+2}),$$

that is, $d(u_{2n}, u_{2n+1}) \leq d(u_{2n+1}, u_{2n+2})$, a contradiction.

So for any $n \in N$, either

(c) $d(u_{2n}, u_{2n+1}) < 2d(u_{2n}, z)$, or

(d) $d(u_{2n+1}, u_{2n+2}) < 2d(u_{2n+1}, z)$.
First we assume that (c) is true and consider the following two cases.

**Case 1:** If $d(u_{2n}, Tu_{2n}) < d(z, Sz)$, then

$$\frac{1}{2} \min \{d(u_{2n}, Tu_{2n}), d(z, Sz)\} < \frac{1}{2} d(u_{2n}, Tu_{2n})$$

$$= \frac{1}{2} d(u_{2n}, u_{2n+1})$$

$$< d(u_{2n}, z).$$

So by (7.2.2), $d(Tu_{2n}, Sz) < d(u_{2n}, z)$, that is, $d(u_{2n+1}, Sz) < d(u_{2n}, z)$.

**Case 2:** If $d(z, Sz) < d(u_{2n}, Tu_{2n})$, then

$$\frac{1}{2} \min \{d(u_{2n}, Tu_{2n}), d(z, Sz)\} < \frac{1}{2} d(z, Sz)$$

$$< \frac{1}{2} d(u_{2n}, Tu_{2n})$$

$$< d(u_{2n}, z).$$

So by (7.2.2), $d(Tu_{2n}, Sz) < d(u_{2n}, z)$, that is, $d(u_{2n+1}, Sz) < d(u_{2n}, z)$.

So in either of the two cases (1 and 2), we obtain

$$d(u_{2n+1}, Sz) < d(u_{2n}, z)$$

holds for all $n \in N$.

Passing to the limit, this yields $Sz = z$. Analogously, $Tz = z$.

Now we suppose (d) is true. Proceeding as in case (c), one can show that $z$ is fixed point of $T$, and $z$ is a fixed point of $S$ as well.

Thus this is completely proved that $z$ is common fixed point of $S$ and $T$.

Now we prove the uniqueness of the common fixed point. Suppose $y$ is another common fixed point of $S$ and $T$. Then

$$\frac{1}{2} \min \{d(z, Sz), d(y, Ty)\} = 0 < d(z, y) \implies d(Sz, Ty) < d(z, y),$$

that is, $d(z, y) = d(Sz, Ty) < d(z, y)$. This contradiction proves that $y = z$. 

We remark that Suzuki-Meir-Keeler contraction $S$, viz. $S$ satisfying (7.1.2) and (7.1.3) is obtained from the conditions (7.2.1) and (7.2.2) with $S = T$. Hence we have the following.

**Corollary 7.2.2.** **Theorem 7.1.2**