Chapter 6

A Suzuki Type Fixed Point Theorem for Generalized Weak Contractions

In this chapter we obtain a fixed point theorem for a generalized weak contractive map in a metric space generalizing recent results of Dorić [58, Theorem 2.2], Zhang and Song [265, Corollary 2.2] and others.
6.1 Introduction

For the sake of brevity, we follow the following notations, wherein $T$ is a map to be defined specifically in a particular context, while $x$ and $y$ are elements of specific domain:

$$m_4(Tx,Ty) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2} \right\},$$

$$M''(Tx,Ty) = \max \{ d(x,y), d(x,Tx), d(y,Ty), d(y,Tx) \},$$

$$M(Sx,Ty) = \max \left\{ d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty) + d(y,Sx)}{2} \right\}.$$

The classical Banach contraction theorem has numerous extensions and generalizations (see, for instance, [14], [15], [30], [31], [41], [42], [49], [51], [53], [58], [59], [61], [62], [63], [66], [69], [70], [94], [111], [112], [113], [114], [115], [147], [177], [191], [194], [200], [203], [207], [208], [216], [222], [230], [231], [246], [248] and others). The following important generalization is due to Ćirić [41].

**Theorem 6.1.1.** Let $X$ be a complete metric space and $T : X \to X$. Assume there exists $r \in [0,1)$ such that for every $x,y \in X$,

$$d(Tx,Ty) \leq r \ m_4(Tx,Ty).$$

Then $T$ has a unique fixed point.

A map $T$ satisfying (6.1.1) is called a generalized contraction.

The following result is quasi-contractions theorem given by Ćirić [42] and is considered the most general contraction theorem in metric fixed point theory (cf. [115], [191], [194] and [200]).

**Theorem 6.1.2.** Let $X$ be a complete metric space and $T : X \to X$. Assume there exists $r \in [0,1)$ such that for every $x,y \in X$,

$$d(Tx,Ty) \leq r \ M''(Tx,Ty).$$

Then $T$ has a unique fixed point.

Notice that (6.1.1) implies (6.1.2), that is, $T$ satisfying the condition (6.1.1) also satisfies (6.1.2). We remark that (6.1.1) is listed as the condition (21') and (6.1.2) as the condition (24) in a comprehensive comparison of contractive conditions given by Rhoades [191] (see also [115]).

Recently, Suzuki [247, Theorem 2], obtained a powerful generalization of the Banach contraction theorem, and the same has been extended in various ways (see, for
instance, [2], [4], [11], [12], [28], [29], [54], [57], [59], [65], [111], [112], [113], [114], [139], [148], [177], [228], [229], [230], [232], [247], [248], [249] and others). Using the idea of the Suzuki contraction [246] (see also [230]) and the generalized contraction (6.1.1), Dorić and Lazović [59, Cor. 2.3], obtain the following generalization of Theorem 6.1.1 in the following manner. 

\textbf{Theorem 6.1.3.} Let \(X\) be a complete metric space and \(T : X \to X\) such that for every \(x, y \in X\),

\[ \varphi(r)d(x, Tx) \leq d(x, y) \] \(\implies\) \(d(Tx, Ty) \leq r m_4(Tx, Ty). \) (6.1.3)

Then \(T\) has a unique fixed point.

The well known Banach contraction theorem and its several extensions have been generalized using recently developed notion of weakly contractive maps. For weak and generalized weak contractions, one may refer to [3], [14], [15], [16], [23], [28], [30], [58], [178], [194], [197], [236], [265] and others. The following basic result is due to Rhoades [194].

\textbf{Theorem 6.1.4.} Let \(X\) be a complete metric space and \(T : X \to X\) such that for every \(x, y \in X\),

\[ d(Tx, Ty) \leq d(x, y) - \varphi_1(d(x, y)), \] (6.1.4)

where \(\varphi_1 : [0, \infty) \to [0, \infty)\) is a continuous and nondecreasing function with \(\varphi_1(0) = 0\) and \(\varphi_1(t) > 0\) for all \(t > 0\). Then \(T\) has a unique fixed point.

Dutta and Choudhary [62] obtained the following generalization of Theorem 6.1.4.

\textbf{Theorem 6.1.5.} Let \(X\) be a complete metric space and \(T : X \to X\) such that for every \(x, y \in X\),

\[ \psi_1(d(Tx, Ty)) \leq \psi_1(d(x, y)) - \varphi_1(d(x, y)), \] (6.1.5)

where

\(i\) \(\psi_1 : [0, \infty) \to [0, \infty)\) is continuous and monotone nondecreasing function with \(\psi_1(t) = 0\) if and only if \(t = 0\).

\(ii\) \(\varphi_1 : [0, \infty) \to [0, \infty)\) is lower semi-continuous function with \(\varphi_1(t) = 0\) if and only if \(t = 0\).

Then \(T\) has a unique fixed point.

Further, contractive condition (6.1.5) has been found equivalent to some \((\psi_1, \varphi_1)\) - contractive conditions studied by Jachymski (for details, one may refer to [88, Theorem 3]).

Theorem 6.1.1 and Theorem 6.1.5 has generalizes by Dorić [58, Theorem 2.2] in the following manner.
Theorem 6.1.6. Let $X$ be a complete metric space and $T : X \rightarrow X$ such that for every $x, y \in X$,

$$
\psi_1(d(Tx, Ty)) \leq \psi_1(m_4(Tx, Ty)) - \varphi_1(m_4(Tx, Ty)),
$$

(6.1.6)

where $\psi_1$ and $\varphi_1$ are defined as in Theorem 6.1.5. Then $T$ has a unique fixed point.

Now the question is, whether it is possible to further generalize Theorem 6.1.6 [58, Theorem 2.2]? Our main result provides an affirmative answer to this question. Also we present a weakly contractive version of Theorem 6.1.3 and give examples to illustrate superiority over Theorems 6.1.1, 6.1.4, 6.1.5 and 6.1.6.

6.2 Main Result and Examples

The following theorem is the main result of this chapter.

Theorem 6.2.1. Let $X$ be a complete metric space and $T : X \rightarrow X$ such that for every $x, y \in X$,

$$
\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } \psi_1(d(Tx, Ty)) \leq \psi_1(m_4(Tx, Ty)) - \varphi_1(m_4(Tx, Ty)),
$$

(6.2.1)

where $\psi_1$ and $\varphi_1$ are defined as in Theorem 6.1.5. Then $T$ has a unique fixed point.

Proof. Pick $x_0 \in X$. Construct a sequence $\{x_n\}$ in $X$ such that $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$

Notice that for any $n$,

$$
\frac{1}{2}d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n).
$$

(6.2.2)

Therefore by (6.2.1), we have

$$
\psi_1(d(Tx_n, Tx_{n-1})) \leq \psi_1(m_4(Tx_n, Tx_{n-1})) - \varphi_1(m_4(Tx_n, Tx_{n-1})).
$$
By (6.2.1) and definition of $m$, we have

$$
\psi_1(d(x_{n+1}, x_n)) = \psi_1(d(Tx_n, Tx_{n-1}))
$$

$$
\leq \psi_1(m(Tx_n, Tx_{n-1})) - \varphi_1(m(Tx_n, Tx_{n-1}))
$$

$$
= \psi_1\left(\max\left\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{2}\right\}\right)
$$

$$
- \varphi_1\left(\max\left\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{2}\right\}\right)
$$

$$
= \psi_1\left(\max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})}{2}\right\}\right)
$$

$$
- \varphi_1\left(\max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})}{2}\right\}\right)
$$

So $\psi_1(d(x_{n+1}, x_n)) \leq \psi_1(\max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\right\}) - \varphi_1(\max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\right\})$

This yields $\psi_1(d(x_{n+1}, x_n)) \leq \psi_1(d(x_n, x_{n-1})) - \varphi_1(d(x_n, x_{n-1})) \leq \psi_1(d(x_n, x_{n-1}))$.

Consequently $\psi_1(d(x_{n+1}, x_n)) \leq \psi_1(d(x_n, x_{n-1}))$,

and by the property of $\psi_1$,

$$
d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}).
$$

Therefore for any $n$, we obtain $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$.

Hence the sequence $\{d(x_{n+1}, x_n)\}$ is monotonic nonincreasing and bounded below. So there exists $r \geq 0$ such that

$$
\lim_{n \to \infty} d(x_{n+1}, x_n) = r = \lim_{n \to \infty} d(x_n, x_{n-1}). \quad (6.2.3)
$$

So by the lower semi-continuity of $\varphi_1$,

$$
\varphi_1(r) = \lim_{n \to \infty} \inf \varphi_1(d(x_n, x_{n-1})).
$$

We claim that $r = 0$. In fact taking upper limits as $n \to \infty$ on either side of the following inequality:

$$
\psi_1(d(x_{n+1}, x_n)) \leq \psi_1(d(x_n, x_{n-1})) - \varphi_1(d(x_n, x_{n-1})),
$$

and using (6.2.3), this gives

$$
\psi_1(r) \leq \psi_1(r) - \varphi_1(r).
$$

So $\varphi_1(r) \leq 0$. Hence by the property of the function $\varphi_1$, $\varphi_1(r) = 0$. But $\varphi_1(r) = 0$
implies \( r = 0 \). So
\[
\lim_{n \to \infty} d(x_{n+1}, x_n) = r = 0.
\]

Next we show that \( \{ x_n \} \) is a Cauchy sequence. Let
\[
C_n = \sup \{ d(x_j, x_k) : j, k \geq n \}.
\]
Then \( \{ C_n \} \) is decreasing. If \( \lim_{n \to \infty} C_n = 0 \), then we are done.
So, assume that \( \lim_{n \to \infty} C_n = C > 0 \).
Choose \( \varepsilon < \frac{C}{8} \) small enough and select \( N \) such that for all \( n \geq N \),
\[
d(x_{n+1}, x_n) < \varepsilon \quad \text{and} \quad C_n < C + \varepsilon. \tag{6.2.4}
\]
By the definition of \( C_{N+1} \), there exist \( m, n \geq N + 1 \) such that
\[
d(x_m, x_n) > C_n - \varepsilon \geq C - \varepsilon.
\]
Replacing \( x_m \) by \( x_{m+1} \), we have
\[
d(x_n, x_{m+1}) > C - \varepsilon,
\]
that is, \( d(x_n, x_{m+1}) - d(x_{m+1}, x_m) > C - \varepsilon - d(x_{m+1}, x_m) \),
that is, \( d(x_n, x_m) \geq d(x_n, x_{m+1}) - d(x_{m+1}, x_m) > C - \varepsilon - d(x_{m+1}, x_m) \),
that is, \( d(x_m, x_n) > C - \varepsilon - d(x_{m+1}, x_m) \).
So by (6.2.4),
\[
d(x_m, x_n) > C - \varepsilon - \varepsilon,
\]
that is, \( d(x_m, x_n) > C - 2\varepsilon \).
Then \( d(x_{m-1}, x_{n-1}) > C - 4\varepsilon \) and,
since \( d(x_{m-1}, x_m) \leq d(x_{m-1}, x_{n-1}) \) and \( d(x_{n-1}, x_n) \leq d(x_{m-1}, x_{n-1}) \),
we obtain \( \frac{1}{2} d(x_{m-1}, x_m) \leq d(x_{m-1}, x_{n-1}) \).
So by the assumption,
\[
\psi_1(d(Tx_{m-1}, Tx_{n-1})) \leq \psi_1(m_4(Tx_{m-1}, Tx_{n-1})) - \varphi_1(m_4(Tx_{m-1}, Tx_{n-1})),
\]
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that is,

\[ \psi_1(d(x_m, x_n)) \]
\[ = \psi_1(d(Tx_{m-1}, Tx_{n-1})) \]
\[ \leq \psi_1 \left( \max \left\{ d(x_{m-1}, x_{n-1}), d(x_{m-1}, Tx_{m-1}), d(x_{n-1}, Tx_{n-1}), \frac{d(x_{m-1}, Tx_{n-1}) + d(x_{n-1}, Tx_{m-1})}{2} \right\} \right) \]
\[ - \varphi_1 \left( \max \left\{ d(x_{m-1}, x_{n-1}), d(x_{m-1}, Tx_{m-1}), d(x_{n-1}, Tx_{n-1}), \frac{d(x_{m-1}, Tx_{n-1}) + d(x_{n-1}, Tx_{m-1})}{2} \right\} \right) \]
\[ \leq \psi_1 \left( \max \left\{ d(x_{m-1}, x_{n-1}), d(x_{m-1}, x_m), d(x_{n-1}, x_n), \frac{d(x_{m-1}, x_n) + d(x_{n-1}, x_m)}{2} \right\} \right) \]
\[ - \varphi_1 \left( \max \left\{ d(x_{m-1}, x_{n-1}), d(x_{m-1}, x_m), d(x_{n-1}, x_n), \frac{d(x_{m-1}, x_n) + d(x_{n-1}, x_m)}{2} \right\} \right), \]

that is,

\[ \psi_1(d(x_m, x_n)) \leq \psi_1(d(x_{m-1}, x_{n-1})) - \varphi_1(d(x_{m-1}, x_{n-1})). \]

Therefore

\[ \limsup_{N \to \infty} \psi_1(C_{N+1}) \leq \limsup_{N \to \infty} \psi_1(C_N) - \liminf_{m,n \to \infty} \varphi_1(d(x_{m-1}, x_{n-1})) \]
\[ \Rightarrow \left( 0 = \limsup_{N \to \infty} \psi_1(C_{N+1}) - \limsup_{N \to \infty} \psi_1(C_N) \right) \]
\[ \leq - \liminf_{m,n \to \infty} \varphi_1(d(x_{m-1}, x_{n-1})) \]
\[ \Rightarrow \left( \frac{C}{2} \leq \liminf_{m,n \to \infty} d(x_m, x_n) = \liminf_{m,n \to \infty} d(x_{m-1}, x_{n-1}) \right. \]
\[ \left. \leq \liminf_{m,n \to \infty} \varphi_1(d(x_{m-1}, x_{n-1})) \leq 0 \right) \Rightarrow (C = 0) \]

Consequently, the sequence \( \{x_n\} \) is Cauchy. Since \( X \) is complete, it has a limit in \( X \).
Call it $z$.
Now we show that $z$ is a fixed point of $T$.
We claim that $\frac{1}{2}d(x_{2n},Tx_{2n}) \leq d(x_{2n},z)$ or $\frac{1}{2}d(x_{2n+1},Tx_{2n+1}) \leq d(x_{2n+1},z)$. Otherwise
\[
d(x_{2n},x_{2n+1}) \leq \frac{1}{2}d(x_{2n},Tx_{2n}) + \frac{1}{2}d(x_{2n+1},Tx_{2n+2}) \\
\leq \frac{1}{2}[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\
\leq \frac{1}{2}(1 + 1)d(x_{2n},x_{2n+1}) \\
= d(x_{2n},x_{2n+1})
\]
That is, $d(x_{2n},x_{2n+1}) < d(x_{2n},x_{2n+1})$ which is a contradiction.
So there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\frac{1}{2}d(x_{n_k},x_{n_k+1}) \leq d(x_{n_k},z)$. Therefore by (6.2.1),
\[
\psi_1(d(Tx_{n_k},Tz)) \leq \psi_1(m_4(Tx_{n_k},Tz)) - \varphi_1(m_4(Tx_{n_k},Tz)) \\
= \psi\left(\max\left\{d(z,x_{n_k}),d(z,Tz),d(x_{n_k},Tx_{n_k}),\frac{d(x_{n_k},Tz) + d(z,Tx_{n_k})}{2}\right\}\right) \\
- \varphi_1\left(\max\left\{d(z,x_{n_k}),d(z,Tz),d(x_{n_k},Tx_{n_k}),\frac{d(x_{n_k},Tz) + d(z,Tx_{n_k})}{2}\right\}\right).
\]
Making $n \to \infty$,
\[
\psi_1(d(z,Tz)) \leq \psi_1(d(z,Tz)) - \varphi_1(d(z,Tz)).
\]
This yields, $z = Tz$.

In order to prove the uniqueness of the fixed point $z$, suppose that $y \in X$ is another fixed point of $T$. Then
\[
\frac{1}{2}d(z,Tz) = 0 \leq d(y,z)
\]
implies
\[
\psi_1(d(y,z)) = \psi_1(d(Ty,Tz)) \leq \psi_1(m_4(Ty,Tz)) - \varphi_1(m_4(Ty,Tz)) \\
= \psi_1(d(y,z)) - \varphi_1(d(y,z)).
\]
This gives $\varphi_1(d(y,z)) \leq 0$. Hence $y = z$.
This completes the proof. \qed
Obviously, the following result is derived from Theorem 6.2.1.

**Corollary 6.2.2.** Let $X$ be a complete metric space and $T : X \rightarrow X$ such that for every $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } \psi_1(d(Tx, Ty)) \leq \psi_1(d(x, y)) - \varphi_1(d(x, y)),$$

where $\psi_1$ and $\varphi_1$ are defined as in Theorem 6.1.5.
Then $T$ has a unique fixed point.

**Corollary 6.2.3.** Let $X$ be a complete metric space and $T : X \rightarrow X$ such that for every $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y) - \varphi_1(d(x, y)),$$

where $\varphi_1$ is defined as in Theorem 6.1.5.
Then $T$ has a unique fixed point.

**Corollary 6.2.4.** Let $X$ be a complete metric space and $T : X \rightarrow X$ such that for every $x, y \in X$,

$$\varphi_1\left(\psi_1(d(Tx, Ty))\right) \leq \varphi_1\left(\psi_4(Tx, Ty)\right) - \psi_1\left(\varphi_4(Tx, Ty)\right),$$

where $\psi_1$ and $\varphi_1$ are defined as in Theorem 6.1.5.
Then $T$ has a unique fixed point.

We remark that Corollary 6.2.4 is a particular case of [222, Corollary 2.9]. Further, a result of Zhang and Song [265, Corollary 2.2] is obtained from Corollary 6.2.4 when $\psi_1(t) = t$.

The following example shows the generality of Theorem 6.2.1 over Theorems 6.1.5. Further, it is interesting to note that the map $T$ of Example 6.2.5 does not satisfy the hypothesis of Theorem 6.1.2.

**Example 6.2.5.** Let $X = \{(0, 0), (0, 4), (4, 0), (0, 5), (5, 0), (4, 5), (5, 4)\}$ be endowed with the metric $d$ defined by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Let $T$ be such that

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases}$$

Then $T$ do not satisfy the condition (6.1.2) of Theorem 6.1.2 at $x = (4, 5), y = (5, 4)$. Choose $\psi_1(t) = t$ and $\varphi_1(t) = \frac{1}{4}t$, it is readily verified that the condition (6.1.5) of
Theorem 6.1.5 is not satisfied at \( x = (4, 5), y = (5, 4) \). However, all the hypotheses of Theorem 6.2.1 are easily verified for the map \( T \).

The following Example 6.2.6 shows the generality of Theorem 6.2.1 over Theorems 6.1.6. Further, it is interesting to note that the map \( T \) of Example 6.2.6 does not satisfy the hypothesis of Theorem 6.1.3.

**Example 6.2.6.** Let \( X = \{(1, 1), (1, 5), (5, 1), (5, 6), (6, 5)\} \) be endowed with the metric \( d \) defined by

\[
d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|.
\]

Let \( T \) be such that

\[
T(x_1, x_2) = \begin{cases} 
(x_1, 1) & \text{if } x_1 \leq x_2 \\
(1, x_2) & \text{if } x_1 > x_2.
\end{cases}
\]

Choose \( \psi_1(t) = \frac{3}{4} t \) and \( \varphi_1(t) = \frac{1}{8} t \), it is easily verify that \( T \) does not satisfy the condition (6.1.6) of Theorem 6.1.6 at \( x = (5, 6), y = (6, 5) \) and \( x = (6, 5), y = (5, 6) \). Also \( T \) does not satisfy the condition (6.1.3) of Theorem 6.1.3 at \( x = (5, 6), y = (6, 5) \). However, all the hypotheses of Theorem 6.2.1 are easily verified for the map \( T \).

The following Example 6.2.7 shows the generality of Corollary 6.2.2 and 6.2.3 over Theorem 6.1.4 and 6.1.5.

**Example 6.2.7.** Let \( X = \{(0, 0), (0, 4), (4, 0), (4, 5), (5, 4)\} \) be endowed with the metric \( d \) defined by

\[
d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|.
\]

Let \( T \) be such that

\[
T(x_1, x_2) = \begin{cases} 
(x_1, 0) & \text{if } x_1 \leq x_2 \\
(0, x_2) & \text{if } x_1 > x_2.
\end{cases}
\]

Choose \( \psi_1(t) = t \) and \( \varphi_1(t) = \frac{1}{8} t \), it is readily verified that the condition (6.1.4) and (6.1.5) of Theorem 6.1.4 and Theorem 6.1.5 are not satisfied at \( x = (4, 5), y = (5, 4) \). However, all the hypotheses of Corollary 6.2.2 and Corollary 6.2.3 are easily verified for the map \( T \).

**Question 1.** Can we extend Theorem 6.2.1 for a pair of maps? Indeed, we conjecture the following:

**Theorem 6.2.8.** Let \( X \) be a complete metric space and \( S, T : X \rightarrow X \) such that for
every $x, y \in X$,

\[
\frac{1}{2} \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)
\]

implies

\[
\psi_1(d(Sx, Ty)) \leq \psi_1(M(Sx, Ty)) - \varphi_1(M(Sx, Ty)),
\]

where $\psi_1$ and $\varphi_1$ are defined as in Theorem 6.1.5. Then $S$ and $T$ have a unique common fixed point.